Optimal Assignment of Servers to Tasks when Collaboration is Inefficient

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Abstract

Consider a Markovian system of two stations in tandem with finite intermediate buffer and two servers. The servers are heterogeneous, flexible, and more efficient when they work on their own than when they collaborate. We determine how the servers should be assigned dynamically to the stations with the goal of maximizing the system throughput. We show that the optimal policy depends on whether or not one server is dominant (i.e., faster at both stations) and on the magnitude of the efficiency loss of collaborating servers. In particular, if one server is dominant then he must divide his time between the two stations and we identify the threshold policy the dominant server should use; otherwise each server should focus on the station where he is the faster server. In all cases, servers only collaborate to avoid idleness when the first station is blocked or the second station is starved, and we determine when collaboration is preferable to idleness as a function of the efficiency loss of collaborating servers.

1 Introduction

In recent years, queueing systems with flexible servers have received a lot of attention in the operations research community. Several authors have focused on the dynamic assignment of servers to tasks in order to optimize system performance (such as throughput or holding costs). Most of the literature in this area has assumed that when multiple servers are assigned to the same task, their combined service rate is additive. However, this assumption does not take into account the fact that server collaboration may or may not be synergistic. Andradóttir, Ayhan, and Down [9] have obtained the optimal server assignment policy when server collaboration is synergistic. By contrast, this paper focuses on the case where the servers lose efficiency when they work together in a team (e.g., due to bad team dynamics, lack of space or tools, etc.). Our objective is to understand how the optimal dynamic server assignment policy depends on how inefficiently the servers work together.

We focus on a queueing network with $N = 2$ stations and $M = 2$ flexible servers. There is an infinite amount of raw material in front of station 1, infinite room for departing jobs after station 2, and a finite buffer between stations 1 and 2, whose size is denoted by B. The system operates under manufacturing blocking (under which a station gets blocked at the time of a service completion if the downstream buffer is full). At any given time, there can be at most one job at each station and each server can work on at most one job. We assume that server $i \in \{1, 2\}$ works at a deterministic rate $\mu_{ij} \in [0, \infty)$ at station $j \in \{1, 2\}$. Thus, server i is trained to work at station j if $\mu_{ij} > 0$ and the server's skill at station j is measured by the magnitude of μ_{ij}. Without loss of generality, we assume that $\mu_{i1} + \mu_{i2} > 0$ (otherwise the problem reduces to having a single server) and $\mu_{1j} + \mu_{2j} > 0$ (otherwise the throughput is zero under any policy) for all $i, j \in \{1, 2\}$. Both servers can work together on a single job, in which case the combined rate of a server team is proportional to the sum of the rates of the individual servers. Thus, if both servers are simultaneously assigned to station $j \in \{1, 2\}$, their combined service rate is equal to $\alpha(\mu_{1j} + \mu_{2j})$. We assume that $0 \leq \alpha \leq 1$, which implies that the servers lose efficiency when they work in a team. The service requirements of different jobs at each station $j \in \{1, 2\}$ are independent and identically distributed (i.i.d.) exponential random variables whose rate we take to be equal to 1 without loss of generality, and the service requirements at different stations are independent of each other. We assume that travel and set-up times are negligible. Under these assumptions, our objective is to determine the dynamic server assignment policy that maximizes the long-run average throughput.

For the system described in the previous paragraph, we completely characterize the optimal server assignment policy. As described in the remainder of the paper, in this case, the optimal policy is more complicated than the one for systems with synergistic servers (where $\alpha \geq 1$, see Andradóttir, Ayhan, and Down [9]) and the structure of the optimal policy depends on whether one server is dominant (faster at both stations) or not. In particular, we show that when $\alpha \leq 1$, if there is no dominant server (i.e., each server is better than the other one at one of the stations), then servers have *primary assignments* and only leave their primary assignments if α is large enough and they have no work to do at their primary assignments. (We say that server $i \in \{1, 2\}$ has a primary assignment at station $j \in \{1, 2\}$ if server i works at station j unless he has no work at station j.) On the other hand, if there is a dominant server, the optimal policy is of threshold type (i.e., the dominant server moves from station 1 to station 2 when the number of jobs waiting in the buffer reaches a certain value). The results of this paper, together with the results of Andradóttir, Ayhan, and Down [9], provide a complete characterization of the optimal server assignment policy in Markovian systems with two stations and two servers for all $0 \leq \alpha < \infty$. In particular, Andradóttir, Ayhan, and Down [9] show that when $\alpha \geq 1$, depending on the value of α, the optimal policy switches from one that takes full advantage of servers' skills (in that both servers have primary assignments and only collaborate when they have no work at their primary assignments), to one that takes full advantage of server synergy (in that servers collaborate at all times).

There is a significant amount of literature on queues with flexible servers. In the interest of space, we do not provide a complete literature review here, but refer the interested reader to Hopp and Van Oyen [13] for a comprehensive review of the literature in this area, and to Akşin, Armony, and Mehrotra [4], Akşin, Karaesmen, and Örmeci [5], and Gans, Koole, and Mandelbaum [12] for thorough reviews of the literature on flexible servers in call centers. This paper is most closely related to other works that employ Markov decision process techniques and sample path analysis in determining effective server allocation schemes, see for example Ahn, Duenyas, and Zhang [1], Ahn and Lewis [2], Ahn and Righter [3], Andradóttir and Ayhan [6], Andradóttir, Ayhan, and Down [7, 8], Kaufman, Ahn, and Lewis [15], Örmeci [17], Sennott, Van Oyen, and Iravani [19], Van Oyen, Gel, and Hopp [20], and Wu, Lewis, and Veatch [21]. However, these papers only consider cases where the combined rate of a set of collaborating servers is additive (i.e., $\alpha = 1$).

To the best of our knowledge, Ahn and Lewis [2], Andradóttir, Ayhan, and Down [9], Argon and Andradóttir [10], and Buzacott [11] are the only papers that study systems with non-additive service rates. Also, Işık, Andradóttir, and Ayhan [14] develop optimal server assignment policies in tandem lines with non-collaborative servers (which could be considered as a form of sub-additive server rates). More specifically, in a recent paper, Ahn and Lewis [2] consider joint routing and allocation policies in a two-station parallel queueing network. When the service rates are superadditive, they completely characterize the optimal policy. However, when the rates are subadditive, they conclude that the problem is more complicated and, hence, characterize the optimal policy for special cases and develop effective heuristics. Andradóttir, Ayhan, and Down [9] are concerned with the optimal assignment of servers to tasks when server collaboration is synergistic. They investigate when it is better to take advantage of synergy among servers, rather than exploiting the servers' special skills, to achieve the best possible system throughput, and completely characterize the optimal policy for Markovian systems with two stations and two servers. On the contrary, in this paper, we focus on queues where servers lose efficiency when they work together (i.e., sub-additive service rates). Argon and Andradóttir [10] provide sufficient conditions for partial pooling of multiple adjacent queueing stations to be beneficial in tandem lines, allowing the service rate of a team of pooled servers to be additive, sub-additive, or super-additive. Finally, Buzacott [11] considers team work involving task partitioning (with the team completing work when all servers have completed their assigned subtasks) in a single stage queue with identical servers. These last two papers do not address how system performance can be optimized by dynamically assigning flexible servers to tasks.

The remainder of the paper is organized as follows. In Section 2, we provide a rigorous description of our problem and introduce the notation that will be used throughout the paper. Section 3 provides the optimal policy for Markovian systems with two stations and two servers when there is no dominant server for all $0 \leq \alpha \leq 1$. On the other hand, in Section 4, the optimal server assignment policy is provided for systems where there is a dominant server for all $0 \leq \alpha \leq 1$. Section 5 summarizes our findings and concludes the paper. Finally, the proofs of some of our results are provided in an appendix.

2 Problem Description

In this section, we define the throughput maximization problem and describe the stochastic process formulation of our model. We use Π to denote the set of all server assignment policies under consideration (defined later) and $D_\pi(t)$ to denote the number of departures under policy π by time $t \geq 0$. Define

$$T_\pi = \limsup_{t\to\infty} \frac{\mathbb{E}[D_\pi(t)]}{t}$$

as the long-run average throughput corresponding to the server assignment policy $\pi \in \Pi$. Our objective is to solve the following optimization problem

$$\max_{\pi\in\Pi} T_\pi. \tag{1}$$

For all $\pi \in \Pi$ and $t \geq 0$, let $X_\pi(t)$ denote the number of jobs that have been processed at station 1 at time t but are either waiting to be processed by station 2 or in process at station 2 at time t. Let $S = \{0, \ldots, B+2\}$ denote the state space of $\{X_\pi(t)\}$. For the remainder of this paper, we assume that the class Π of server assignment policies under consideration consists of Markovian stationary deterministic policies corresponding to the state space S. Then it is clear that for $\pi \in \Pi$, $\{X_\pi(t)\}$ is a birth-death process with state space S and that there exists a scalar $q_\pi \leq \sum_{i=1}^{2} \max_{1\leq j\leq 2} \mu_{ij} < \infty$ such that the transition rates $\{q_\pi(s, s')\}$ of $\{X_\pi(t)\}$ satisfy $\sum_{s'\in S, s'\neq s} q_\pi(s, s') \leq q_\pi$ for all $s \in S$. Hence, $\{X_\pi(t)\}$ is uniformizable. Let $\{Y_\pi(k)\}$ be the corresponding discrete time Markov chain, so that $\{Y_\pi(k)\}$ has state space S and transition probabilities $p_\pi(s, s') = q_\pi(s, s')/q_\pi$ if $s' \neq s$ and $p_\pi(s, s) = 1 - \sum_{s'\in S, s'\neq s} q_\pi(s, s')/q_\pi$ for all $s \in S$. Using the analysis in Section 3 of Andradóttir, Ayhan, and Down [7], one can show that the original optimization problem in (1) can be translated into an equivalent (discrete time) Markov decision problem. More specifically, for all $i \in S$, let

$$R_\pi(i) = \begin{cases} q_\pi(i, i-1) & \text{if } i \in \{1, \ldots, B+2\}, \\ 0 & \text{if } i = 0, \end{cases}$$

be the rate at which customers depart when the state is i under policy π. Then the optimization problem (1) has the same solution as the Markov decision problem

$$\max_{\pi\in\Pi} \lim_{K\to\infty} \mathbb{E}\left[\frac{1}{K}\sum_{k=1}^{K} R_\pi(Y_\pi(k-1))\right].$$

In other words, maximizing the steady-state throughput of the original queueing system is equivalent to maximizing the steady-state departure rate for the associated embedded (discrete time) Markov chain.

As described in Section 1, the optimal policy depends on whether one server is dominant (faster at both stations) or not. In what follows, we will describe the optimal policy in these two cases separately, starting with the case where different servers are faster at different stations.

3 Optimal Policy for Systems with No Dominant Server

Without loss of generality, assume that server 1 is faster at station 1 and server 2 is faster at station 2. Thus, $\mu_{11} \geq \mu_{21}$ and $\mu_{22} \geq \mu_{12}$. Note that this assumption on the rates implies that $\mu_{11}\mu_{22} \geq \mu_{21}\mu_{12}$. In the next four theorems, we will give a complete characterization of the optimal policies for all possible values of $0 \leq \alpha \leq 1$. Since the proofs of Theorems 3.1 to 3.4 are similar, we will only provide the proof for the most complicated case, which is given in Theorem 3.4.

We first consider the case where the combined service rate of collaborating servers at each station is slower than the rate of the faster server at that station. The next theorem states that in this case, the optimal policy does not allow collaboration and each server always works at the station that he is better trained for. Thus, the optimal policy for small α agrees with the optimal policy for non-collaborative servers identified by Işık, Andradóttir, and Ayhan [14]. Throughout the paper, δ^* denotes the optimal decision rule (i.e., $\delta^*(s)$ prescribes the optimal action in state s for all $s \in \{0, \ldots, B+2\}$) and $(\delta^*)^\infty$ is the corresponding stationary optimal policy.

Theorem 3.1 *Suppose* $0 \leq \alpha \leq \min\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{22}}{\mu_{12}+\mu_{22}}\}$. *Let*

$$\delta^*(s) = \textit{ server 1 works at station 1, server 2 works at station 2 } \quad \textit{for } 0 \leq s \leq B+2.$$

Then $(\delta^*)^\infty$ *is optimal. Moreover, this is the unique optimal policy in the class of Markovian stationary deterministic policies if all the inequalities on the service rates (i.e.,* $\mu_{11} \geq \mu_{21}$ *and* $\mu_{22} \geq \mu_{12}$*) and the upper bound on* α *are strict.*

Note that $\frac{\mu_{11}}{\mu_{11}+\mu_{21}} \leq (\geq) \frac{\mu_{22}}{\mu_{12}+\mu_{22}}$ if and only if $\mu_{11}\mu_{12} \leq (\geq)\ \mu_{21}\mu_{22}$, which can be interpreted as server 2 (1) being more effective overall. Next we assume that α is large enough to allow collaboration at one station but too small to allow collaboration at the other station. In this case, the optimal policy assigns servers to the stations where they are faster unless the station where they cannot collaborate effectively is blocked or starved, in which case both servers work at the other station. Theorem 3.2 focuses on the case where α is large enough to allow collaboration at station 2 only. Theorem 3.3 considers the case where α is large enough to allow collaboration at station 1 only. Moreover, Theorem 3.3 (Theorem 3.2) follows from Theorem 3.2 (Theorem 3.3) using the result on the reversibility of two-station tandem lines with exponential service times and flexible servers proved in Section 5 of Andradóttir and Ayhan [6]. In particular, Andradóttir and

Ayhan [6] prove that in two-station Markovian tandem lines with flexible servers, the stochastic process corresponding to the reversed line under a reversal of a policy π is stochastically equivalent to $B+2-X^{\pi}(\cdot)$.

Theorem 3.2 *Suppose* $\mu_{21}\mu_{22} \le \mu_{11}\mu_{12}$ *and* $\frac{\mu_{22}}{\mu_{12}+\mu_{22}} \le \alpha \le \frac{\mu_{11}}{\mu_{11}+\mu_{21}}$. *Let*

$$\delta^*(s) = \begin{cases} \textit{server 1 works at station 1, server 2 works at station 2} & \textit{for } 0 \le s \le B+1, \\ \textit{both servers work at station 2} & \textit{for } s = B+2. \end{cases}$$

Then $(\delta^*)^\infty$ *is optimal. Moreover, this is the unique optimal policy in the class of Markovian stationary deterministic policies if all the inequalities on the service rates (i.e.,* $\mu_{11} \ge \mu_{21}$, $\mu_{22} \ge \mu_{12}$, *and* $\mu_{21}\mu_{22} \le \mu_{11}\mu_{12}$*) and* α *are strict.*

Theorem 3.3 *Suppose* $\mu_{11}\mu_{12} \le \mu_{21}\mu_{22}$ *and* $\frac{\mu_{11}}{\mu_{11}+\mu_{21}} \le \alpha \le \frac{\mu_{22}}{\mu_{12}+\mu_{22}}$. *Let*

$$\delta^*(s) = \begin{cases} \textit{both servers work at station 1} & \textit{for } s = 0 \\ \textit{server 1 works at station 1, server 2 works at station 2} & \textit{for } 1 \le s \le B+2. \end{cases}$$

Then $(\delta^*)^\infty$ *is optimal. Moreover, this is the unique optimal policy in the class of Markovian stationary deterministic policies if all the inequalities on the service rates (i.e.,* $\mu_{11} \ge \mu_{21}$, $\mu_{22} \ge \mu_{12}$, *and* $\mu_{11}\mu_{12} \le \mu_{21}\mu_{22}$*) and* α *are strict.*

Note that in Theorems 3.2 and 3.3, the optimal policy does not allow the more effective server to idle. Moreover, collaboration only occurs to avoid idling the more effective server.

Finally, assume that when servers collaborate, their combined service rate at each station is faster than the rate of the faster server at that station. In this case, the optimal policy assigns server 1 (2) to station 1 (2) unless station 1 (2) is blocked (starved), in which case both servers work at station 2 (1). Thus, the servers work at the stations for which they are better trained unless they have nothing to do there, in which case they collaborate.

Theorem 3.4 *Suppose* $\max\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{22}}{\mu_{12}+\mu_{22}}\} \le \alpha \le 1$. *Let*

$$\delta^*(s) = \begin{cases} \textit{both servers work at station 1} & \textit{for } s = 0, \\ \textit{server 1 works at station 1, server 2 works at station 2} & \textit{for } 1 \le s \le B+1, \\ \textit{both servers work at station 2} & \textit{for } s = B+2. \end{cases}$$

Then $(\delta^*)^\infty$ *is optimal. Moreover, this is the unique optimal policy in the class of Markovian stationary deterministic policies if all the inequalities on service rates (i.e.,* $\mu_{11} \ge \mu_{21}$ *and* $\mu_{22} \ge \mu_{12}$*) and the lower bound on* α *are strict.*

The proof of Theorem 3.4 is given in the Appendix. Note that Theorem 3.4 indicates that the optimal server assignment policy is continuous in α as $\alpha \nearrow 1$. That is, if $\max\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{22}}{\mu_{12}+\mu_{22}}\} < 1$, which occurs if $\mu_{21}, \mu_{12} > 0$, so that both servers are capable of working at both stations, the optimal policy for $\max\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{22}}{\mu_{12}+\mu_{22}}\} \leq \alpha < 1$ is the same as the optimal policy for $\alpha = 1$ that is described in Andradóttir, Ayhan, and Down [7]. Moreover, Andradóttir, Ayhan, and Down [9] show that this policy remains to be optimal for all $1 \leq \alpha \leq \max\left\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}} + \frac{\mu_{22}}{\mu_{12}+\mu_{22}}, \frac{\mu_{21}}{\mu_{11}+\mu_{21}} + \frac{\mu_{12}}{\mu_{12}+\mu_{22}}\right\}$. On the other hand, when $\alpha \geq \max\left\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}} + \frac{\mu_{22}}{\mu_{12}+\mu_{22}}, \frac{\mu_{21}}{\mu_{11}+\mu_{21}} + \frac{\mu_{12}}{\mu_{12}+\mu_{22}}\right\}$, any fully collaborative policy that has both servers in a team at all times is optimal.

In systems with generalist servers, the service rate of each server at each station can be expressed as the product of two constants, one representing the server's speed at every task and the other representing the intrinsic difficulty of the task at the station. Thus, $\mu_{ij} = \mu_i \gamma_j$ for all $i \in \{1, \ldots, M\}$ and $j \in \{1, \ldots, N\}$.

Remark 3.1 *Andradóttir, Ayhan, and Down [8] have shown that when the servers are generalists and the combined service rate of multiple servers assigned to the same task is additive, then any non-idling policy is throughput optimal for tandem lines with finite buffers. Similarly, Andradóttir, Ayhan, and Down [9] proved that when the servers are generalists and synergistic (i.e., they work more effectively in teams than on their own), any fully collaborative policy that has all servers work in teams of two or more at all times is optimal. Unfortunately, when $\alpha < 1$, the optimal policy does not present a similarly simple form for systems with generalist servers. However, note that when the servers are generalists, the assumption that there is no dominant server implies that $\mu_1 = \mu_2$. Furthermore, $\frac{\mu_{11}}{\mu_{11}+\mu_{21}} = \frac{\mu_{22}}{\mu_{12}+\mu_{22}} = \frac{\mu_1}{\mu_1+\mu_2} = \frac{1}{2}$. Thus, we have two identical servers and the optimal policy is given in Theorem 3.1 if $\alpha \leq \frac{1}{2}$ and in Theorem 3.4 if $\alpha \geq \frac{1}{2}$. Hence, the optimal policy for generalist servers either allows no collaboration or involves collaboration at both stations, depending on the value of α.*

4 Optimal Policy for Systems with a Dominant Server

Without loss of generality, assume that server 1 is the dominant server. Thus, $\mu_{11} \geq \mu_{21}$ and $\mu_{12} \geq \mu_{22}$. Note that this case will reduce to the situation discussed in Section 3 only if $\mu_{12} = \mu_{22}$. In what follows, we first characterize the optimal policy in Section 4.1, provide some properties of the optimal policy in Section 4.2, and present numerical examples to illustrate the optimal policy and its properties in Section 4.3.

4.1 Characterization of the Optimal Policy

In the next four theorems we will give a complete characterization of the optimal policies for all possible values of $0 \leq \alpha \leq 1$. Since the proofs of Theorems 4.1 to 4.4 are similar, we will only provide the proof for the most complicated case, which is given in Theorem 4.4.

We again start with the case where the combined rate of collaborating servers is slower than the rate of the dominant server (i.e., server 1) at both stations. For all $i \in \{1, 2, \ldots, B+3\}$, define

$$
\begin{aligned}
f_1(i) &= \mu_{11}\mu_{22}^{i-1}\sum_{j=0}^{B-i+3}\mu_{21}^{j}\mu_{12}^{B-i+3-j} + \mu_{21}^{B-i+3}\sum_{j=0}^{i-3}\mu_{22}^{j+1}\mu_{11}^{i-1-j} \\
&\quad -\mu_{12}\mu_{21}^{B-i+3}\sum_{j=0}^{i-1}\mu_{22}^{j}\mu_{11}^{i-1-j} - \mu_{22}^{i-1}\sum_{j=0}^{B-i+1}\mu_{21}^{j+1}\mu_{12}^{B-i+3-j}.
\end{aligned}
$$

We will use the convention that the summation over an empty set equals zero throughout. An interpretation of the function f_1 is provided after Theorem 4.1. Furthermore, let

$$S_1^* = \Big\{s \in S\backslash\{0\} : f_1(s) \geq 0 \text{ and } f_1(s+1) \leq 0\Big\}.$$

We will need the following lemma, whose proof is provided by Işık, Andradóttir, and Ayhan [14] where the function f_1 appears in a different setting.

Lemma 4.1 $S_1^* \neq \emptyset$.

We now provide the optimal policy when $\alpha \leq \min\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{12}}{\mu_{12}+\mu_{22}}\}$, so that the combined rate of servers 1 and 2 at each station is smaller than the rate of server 1 at that station. The next proposition states that in this case, the optimal policy does not allow collaboration and the dominant server switches from station 1 to station 2 when the number of customers in the buffer reaches a certain threshold.

Theorem 4.1 *Suppose* $0 \leq \alpha \leq \min\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{12}}{\mu_{12}+\mu_{22}}\}$ *and* $s^* \in S_1^*$. *Let*

$$\delta^*(s) = \begin{cases} \textit{server 1 works at station 1, server 2 works at station 2} & \textit{for } 0 \leq s \leq s^*-1, \\ \textit{server 2 works at station 1, server 1 works at station 2} & \textit{for } s^* \leq s \leq B+2. \end{cases}$$

Then $(\delta^*)^\infty$ *is optimal. Moreover, this is the unique optimal policy in the class of Markovian stationary deterministic policies if* $S_1^* = \{s^*\}$ *and all inequalities on service rates (i.e.,* $\mu_{11} \geq \mu_{21}$ *and* $\mu_{12} \geq \mu_{22}$*) and the upper bound on* α *are strict.*

Theorem 4.1 illustrates that when $\alpha \leq \min\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{12}}{\mu_{12}+\mu_{22}}\}$, the optimal switch point does not depend on α. Moreover, as in Section 3, the optimal policy for small α is identical to the optimal policy for non-collaborative servers and involves a similar proof (see Işık, Andradóttir, and Ayhan [14]).

Note that S_1^* characterizes the set of optimal switch points in Theorem 4.1 and is defined using the function f_1, where for $i = 2, \ldots, B+2$, $f_1(i)$ is proportional to the difference between the throughputs of two policies of the form given in Theorem 4.1 that have server 1 move to station 2 at state i versus state $i-1$. Similarly, S_2^*, S_3^*, and S_4^* characterize the sets of optimal switch points in Theorems 4.2, 4.3, and 4.4 below, which are defined using the functions $f_2(\cdot,\cdot)$, $f_3(\cdot,\cdot)$, and $f_4(\cdot,\cdot)$, respectively. Furthermore, $f_2(i,\alpha)$, $f_3(i,\alpha)$, and $f_4(i,\alpha)$ are proportional to the difference between the throughputs of two policies of the form given in Theorems 4.2, 4.3, and 4.4, respectively.

Next we assume that α is large enough to allow collaboration at one station but still too small to allow collaboration at the other station. Note that $\frac{\mu_{12}}{\mu_{12}+\mu_{22}} \leq (\geq) \frac{\mu_{11}}{\mu_{11}+\mu_{21}}$ if and only if $\mu_{11}\mu_{22} \geq (\leq)\ \mu_{12}\mu_{21}$. Let $\mu_{11}\mu_{22} \geq \mu_{12}\mu_{21}$ and define for all $i \in \{1, 2, \ldots, B+3\}$ and $\frac{\mu_{12}}{\mu_{12}+\mu_{22}} \leq \alpha \leq \frac{\mu_{11}}{\mu_{11}+\mu_{21}}$,

$$
\begin{aligned}
f_2(i,\alpha) \quad = \quad & \alpha(\mu_{12}+\mu_{22})\mu_{21}^{B-i+2}\mathbb{I}(i \leq B+2)\sum_{j=0}^{i-2}\mu_{22}^{j}\mu_{11}^{i-j} \\
& +\alpha\mu_{11}(\mu_{12}+\mu_{22})\mu_{22}^{i-1}\sum_{j=0}^{B-i+2}\mu_{21}^{j}\mu_{12}^{B-i+2-j} + \mu_{21}^{B-i+3}\sum_{j=0}^{i-2}\mu_{22}^{j+1}\mu_{11}^{i-1-j} \\
& -\alpha(\mu_{12}+\mu_{22})\mu_{21}^{B-i+3}\sum_{j=0}^{i-1}\mu_{22}^{j}\mu_{11}^{i-1-j} - \alpha(\mu_{12}+\mu_{22})\mu_{22}^{i-1}\sum_{j=0}^{B-i+1}\mu_{21}^{j+1}\mu_{12}^{B-i+2-j} \\
& -\mu_{12}\mu_{21}^{B-i+2}\mathbb{I}(i \leq B+2)\sum_{j=0}^{i-2}\mu_{22}^{j}\mu_{11}^{i-j}.
\end{aligned}
$$

An interpretation of the function f_2 is provided in the previous paragraph. Let

$$S_2^* = \Big\{ s \in S\backslash\{0\} : f_2(s,\alpha) \geq 0 \text{ and } f_2(s+1,\alpha) \leq 0 \Big\}.$$

We can now state and prove the following lemma.

Lemma 4.2 $S_2^* \neq \emptyset$.

Proof: We have

$$f_2(1,\alpha) \quad = \quad \alpha(\mu_{12}+\mu_{22})(\mu_{11}-\mu_{21})\sum_{j=0}^{B+1}\mu_{21}^{j}\mu_{12}^{B+1-j} \geq 0$$

and

$$f_2(B+3,\alpha) \quad = \quad \mu_{22}\sum_{j=0}^{B+1}\mu_{22}^{j}\mu_{11}^{B+2-j} - \alpha(\mu_{12}+\mu_{22})\sum_{j=0}^{B+2}\mu_{22}^{j}\mu_{11}^{B+2-j} \leq 0,$$

which immediately imply that $S_2^* \neq \emptyset$. $\square$

Next we state the optimal policy when $\mu_{11}\mu_{22} \geq \mu_{12}\mu_{21}$ and $\frac{\mu_{12}}{\mu_{12}+\mu_{22}} \leq \alpha \leq \frac{\mu_{11}}{\mu_{11}+\mu_{21}}$. In this case, the optimal policy has servers work together at station 2 when station 1 is blocked and otherwise has the faster (slower) server at station 1 (2) until the number of customers in the buffer reaches a certain value and then the servers switch their assignments.

Theorem 4.2 *Suppose $\mu_{11}\mu_{22} \geq \mu_{12}\mu_{21}$, $\frac{\mu_{12}}{\mu_{12}+\mu_{22}} \leq \alpha \leq \frac{\mu_{11}}{\mu_{11}+\mu_{21}}$, and $s^* \in S_2^*$. Let*

$$\delta^*(s) = \begin{cases} \textit{server 1 works at station 1, server 2 works at station 2} & \textit{for } 0 \leq s \leq s^*-1, \\ \textit{server 2 works at station 1, server 1 works at station 2} & \textit{for } s^* \leq s \leq B+1, \\ \textit{both servers work at station 2} & \textit{for } s = B+2. \end{cases}$$

Then $(\delta^*)^\infty$ is optimal. Moreover, this is the unique optimal policy in the class of Markovian stationary deterministic policies if $S_2^* = \{s^*\}$ and all inequalities on service rates (i.e., $\mu_{11} \geq \mu_{21}$, $\mu_{12} \geq \mu_{22}$, and $\mu_{11}\mu_{22} \geq \mu_{12}\mu_{21}$) and α are strict.

In the third case, we assume that $\mu_{12}\mu_{21} \geq \mu_{11}\mu_{22}$ and that α is large enough to allow collaboration at one station but not at the other station. Define for all $i \in \{1, 2, \ldots, B+3\}$ and $\frac{\mu_{11}}{\mu_{11}+\mu_{21}} \leq \alpha \leq \frac{\mu_{12}}{\mu_{12}+\mu_{22}}$,

$$\begin{aligned} f_3(i,\alpha) &= \alpha(\mu_{11}+\mu_{21})\mu_{22}^{i-1}\sum_{j=0}^{B+3-i}\mu_{21}^{j}\mu_{12}^{B+3-i-j} + \alpha(\mu_{11}+\mu_{21})\mu_{21}^{B+3-i}\sum_{j=0}^{i-3}\mu_{22}^{j+1}\mu_{11}^{i-2-j} \\ &\quad + \mu_{11}\mu_{22}^{i-2}\mathbb{I}(i \geq 2)\sum_{j=0}^{B+2-i}\mu_{21}^{j}\mu_{12}^{B+4-i-j} - \alpha(\mu_{11}+\mu_{21})\mu_{22}^{i-2}\mathbb{I}(i \geq 2)\sum_{j=0}^{B+2-i}\mu_{21}^{j}\mu_{12}^{B+4-i-j} \\ &\quad - \alpha(\mu_{11}+\mu_{21})\mu_{12}\mu_{21}^{B+3-i}\sum_{j=0}^{i-2}\mu_{22}^{j}\mu_{11}^{i-2-j} - \mu_{22}^{i-1}\sum_{j=0}^{B+2-i}\mu_{21}^{j+1}\mu_{12}^{B+3-i-j}. \end{aligned}$$

An interpretation of the function f_3 is provided following Theorem 4.1. Let

$$S_3^* = \Big\{ s \in S\backslash\{0\} : f_3(s,\alpha) \geq 0 \text{ and } f_3(s+1,\alpha) \leq 0 \Big\}.$$

We are ready for the following lemma.

Lemma 4.3 $S_3^* \neq \emptyset$.

Proof: We have

$$f_3(1,\alpha) = \alpha(\mu_{11}+\mu_{21})\sum_{j=0}^{B+2}\mu_{21}^{j}\mu_{12}^{B+2-j} - \mu_{21}\sum_{j=0}^{B+1}\mu_{21}^{j}\mu_{12}^{B+2-j} \geq 0$$

and

$$f_3(B+3,\alpha) = \alpha(\mu_{11}+\mu_{21})(\mu_{22}-\mu_{12})\sum_{j=0}^{B+1}\mu_{22}^{j}\mu_{11}^{B+1-j} \leq 0,$$

which immediately imply that $S_3^* \neq \emptyset$. $\square$

We are now ready to characterize the optimal policy when $\mu_{12}\mu_{21} \geq \mu_{11}\mu_{22}$ and $\frac{\mu_{11}}{\mu_{11}+\mu_{21}} \leq \alpha \leq \frac{\mu_{12}}{\mu_{12}+\mu_{22}}$. In this case, the optimal policy has the servers work together at station 1 when station 2 is starved and otherwise has the faster (slower) server at station 1 (2) until the number of customers in the buffer reaches a certain value and then the servers switch their assignments.

Theorem 4.3 *Suppose $\mu_{12}\mu_{21} \geq \mu_{11}\mu_{22}$, $\frac{\mu_{11}}{\mu_{11}+\mu_{21}} \leq \alpha \leq \frac{\mu_{12}}{\mu_{12}+\mu_{22}}$ and $s^* \in S_3^*$. Let*

$$\delta^*(s) = \begin{cases} \text{both servers work at station 1} & \text{for } s = 0, \\ \text{server 1 works at station 1, server 2 works at station 2} & \text{for } 1 \leq s \leq s^*-1, \\ \text{server 2 works at station 1, server 1 works at station 2} & \text{for } s^* \leq s \leq B+2. \end{cases}$$

Then $(\delta^*)^\infty$ is optimal. Moreover, this is the unique optimal policy in the class of Markovian stationary deterministic policies if $S_3^* = \{s^*\}$ and all inequalities on service rates (i.e., $\mu_{11} \geq \mu_{21}$, $\mu_{12} \geq \mu_{22}$ and $\mu_{11}\mu_{22} \leq \mu_{12}\mu_{21}$) and α are strict.

The optimal policies defined in Theorems 4.2 and 4.3 avoid idling the slower server at the station that he is relatively better at than server 1, which is station 2 in Theorem 4.2 (since $\frac{\mu_{22}}{\mu_{21}} \geq \frac{\mu_{12}}{\mu_{11}}$) and station 1 in Theorem 4.3 (since $\frac{\mu_{21}}{\mu_{22}} \geq \frac{\mu_{11}}{\mu_{12}}$). Moreover, as in Section 3, Theorem 4.3 (Theorem 4.2) follows from Theorem 4.2 (Theorem 4.3) using the reversibility result of Andradóttir and Ayhan [6]. Furthermore, $f_3(i,\alpha)$ ($f_2(i,\alpha)$) can be obtained from $-f_2(i,\alpha)$ ($-f_3(i,\alpha)$) via replacing μ_{11} by μ_{12}, μ_{12} by μ_{11}, μ_{21} by μ_{22}, μ_{22} by μ_{21}, and i by $B-i+4$.

Finally, assume that when servers collaborate, their combined service rate is faster than the rate of the dominant server (server 1) at both stations. Define for all $i \in \{1,2,\ldots,B+3\}$ and $\max\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{12}}{\mu_{12}+\mu_{22}}\} \leq \alpha \leq 1$,

$$\begin{aligned} f_4(i,\alpha) \;=\; & \sum_{j=0}^{B+2-i} \mu_{21}^{j}\mu_{12}^{B+2-i-j}\mu_{22}^{i-2}\Big[\alpha\mu_{22}^2(\mu_{11}+\mu_{21}) - \mathbb{I}(i\geq 2)\alpha\mu_{12}^2(\mu_{11}+\mu_{21}) - \mu_{22}\mu_{21}\mu_{12} \\ & -\mu_{22}^2\mu_{21} + \mathbb{I}(i\geq 2)\mu_{11}\mu_{12}^2 + \mu_{11}\mu_{22}\mu_{12}\Big] \\ & + \sum_{j=0}^{i-2} \mu_{22}^{j}\mu_{11}^{i-2-j}\mu_{21}^{B+2-i}\Big[-\alpha\mu_{21}^2(\mu_{12}+\mu_{22}) + \mathbb{I}(i\leq B+2)\alpha(\mu_{12}+\mu_{22})\mu_{11}^2 \\ & -\mathbb{I}(i\leq B+2)\mu_{11}^2\mu_{12} + \mu_{22}\mu_{21}^2 + \mu_{11}\mu_{22}\mu_{21} - \mu_{12}\mu_{21}\mu_{11}\Big]. \end{aligned}$$

An interpretation of the function f_4 is provided after Theorem 4.1. Let

$$S_4^* = \Big\{s \in S\backslash\{0\} : f_4(s,\alpha) \geq 0 \text{ and } f_4(s+1,\alpha) \leq 0\Big\}.$$

We will need the following lemma.

Lemma 4.4 $S_4^* \neq \emptyset$.

Proof: We have

$$f_4(1,\alpha) \;=\; [(\alpha(\mu_{11}+\mu_{21}) - \mu_{21})\mu_{22} + \mu_{12}(\mu_{11}-\mu_{21})] \sum_{j=0}^{B+1} \mu_{21}^{j}\mu_{12}^{B+1-j} \geq 0$$

and

$$f_4(B+3,\alpha) \;=\; [(-\alpha(\mu_{12}+\mu_{22}) + \mu_{22})\mu_{21} + \mu_{11}(\mu_{22}-\mu_{12})] \sum_{j=0}^{B+1} \mu_{22}^{j}\mu_{11}^{B+1-j} \leq 0,$$

which immediately imply that $S_4^* \neq \emptyset$. $\square$

We are now ready to state the optimal server assignment policy when $\max\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{12}}{\mu_{12}+\mu_{22}}\} \leq \alpha \leq 1$. The next proposition states that the optimal policy has the servers work together at station 1 (2) when station 2 (1) is starved (blocked) and otherwise has the faster (slower) server at station 1 (2) until the number of customers in the buffer reaches a certain value and then the servers switch their assignments.

Theorem 4.4 *Suppose* $\max\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{12}}{\mu_{12}+\mu_{22}}\} \le \alpha \le 1$ *and* $s^* \in S_4^*$. *Let*

$$\delta^*(s) = \begin{cases} \textit{both servers work at station 1} & \textit{for } s = 0, \\ \textit{server 1 works at station 1, server 2 works at station 2} & \textit{for } 1 \le s \le s^*-1, \\ \textit{server 2 works at station 1, server 1 works at station 2} & \textit{for } s^* \le s \le B+1, \\ \textit{both servers work at station 2} & \textit{for } s = B+2. \end{cases}$$

Then $(\delta^*)^\infty$ *is optimal. Moreover, this is the unique optimal policy in the class of Markovian stationary deterministic policies if* $S_4^* = \{s^*\}$ *and all inequalities on service rates (i.e.,* $\mu_{11} \ge \mu_{21}$, $\mu_{12} \ge \mu_{22}$, *and* $\mu_{11}\mu_{22} \ne \mu_{12}\mu_{21}$*) and the lower bound on* α *are strict.*

The proof of Theorem 4.4 is given in the Appendix.

Remark 4.1 *When the servers are generalists, since* $\frac{\mu_{11}}{\mu_{11}+\mu_{21}} = \frac{\mu_{12}}{\mu_{12}+\mu_{22}} = \frac{\mu_1}{\mu_1+\mu_2}$, *depending on the value of* α, *the optimal policy can be determined using either Theorem 4.1 or Theorem 4.4. Hence, the optimal policy either allows no collaboration or collaboration at both stations depending on the value of* α. *Moreover, the optimal switch point* s^* *can attain any value in* $\{1, \dots, B+2\}$. *Note that in this case, the lower bound on* α *in Theorem 4.4 is greater than or equal to* $\frac{1}{2}$, *with equality only when* $\mu_1 = \mu_2$ *(which is discussed in Remark 3.1). Thus, server collaboration is optimal for a smaller range of* α *values when there is a dominant server. This is reasonable because as the servers become more different, ensuring that the dominant server works at full capacity becomes more important than keeping the slower server busy.*

4.2 Properties of the Optimal Policy

In this section, we investigate properties of the optimal switch point s^* in Theorems 4.1 through 4.4. The following lemma implies that when $0 \le \alpha \le \min\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{12}}{\mu_{12}+\mu_{22}}\}$, if there are multiple optimal switch points, then they are consecutive states. The proof of the lemma is given in Işık, Andradóttir, and Ayhan [14] where the f_1 function appears in a different setting.

Lemma 4.5 *The function* $f_1(i)$ *is non-increasing in* $i \in S \setminus \{0\}$.

The next lemma provides properties of f_2 that will allow us to study the optimal switch point when $\mu_{11} > \mu_{21}$, $\mu_{11}\mu_{22} \ge \mu_{21}\mu_{12}$, and $\frac{\mu_{12}}{\mu_{12}+\mu_{22}} \le \alpha \le \frac{\mu_{11}}{\mu_{11}+\mu_{21}}$. Note that $\frac{\mu_{12}}{\mu_{12}+\mu_{22}} \le \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})} \le 1$ when $\mu_{11}\mu_{22} \ge \mu_{21}\mu_{12}$ and $\mu_{11} \ne \mu_{21}$. If $\mu_{11} = \mu_{21}$, then $\mu_{12} \ge \mu_{22}$ and $\mu_{11}\mu_{22} \ge \mu_{12}\mu_{21}$ imply that $\mu_{12} = \mu_{22}$, and hence, Remark 3.1 applies. Moreover, when $\mu_{11} = \mu_{21}$ and $\mu_{12} = \mu_{22}$, then $\frac{\mu_{12}}{\mu_{12}+\mu_{22}} \le \alpha \le \frac{\mu_{11}}{\mu_{11}+\mu_{21}}$ implies that $\alpha = \frac{\mu_{12}}{\mu_{12}+\mu_{22}} = \frac{\mu_{11}}{\mu_{11}+\mu_{21}} = \frac{1}{2}$, and the proof of part (i) of Lemma 4.6 shows that $S_2^* = S \setminus \{0\}$ when $\alpha = \frac{1}{2}$.

Lemma 4.6 *(i) If* $\frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})} \le \alpha \le 1$, $f_2(i,\alpha) \ge 0$ *for all* $i \in \{1, \dots, B+2\}$.
(ii) If $\frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})} \le \frac{\mu_{11}}{\mu_{11}+\mu_{21}}$, *then* $f_2(i,\alpha)$ *is non-increasing in* $i \in S \setminus \{0\}$ *for all* $\frac{\mu_{12}}{\mu_{12}+\mu_{22}} \le \alpha \le \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})}$.

(iii) If $\frac{\mu_{11}}{\mu_{11}+\mu_{21}} \leq \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})}$*, then* $f_2(i,\alpha)$ *is non-increasing in* $i \in S \setminus \{0\}$ *for all* $\frac{\mu_{12}}{\mu_{12}+\mu_{22}} \leq \alpha \leq \frac{\mu_{11}}{\mu_{11}+\mu_{21}}$.

Proof: (i) Note that for $i \in \{1, \ldots, B+2\}$, we have

$$f_2(i,\alpha) = \sum_{j=0}^{i-2} \mu_{22}^j \mu_{11}^{i-1-j} \mu_{21}^{B-i+2} \beta_2(\alpha) + \sum_{j=0}^{B-i+2} \mu_{21}^j \mu_{12}^{B-i+2-j} \mu_{22}^{i-1} \alpha(\mu_{12}+\mu_{22})(\mu_{11}-\mu_{21}) \quad (2)$$

where $\beta_2(\alpha) = \alpha(\mu_{12}+\mu_{22})(\mu_{11}-\mu_{21}) - \mu_{11}\mu_{12} + \mu_{21}\mu_{22}$. The second summation above is always non-negative since $\mu_{11} > \mu_{21}$ and the first summation is non-negative as long as $\beta_2(\alpha) \geq 0$, which holds if $\alpha \geq \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})}$.
(ii) Note that for all $i \in \{1, \ldots, B+1\}$

$$\begin{aligned} f_2(i+1,\alpha) - f_2(i,\alpha) &= \sum_{j=0}^{i-2} \mu_{22}^j \mu_{11}^{i-1-j} \mu_{21}^{B-i+1} (\mu_{11}-\mu_{21}) \beta_2(\alpha) \\ &\quad + \sum_{j=0}^{B-i+1} \mu_{21}^j \mu_{12}^{B-i+1-j} \mu_{22}^{i-1} \alpha(\mu_{12}+\mu_{22})(\mu_{11}-\mu_{21})(\mu_{22}-\mu_{12}) \\ &\quad + \mu_{22}^{i-1} \mu_{21}^{B-i+1} [\mu_{11}\beta_2(\alpha) - \mu_{21}\alpha(\mu_{12}+\mu_{22})(\mu_{11}-\mu_{21})] \\ &\leq 0, \end{aligned} \quad (3)$$

where the inequality follows because $\beta_2(\alpha) \leq 0$ when $\alpha \leq \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})}$ as seen in the proof of part (i), the expression in the second line is non-positive since $\mu_{11} > \mu_{21}$ and $\mu_{22} \leq \mu_{12}$, and finally $\mu_{11}\beta_2(\alpha) - \mu_{21}\alpha(\mu_{12}+\mu_{22})(\mu_{11}-\mu_{21}) \leq (\mu_{11}-\mu_{21})\beta_2(\alpha) \leq 0$ (the first inequality follows from $\mu_{11}\mu_{12} \geq \mu_{21}\mu_{22}$).
(iii) follows immediately from the proof of part (ii). □

The next lemma identifies properties of f_3 that are useful for studying the optimal switch point when $\mu_{12} > \mu_{22}$, $\mu_{21}\mu_{12} \geq \mu_{11}\mu_{22}$, and $\frac{\mu_{11}}{\mu_{11}+\mu_{21}} \leq \alpha \leq \frac{\mu_{12}}{\mu_{12}+\mu_{22}}$. Note that $\frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})} \leq 1$ when $\mu_{21}\mu_{12} \geq \mu_{11}\mu_{22}$ and $\mu_{12} \neq \mu_{22}$. If $\mu_{12} = \mu_{22}$, then we must have that $\mu_{11} = \mu_{21}$, Remark 3.1 applies, and $S_3^* = S \setminus \{0\}$ when $\alpha = \frac{1}{2}$. As discussed in Section 4.1, $f_3(i,\alpha)$ ($f_2(i,\alpha)$) can be obtained from $-f_2(i,\alpha)$ ($-f_3(i,\alpha)$) using the reversibility result of Andradóttir and Ayhan [6]. This implies that Lemma 4.6 (4.7) follows from Lemma 4.7 (4.6). However, we still provide explicit proofs of both lemmas for clarity and to introduce notation that will be used in other proofs.

Lemma 4.7 *(i) If* $\frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})} \leq \alpha \leq 1$, $f_3(i,\alpha) \leq 0$ *for all* $i \in \{2, \ldots, B+2\}$.
(ii) If $\frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})} \leq \frac{\mu_{12}}{\mu_{12}+\mu_{22}}$*, then* $f_3(i,\alpha)$ *is non-increasing in* $i \in S \setminus \{0\}$ *for all* $\frac{\mu_{11}}{\mu_{11}+\mu_{21}} \leq \alpha \leq \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})}$.
(iii) If $\frac{\mu_{12}}{\mu_{12}+\mu_{22}} \leq \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})}$*, then* $f_3(i,\alpha)$ *is non-increasing in* $i \in S \setminus \{0\}$ *for all* $\frac{\mu_{11}}{\mu_{11}+\mu_{21}} \leq \alpha \leq \frac{\mu_{12}}{\mu_{12}+\mu_{22}}$.

Proof: (i) As shown in the proof of Lemma 4.3, $f_3(1,\alpha) \geq 0$. Note that for $i \in \{2, \ldots, B+2\}$, we have

$$f_3(i,\alpha) \;=\; \sum_{j=0}^{i-2} \mu_{22}^{j}\mu_{11}^{i-2-j}\mu_{21}^{B-i+3}\alpha(\mu_{11}+\mu_{21})(\mu_{22}-\mu_{12}) + \sum_{j=0}^{B-i+2} \mu_{21}^{j}\mu_{12}^{B-i+3-j}\mu_{22}^{i-2}\beta_1(\alpha), \quad (4)$$

where $\beta_1(\alpha) = \alpha(\mu_{11}+\mu_{21})(\mu_{22}-\mu_{12}) + \mu_{11}\mu_{12} - \mu_{21}\mu_{22}$. The first summation above is always non-positive since $\mu_{22} < \mu_{12}$ and the second summation is non-positive as long as $\beta_1(\alpha) \leq 0$, which holds if $\alpha \geq \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})}$.
(ii) For $i = 1$

$$\begin{aligned} f_3(i+1,\alpha) - f_3(i,\alpha) \;&=\; \sum_{j=0}^{B} \mu_{21}^{j}\mu_{12}^{B+1-j}\left[\alpha(\mu_{11}+\mu_{21})(\mu_{22}-2\mu_{12}) + \mu_{12}\mu_{11} - \mu_{21}\mu_{22} + \mu_{12}\mu_{21}\right] \\ &\quad +\mu_{21}^{B+1}(\alpha(\mu_{11}+\mu_{21})(\mu_{22}-\mu_{12}-\mu_{21}) + \mu_{12}(\mu_{21}-\alpha(\mu_{11}+\mu_{21})) \\ &\leq\; 0, \end{aligned} \quad (5)$$

where the inequality follows because the expression in the first line is non-positive since $\alpha \geq \frac{\mu_{11}}{\mu_{11}+\mu_{21}} \geq \frac{\mu_{12}\mu_{11}-\mu_{21}\mu_{22}+\mu_{12}\mu_{21}}{(2\mu_{12}-\mu_{22})(\mu_{11}+\mu_{21})}$ (the second inequality follows from $\mu_{11} \geq \mu_{21}$ and $\mu_{12} > \mu_{22}$) and the expression in the second line is non-positive since $\mu_{22} \leq \mu_{12}$ and $\alpha(\mu_{11}+\mu_{21}) \geq \mu_{11} \geq \mu_{21}$. On the other hand, if $i \in \{2, \ldots, B+1\}$

$$\begin{aligned} f_3(i+1,\alpha) - f_3(i,\alpha) \;&=\; \sum_{j=0}^{i-2} \mu_{22}^{j}\mu_{11}^{i-2-j}\mu_{21}^{B-i+2}\alpha(\mu_{11}+\mu_{21})(\mu_{11}-\mu_{21})(\mu_{22}-\mu_{12}) \\ &\quad + \sum_{j=0}^{B-i+1} \mu_{21}^{j}\mu_{12}^{B-i+2-j}\mu_{22}^{i-2}(\mu_{12}-\mu_{22})(-\beta_1(\alpha)) \\ &\quad +\mu_{22}^{i-2}\mu_{21}^{B-i+2}\left[\mu_{22}\alpha(\mu_{11}+\mu_{21})(\mu_{22}-\mu_{12}) - \mu_{12}\beta_1(\alpha)\right] \\ &\leq\; 0, \end{aligned} \quad (6)$$

where the inequality follows because $\mu_{11} \geq \mu_{21}$, $\mu_{12} > \mu_{22}$, $\beta_1(\alpha) \geq 0$ if $\alpha \leq \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})}$, and $\alpha(\mu_{11}+\mu_{21})\mu_{22}(\mu_{22}-\mu_{12}) - \mu_{12}\beta_1(\alpha) \leq (\mu_{22}-\mu_{12})\beta_1(\alpha) \leq 0$ (the first inequality follows from $\mu_{11}\mu_{12} \geq \mu_{21}\mu_{22}$).
(iii) follows immediately from the proof of part (ii). $\Box$

The next lemma provides properties of f_4 that will help us analyze the optimal switch points when $\max\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{12}}{\mu_{12}+\mu_{22}}\} \leq \alpha \leq 1$.

Lemma 4.8 *(i) If $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$ and $\frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})} \leq \alpha \leq 1$, $f_4(i,\alpha) > 0$ for all $i \in \{2, \ldots, B+2\}$.*
(ii) If $\mu_{12}\mu_{21} > \mu_{11}\mu_{22}$ and $\frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})} \leq \alpha \leq 1$, $f_4(i,\alpha) < 0$ for all $i \in \{2, \ldots, B+2\}$.
(iii) If $\mu_{12}\mu_{21} = \mu_{11}\mu_{22}$, $\mu_{11} > \mu_{21}$, $\mu_{12} > \mu_{22}$, and $\alpha = \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})} = \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})} = 1$, then $f_4(i,\alpha) = 0$ for all $i \in \{2, \ldots, B+2\}$.
(iv) If $\mu_{12}\mu_{21} = \mu_{11}\mu_{22}$ and $\mu_{11} = \mu_{21}$ ($\mu_{12} = \mu_{22}$), then $\mu_{12} = \mu_{22}$ ($\mu_{11} = \mu_{21}$) and $f_4(i,\alpha) = 0$ for

all $i \in \{2, \ldots, B+2\}$ and all $\frac{1}{2} \leq \alpha \leq 1$.
(v) If none of the conditions of parts (i), (ii), (iii), and (iv) hold, then $f_4(i,\alpha)$ is non-increasing in $i \in S \setminus \{0\}$.

Proof: (i) As shown in Lemma 4.4, $f_4(1,\alpha) \geq 0$. Note that for $i \in \{2, \ldots, B+2\}$, we have

$$f_4(i,\alpha) = \sum_{j=0}^{B+2-i} \mu_{21}^{j}\mu_{12}^{B+2-i-j}\mu_{22}^{i-2}\Delta_1(\alpha) + \sum_{j=0}^{i-2} \mu_{22}^{j}\mu_{11}^{i-2-j}\mu_{21}^{B+2-i}\Delta_2(\alpha), \tag{7}$$

where

$$\Delta_1(\alpha) = (\mu_{12}+\mu_{22})\beta_1(\alpha)$$

with $\beta_1(\alpha)$ defined in the proof of Lemma 4.7 and

$$\Delta_2(\alpha) = (\mu_{11}+\mu_{21})\beta_2(\alpha)$$

with $\beta_2(\alpha)$ defined in the proof of Lemma 4.6. Note that $\Delta_1(\alpha)$ is non-increasing in α and

$$\Delta_1(1) = (\mu_{12}+\mu_{22})(\mu_{11}\mu_{22} - \mu_{12}\mu_{21}) > 0,$$

where the inequality follows from the assumption that $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$. On the other hand, $\mu_{12} \geq \mu_{22}$ and $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$ imply that $\mu_{11} > \mu_{21}$, and if $\alpha \geq \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})}$, then $\Delta_2(\alpha) \geq 0$. Thus, $f_4(i,\alpha) > 0$ for all $i \in \{2, \ldots, B+2\}$ when $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$ and $\frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})} \leq \alpha \leq 1$ (note that $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$ implies that $\mu_{22} > 0$, and, hence, $\mu_{12} \geq \mu_{22} > 0$).
(ii) Similarly, $\Delta_2(\alpha)$ is non-decreasing in α and

$$\Delta_2(1) = -(\mu_{11}+\mu_{21})(\mu_{12}\mu_{21} - \mu_{11}\mu_{22}) < 0,$$

where the inequality follows from the assumption that $\mu_{12}\mu_{21} > \mu_{11}\mu_{22}$. On the other hand, $\mu_{11} \geq \mu_{21}$ and $\mu_{12}\mu_{21} > \mu_{11}\mu_{22}$ imply that $\mu_{12} > \mu_{22}$, and if $\alpha \geq \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})}$, then $\Delta_1(\alpha) \leq 0$. Thus, $f_4(i,\alpha) < 0$ for all $i \in \{2, \ldots, B+2\}$ when $\mu_{12}\mu_{21} > \mu_{11}\mu_{22}$ and $\frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})} \leq \alpha \leq 1$ (note that $\mu_{12}\mu_{21} > \mu_{11}\mu_{22}$ implies that $\mu_{21} > 0$, and, hence, $\mu_{11} \geq \mu_{21} > 0$).
(iii) Immediately follows from parts (i) and (ii).
(iv) Follows from (7) after noting that in this case the servers are identical, Remark 3.1 applies, and $\beta_1(\alpha) = \beta_2(\alpha) = 0$.
(v) For $i = 1$,

$$\begin{aligned} f_4(i+1,\alpha) - f_4(i,\alpha) &= \sum_{j=0}^{B} \mu_{21}^{j}\mu_{12}^{B-j}\Big[(\mu_{12}^2 - \mu_{22}^2)(-\alpha(\mu_{11}+\mu_{21}) + \mu_{21}) + \mu_{12}\mu_{22}(-\alpha(\mu_{11}+\mu_{21}) + \mu_{11})\Big] \\ &\quad + \mu_{21}^{B+1}\big[\mu_{22}(\mu_{21} - \alpha(\mu_{11}+\mu_{21})) - \mu_{12}(\mu_{11}-\mu_{21})\big] + \mu_{21}^{B}\Delta_2(\alpha) \\ &\leq 0, \end{aligned} \tag{8}$$

where the inequality follows because the first two expressions are non-positive (since $\alpha(\mu_{11}+\mu_{21}) \geq \mu_{11} \geq \mu_{21}$ and $\mu_{12} \geq \mu_{22}$) and the last expression is non-positive (since $\Delta_2(\alpha) \geq 0$ if and only if the conditions of (i), (iii), or (iv) are satisfied). If $i \in \{2, \ldots, B+1\}$

$$\begin{aligned} f_4(i+1,\alpha) - f_4(i,\alpha) &= \mu_{22}^{i-2}(-\Delta_1(\alpha))\Big[(\mu_{12}-\mu_{22})\sum_{j=0}^{B+1-i}\mu_{21}^{j}\mu_{12}^{B+1-i-j} + \mu_{21}^{B+2-i}\Big] \\ &\quad +\mu_{21}^{B+1-i}\Delta_2(\alpha)\Big[(\mu_{11}-\mu_{21})\sum_{j=0}^{i-2}\mu_{22}^{j}\mu_{11}^{i-2-j} + \mu_{22}^{i-1}\Big] \\ &\leq 0, \end{aligned} \tag{9}$$

where the inequality follows since $\Delta_1(\alpha) \leq 0$ if and only if the conditions of (ii), (iii), or (iv) are satisfied and $\Delta_2(\alpha) \geq 0$ if and only if the conditions of (i), (iii), or (iv) are satisfied. $\square$

Remark 4.2 *Note that when $\mu_{12} = \mu_{22}$, the optimal policies in Theorems 4.1 through 4.4 reduce to the ones in Theorems 3.1 through 3.4. In particular, if $\mu_{12} = \mu_{22}$, then $\mu_{11}\mu_{22} \geq \mu_{12}\mu_{21}$ and $\mu_{11}\mu_{12} \geq \mu_{21}\mu_{22}$, so the conditions of Theorems 3.2 and 4.2 hold. Moreover,*

- $f_1(i) = (\mu_{11}-\mu_{21})\sum_{j=0}^{B-i+2}\mu_{21}^{j}\mu_{22}^{B+2-j} \geq 0$ *(with equality only when $\mu_{11} = \mu_{21}$ or $\mu_{21} = \mu_{22} = 0$) for all $i \in \{1, \ldots, B+2\}$, implying that $s^* = B+2$ in Theorem 4.1, which is consistent with the result in Theorem 3.1 since when station 1 is blocked ($s = B+2$), it does not matter which server works at station 2;*

- $\frac{\mu_{12}}{\mu_{12}+\mu_{22}} = \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})}$ *and part (i) of Lemma 4.6 implies that $s^* = B+2$ in Theorem 4.2, which is consistent with the result in Theorem 3.2;*

- $\max\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{12}}{\mu_{12}+\mu_{22}}\} \geq \frac{1}{2} = \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})}$*, and the proof of part (i) of Lemma 4.8 implies that $s^* = B+2$ in Theorem 4.4, which is consistent with Theorem 3.4.*

The next proposition shows that s^* is a non-decreasing function of α if $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$. Thus, the fast server (server 1) focuses more on the station he is relatively better at (station 1 since $\frac{\mu_{11}}{\mu_{12}} > \frac{\mu_{21}}{\mu_{22}}$) for large α. The proof of Proposition 4.1 is given in the Appendix.

Proposition 4.1 *If $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$, then s^* can take at most two adjacent integer values for each $0 \leq \alpha \leq 1$. Moreover, as long as s^* is chosen in a consistent manner when $\alpha < \frac{\mu_{12}}{\mu_{12}+\mu_{22}}$ and two values are possible, then s^* is a non-decreasing function of α, taking jumps of size one, and eventually reaching $B+2$ at $\alpha = \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})} < 1$.*

Similarly, when $\mu_{12}\mu_{21} > \mu_{11}\mu_{22}$, so that the dominant server is relatively better at station 2, then his focus on station 2 increases with α. The proof of Proposition 4.2 is omitted since Proposition 4.2 (4.1) follows from Proposition 4.1 (4.2) using the reversibility result of Andradóttir and Ayhan [6]. Recall that $f_3(i,\alpha)$ ($f_2(i,\alpha)$) can be obtained from $-f_2(i,\alpha)$ ($-f_3(i,\alpha)$) via replacing μ_{11} by μ_{12}, μ_{12} by μ_{11}, μ_{21} by μ_{22}, μ_{22} by μ_{21}, and i by $B-i+4$. Similarly, the condition on the service rates and the threshold α value of Proposition 4.2 (4.1) can be obtained from the corresponding quantities of Proposition 4.2 (4.1) via replacing μ_{11} by μ_{12}, μ_{12} by μ_{11}, μ_{21} by μ_{22}, and μ_{22} by μ_{21}. Note that when $\mu_{12}\mu_{21} = \mu_{11}\mu_{22}$, the servers are generalists and Remark 4.1 applies.

Proposition 4.2 *If $\mu_{12}\mu_{21} > \mu_{11}\mu_{22}$, then s^* can take at most two adjacent integer values for each $0 \leq \alpha \leq 1$. Moreover, as long as s^* is chosen in a consistent manner when $\alpha < \frac{\mu_{11}}{\mu_{11}+\mu_{21}}$ and two values are possible, then s^* is a non-increasing function of α, taking jumps of size one, and eventually reaching 1 at $\alpha = \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})} < 1$.*

It immediately follows from Propositions 4.1 and 4.2 that if $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$ and

$$f_1(B+2) \quad = \quad \mu_{11}\mu_{12}\mu_{22}^{B+1} + \mu_{21}\sum_{j=0}^{B}\mu_{22}^{j+1}\mu_{11}^{B+1-j} - \mu_{12}\mu_{21}\sum_{j=0}^{B+1}\mu_{22}^{j}\mu_{11}^{B+1-j} > 0,$$

then $s^* = B+2$ for all $0 \leq \alpha \leq 1$. Similarly, if $\mu_{12}\mu_{21} > \mu_{11}\mu_{22}$ and

$$f_1(2) \quad = \quad \mu_{11}\mu_{22}\sum_{j=0}^{B+1}\mu_{21}^{j}\mu_{12}^{B+1-j} - \mu_{22}\sum_{j=0}^{B}\mu_{21}^{j+1}\mu_{12}^{B+1-j} - \mu_{12}\mu_{21}^{B+1}\mu_{11} < 0,$$

then $s^* = 1$ for all $0 \leq \alpha \leq 1$. Thus, in these two cases, there is an optimal policy that does not depend on how ineffectively the servers collaborate. One can observe that for $B = 0$ the above expressions coincide and reduce to

$$f_1(2) \quad = \quad \mu_{11}\mu_{22}(\mu_{12}+\mu_{21}) - \mu_{12}\mu_{21}(\mu_{11}+\mu_{22}).$$

Define

$$C \quad = \quad \max\Big\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{12}}{\mu_{12}+\mu_{22}}, \min\big\{\frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})}, \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})}\big\}\Big\},$$

where $\frac{x}{0}$ is defined to be infinity for all $x \geq 0$. It is easy to see that if $\mu_{11}\mu_{22} \geq (\leq)\mu_{12}\mu_{21}$, then $\frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})} \leq (\geq)\frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})}$. Thus, if $\mu_{11}\mu_{22} \geq \mu_{12}\mu_{21}$, then

$$C \quad = \quad \max\Big\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})}\Big\}$$

and $C < 1$ when $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$ and $\mu_{21} > 0$. Similarly, if $\mu_{12}\mu_{21} \geq \mu_{11}\mu_{22}$, then

$$C \quad = \quad \max\Big\{\frac{\mu_{12}}{\mu_{12}+\mu_{22}}, \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})}\Big\}$$

and $C < 1$ when $\mu_{12}\mu_{21} > \mu_{11}\mu_{22}$ and $\mu_{22} > 0$. The next corollary states that the optimal server assignment policy is again continuous in α as $\alpha \nearrow 1$, as in the case discussed in Section 3. That is if $C \leq \alpha \leq 1$, the optimal policy is the same as the one for $\alpha = 1$, which is described in Andradóttir, Ayhan, and Down [7]. The proof of Corollary 4.1 follows immediately from parts (i) and (ii) of Lemma 4.8 and the definition of s^* in Theorem 4.4.

Corollary 4.1 *If $C \leq \alpha \leq 1$ and $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$ ($\mu_{12}\mu_{21} < \mu_{11}\mu_{22}$), then the policy where server 1 (2) has primary assignment at station 1 and server 2 (1) has primary assignment at station 2 and both servers work at station 1 (2) when station 2 (1) is starved (blocked) is optimal.*

As was mentioned in Section 3, Andradóttir, Ayhan, and Down [9] show that the policy described in Corollary 4.1 remains optimal for all $1 \leq \alpha \leq \max\left\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}} + \frac{\mu_{22}}{\mu_{12}+\mu_{22}}, \frac{\mu_{21}}{\mu_{11}+\mu_{21}} + \frac{\mu_{12}}{\mu_{12}+\mu_{22}}\right\}$. On the other hand, when $\alpha \geq \max\left\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}} + \frac{\mu_{22}}{\mu_{12}+\mu_{22}}, \frac{\mu_{21}}{\mu_{11}+\mu_{21}} + \frac{\mu_{12}}{\mu_{12}+\mu_{22}}\right\}$, any fully collaborative policy that has both servers in a team at all times is optimal.

The results of Sections 4.1 and 4.2 show that when there is a dominant server, the optimal server assignment policy is of threshold type in that the dominant server switches from station 1 to station 2 when the number of jobs in the buffer reaches a certain threshold. This has the effect of balancing the two stations and using the dominant server as effectively as possible. When the dominant server is relatively better at station 1 than at station 2, then the optimal threshold is a non-decreasing function of α, taking the largest possible value $B + 2$ before α reaches one. Similarly, when the dominant server is relatively better at station 2, then the optimal threshold is a non-increasing function of α, taking the smallest possible value 1 before α reaches 1. Thus, as α increases and collaboration becomes more effective, there is lesser need for balancing the two stations by moving the dominant server and he can focus more of his time on the station where he is relatively better.

When α is small, there is no collaboration. Thus the dominant server is busy at all times, but the slower server is idle when station 1 is blocked or station 2 is starved. As α increases, so that collaboration is more effective, it becomes desirable to utilize the slower server when station 1 is blocked or station 2 is starved. In particular, for intermediate values of α, the servers collaborate at the station where the slower server is relatively better (as compared to the dominant server) but not at the other station. Finally, for large values of $\alpha \leq 1$, the servers collaborate at both stations when this is needed to avoid idling the slower server. Thus, as α increases, collaboration becomes a desirable way of utilizing the slower server, starting with the station where the slower server is relatively better.

4.3 Numerical Examples

In this section, we provide two numerical examples to illustrate the optimal policy and its properties stated in Propositions 4.1 and 4.2. First assume that $\mu_{11} = 8$, $\mu_{12} = 6$, $\mu_{21} = 5$, $\mu_{22} = 4$, and the buffer size $B = 5$. Hence, the rates of the servers satisfy $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$, the assumptions of Theorem 4.2 and Proposition 4.1 hold, and the state space $S = \{0, 1, \ldots, 7\}$. Figure 1 illustrates how the optimal policy and the optimal switch point s^* change as a function of α for this example (the optimal policy is shown for $\alpha \geq 0.5$; the optimal policy for $\alpha < 0.5$ agrees with the optimal policy for $0.5 \leq \alpha \leq 0.6$). As Figure 1 shows, the optimal policy allows no collaboration when $\alpha < 0.6$ (as in Theorem 4.1) and in this case the dominant server switches from station 1 to station 2 at state 3. On the other hand, when $0.6 \leq \alpha < \frac{8}{13}$, the servers collaborate in state 7 (as in Theorem 4.2) and the optimal switch point remains to be state 3. Finally, when $\alpha > \frac{8}{13}$, the optimal policy allows collaboration in states 0 and 7 (as in Theorem 4.4) and the optimal switch point increases from 3 to 7 taking jumps of size one and reaching 7 at $\alpha = \frac{14}{15}$.

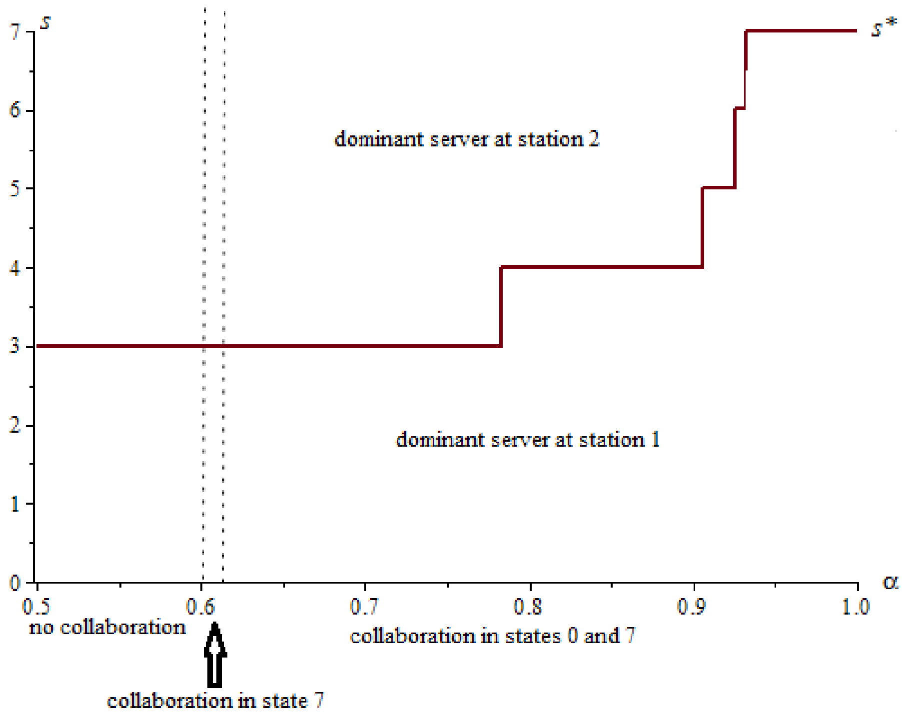

Figure 1: Optimal policy and optimal switch point s^* as a function of α for an example with $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$.

Next assume that $\mu_{11} = 4$, $\mu_{12} = 7$, $\mu_{21} = 3$, $\mu_{22} = 5$, and the buffer size $B = 4$. Hence, the rates of the servers satisfy $\mu_{12}\mu_{21} > \mu_{11}\mu_{22}$, the assumptions of Theorem 4.3 and Proposition 4.2 hold, and the state space $S = \{0, 1, \ldots, 6\}$. Figure 2 indicates how the optimal policy and the optimal switch point s^* depend on α for this example (the optimal policy is shown for $\alpha \geq 0.55$; the optimal policy for $\alpha < 0.55$ agrees with the optimal policy for $0.55 \leq \alpha \leq \frac{4}{7}$). As Figure 2 shows, the optimal policy allows no collaboration when $\alpha < \frac{4}{7}$ (as in Theorem 4.1) and in this case the dominant server switches from station 1 to station 2 at state 6. On the other hand, when $\frac{4}{7} \leq \alpha < \frac{7}{12}$, the servers collaborate in state 0 (as in Theorem 4.3) and the optimal switch point remains at state 6. Finally, when $\alpha > \frac{7}{12}$, the optimal policy allows collaboration in states 0 and 6 (as in Theorem 4.4) and the optimal switch point decreases from 6 to 1 taking jumps of size one and reaching 1 at $\alpha = \frac{13}{14}$.

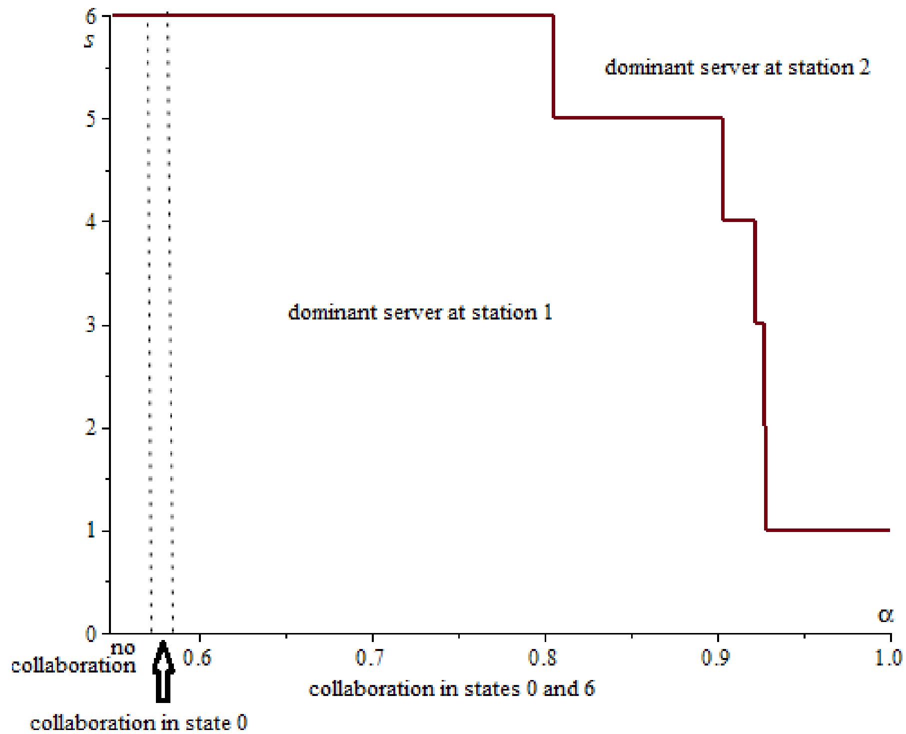

Figure 2: Optimal policy and optimal switch point s^* as a function of α for an example with $\mu_{12}\mu_{21} > \mu_{11}\mu_{22}$.

5 Conclusions

We considered two-station tandem lines with a finite buffer between the stations and two flexible servers who lose efficiency when they work together in a team. We assumed that the combined rate of a server team is proportional to the sum of the rates of the individual servers. For such systems, we completely characterized the server assignment policy that maximizes the long-run average throughput. In particular, we showed that if there is no dominant server, then servers have primary assignments and only leave their primary assignments if the proportionality constant α is large enough and they have no work to do at their primary assignments. On the other hand, if there is a dominant server, the optimal policy is of threshold type (i.e., the dominant server moves from station 1 to station 2 when the number of jobs waiting in the buffer reaches a certain value). We also investigated properties of the optimal switch point where the dominant server moves from station 1 to station 2. We proved that the optimal switch point is either a non-decreasing or non-increasing function of α, eventually reaching $B+2$ (the state representing station 1 being blocked) or 1 (the state representing station 2 being starved) depending on the sign of $\mu_{11}\mu_{22} - \mu_{12}\mu_{21}$.

In both cases (when there is or is not a dominant server), the optimal server assignment allows no collaboration when α is small. When there is no dominant server, for intermediate values of α, the optimal policy allows collaboration only to avoid idling the more effective server (the effectiveness of a server is measured by the product of the server's rates). On the other hand, when there is a dominant server, for intermediate values of α, the servers collaborate at the station where the slower server is relatively better (as compared to the dominant server) but not at the other station. Finally, for large values of $\alpha \leq 1$, the servers collaborate at both stations when this is needed to avoid idling a server. When the servers are generalists, the intermediate range is empty, and hence, there is either no collaboration or collaboration at both stations (in order to avoid idling a server).

The results of this paper, together with the work of Andradóttir, Ayhan, and Down [9], provide a complete characterization of the optimal server assignment policy in Markovian systems with two stations and two servers for all $0 \leq \alpha < \infty$. In particular, Andradóttir, Ayhan, and Down [9] showed that when servers are synergistic (i.e., $\alpha \geq 1$), depending on the value of α, the optimal policy switches from one that takes full advantage of servers' skills (in that both servers have primary assignments and do not collaborate unless they have no work at their primary assignments), to one that takes full advantage of server synergy (in that servers collaborate at all times). Furthermore, the optimal server assignment policy is continuous in α as $\alpha \nearrow 1$ and $\alpha \searrow 1$.

We plan to continue this line of research by investigating whether similar structural results also hold for systems with more general models for the team service rates. For example, we will allow the proportionality constant α to depend on the station where the team is working. Allowing α to depend on the station captures the fact that collaboration may not be equally beneficial for all tasks.

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Appendix

Proof of Theorem 3.4

It follows from our assumptions on service rates that $\mu_{11} > 0$ and $\mu_{22} > 0$. The set of possible actions is given by $A = \{a_{\sigma_1\sigma_2} : \sigma_i \in \{0,1,2\}, \forall i = 1,2\}$, where for all $i \in \{1,2\}$, $\sigma_i = 0$ when server i is idle and $\sigma_i = j \in \{1,2\}$ when server i is assigned to station j.

The set A_s of allowable actions in state s is given as

$$A_s = \begin{cases} \{a_{11}\} & \text{for } s = 0, \\ \{a_{11}, a_{12}, a_{21}, a_{22}\} & \text{for } s \in \{1, \dots, B+1\}, \\ \{a_{22}\} & \text{for } s = B+2, \end{cases}$$

where we use sample path arguments similar to those of Lemma 2.1 and Corollary 2.2 of Kırkızlar, Andradóttir, and Ayhan [16] to eliminate actions that allow servers to idle (exploiting the fact that when servers collaborate, their combined service rate at each station is faster than the rate of the faster server at that station). Since the number of possible states and actions are both finite, the existence of an optimal Markovian stationary deterministic policy follows from Theorem 9.1.8 of Puterman [18], which provides sufficient conditions under which such a policy exists.

Under our assumptions on the service rates, the policy described in Theorem 3.4 corresponds to an irreducible Markov chain, and, hence, we have a communicating Markov decision process. Therefore, we use the policy iteration algorithm for communicating models (see pages 479 and 480 of Puterman [18]) to prove the optimality of the policy.

Let $p(s'|s, d(s))$ be the probability of going to state $s' \in S$ in one step when the action prescribed by decision rule d is taken in state s and P_d be the corresponding $(B+3) \times (B+3)$-dimensional probability transition matrix. Similarly, $r(s, d(s))$ denotes the immediate reward obtained when the action prescribed by decision rule d is taken in state s and r_d denotes the corresponding $(B+3)$-dimensional reward vector.

As the initial policy of the policy iteration algorithm, we choose

$$d_0(s) = \begin{cases} a_{11} & \text{for } s = 0, \\ a_{12} & \text{for } 1 \le s \le B+1, \\ a_{22} & \text{for } s = B+2, \end{cases}$$

corresponding to the policy described in Theorem 3.4. Then

$$r(s, d_0(s)) = \begin{cases} 0 & \text{for } s = 0, \\ \mu_{22} & \text{for } 1 \le s \le B+1, \\ \alpha(\mu_{12} + \mu_{22}) & \text{for } s = B+2, \end{cases}$$

and

$$p(s'|s, d_0(s)) = \begin{cases} \frac{\alpha(\mu_{11}+\mu_{21})}{q} & \text{for } s = 0,\ s' = 1, \\ 1 - \frac{\alpha(\mu_{11}+\mu_{21})}{q} & \text{for } s = s' = 0, \\ \frac{\mu_{22}}{q} & \text{for } 1 \le s \le B+1,\ s' = s-1, \\ 1 - \frac{\mu_{11}+\mu_{22}}{q} & \text{for } 1 \le s \le B+1,\ s' = s, \\ \frac{\mu_{11}}{q} & \text{for } 1 \le s \le B+1,\ s' = s+1, \\ \frac{\alpha(\mu_{12}+\mu_{22})}{q} & \text{for } s = B+2,\ s' = B+1, \\ 1 - \frac{\alpha(\mu_{12}+\mu_{22})}{q} & \text{for } s = s' = B+2, \end{cases}$$

where q is the uniformization constant. Since the policy $(d_0)^\infty$ (corresponding to the decision rule d_0) yields an irreducible Markov chain, we find a scalar g_0 and a vector h_0 solving

$$r_{d_0} - g_0 e + (P_{d_0} - I)h_0 = 0, \tag{10}$$

subject to $h_0(0) = 0$, where e is a column vector of ones and I is the identity matrix. Then

$$g_0 = \frac{\alpha(\mu_{11}+\mu_{21})(\mu_{12}+\mu_{22}) \sum_{j=0}^{B+1} \mu_{11}^j \mu_{22}^{B+1-j}}{\mu_{11}^{B+1}(\mu_{11}+\mu_{21}) + \mu_{22}^{B+1}(\mu_{12}+\mu_{22}) + \alpha(\mu_{11}+\mu_{21})(\mu_{12}+\mu_{22}) \sum_{j=0}^{B} \mu_{11}^j \mu_{22}^{B-j}},$$

$h_0(0) = 0$, and

$$h_0(s) = \frac{qg_0}{\alpha(\mu_{11}+\mu_{21})\mu_{11}^{s-1}}\Big[(\alpha(\mu_{11}+\mu_{21}) - \mu_{11} + \mu_{22})\sum_{j=0}^{s-2}(j+1)\mu_{11}^{j}\mu_{22}^{s-2-j} + s\mu_{11}^{s-1}\Big]$$
$$-\frac{q\mu_{22}}{\mu_{11}^{s-1}}\sum_{j=0}^{s-2}(j+1)\mu_{11}^{j}\mu_{22}^{s-2-j}$$

for $1 \le s \le B+2$ constitute a solution to equation (10).

For the next step of the policy iteration algorithm, we choose

$$d_1(s) \in \arg\max_{a\in A_s}\Big\{r(s,a) + \sum_{j\in S} p(j|s,a)h_0(j)\Big\}, \quad \forall s \in S,$$

setting $d_1(s) = d_0(s)$ if possible. We now show that $d_1(s) = d_0(s)$ for all $s \in S$. For all $s \in S \setminus \{0, B+2\}$ and $a \in A_s \setminus \{d_0(s)\}$, we will compute the differences

$$\epsilon(s,a) = r(s,d_0(s)) + \sum_{j\in S} p(j|s,d_0(s))h_0(j) - \Big(r(s,a) + \sum_{j\in S} p(j|s,a)h_0(j)\Big)$$

and show that the differences are non-negative. For $s = 0$ and $s = B+2$, there is nothing to prove because there is only one possible action in these states, namely $d_0(0) = a_{11}$ and $d_0(B+2) = a_{22}$.

For $s \in \{1,\dots,B+1\}$, we have that $d_0(s) = a_{12}$. We will specify $\epsilon(s,a)$ for actions a_{11}, a_{21}, and a_{22}. With some algebra we obtain

$$\epsilon(s,a_{11}) = \frac{\alpha(\mu_{11}+\mu_{21})\mu_{11}^{B+1-s}\sum_{j=0}^{s-1}\mu_{11}^{j}\mu_{22}^{s-1-j}\Upsilon_1(\alpha)}{\Upsilon},$$

where

$$\Upsilon_1(\alpha) = -\alpha(\mu_{11}+\mu_{21})(\mu_{12}+\mu_{22}) + 2\mu_{11}\mu_{22} + \mu_{11}\mu_{12} + \mu_{21}\mu_{22} \tag{11}$$

and

$$\Upsilon = \mu_{11}^{B+1}(\mu_{11}+\mu_{21}) + \mu_{22}^{B+1}(\mu_{12}+\mu_{22}) + \alpha(\mu_{11}+\mu_{21})(\mu_{12}+\mu_{22})\sum_{j=0}^{B}\mu_{11}^{j}\mu_{22}^{B-j} > 0.$$

Note that $\Upsilon_1(\alpha) \ge 0$ for $\alpha \le 1$ as long as $\mu_{11}\mu_{22} \ge \mu_{12}\mu_{21}$, which follows from our assumptions on the service rates. Hence, $\epsilon(s,a_{11}) \ge 0$ with equality only when $\mu_{11}\mu_{22} = \mu_{12}\mu_{21}$ and $\alpha = 1$.

Similarly,

$$\epsilon(s,a_{21}) = \frac{\mu_{22}^{B}\Delta_1(\alpha) + \sum_{j=0}^{B-s}\mu_{11}^{j+1}\mu_{22}^{B-j-1}\Delta_1(\alpha) + \sum_{j=B-s+1}^{B}\mu_{11}^{j}\mu_{22}^{B-j}\Delta_2(\alpha)}{\Upsilon},$$

where $\Delta_1(\alpha)$ and $\Delta_2(\alpha)$ are defined in the proof of Lemma 4.8. In this case, $\Delta_1(\alpha)$ is non-decreasing in α. Since

$$\Delta_1\Big(\frac{\mu_{11}}{\mu_{11}+\mu_{21}}\Big) = \mu_{22}(\mu_{12}+\mu_{22})(\mu_{11}-\mu_{21}) \ge 0,$$

we can conclude that $\Delta_1(\alpha) \geq 0$ for all $\max\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{22}}{\mu_{12}+\mu_{22}}\} \leq \alpha \leq 1$. Moreover, $\Delta_2(\alpha)$ is non-decreasing in α. Since

$$\Delta_2(\frac{\mu_{22}}{\mu_{12}+\mu_{22}}) \quad = \quad \mu_{11}(\mu_{11}+\mu_{21})(\mu_{22}-\mu_{12}) \geq 0,$$

we can conclude that $\Delta_2(\alpha) \geq 0$ for all $\max\{\frac{\mu_{11}}{\mu_{11}+\mu_{21}}, \frac{\mu_{22}}{\mu_{12}+\mu_{22}}\} \leq \alpha \leq 1$. Hence, $\epsilon(s, a_{21}) \geq 0$ with equality only when $\mu_{11} = \mu_{21}$ and $\mu_{12} = \mu_{22}$.

Finally,

$$\epsilon(s, a_{22}) \quad = \quad \frac{\alpha(\mu_{12}+\mu_{22})\mu_{22}^{s-1}\sum_{j=0}^{B-s+1}\mu_{11}^{j}\mu_{22}^{B-s+1-j}\Upsilon_1(\alpha)}{\Upsilon},$$

which is non-negative since $\Upsilon_1(\alpha) \geq 0$ with equality only when $\mu_{11}\mu_{22} = \mu_{12}\mu_{21}$ and $\alpha = 1$. This proves that $d_1(s) = d_0(s)$ for all $s \in S$. Thus, the policy described in Theorem 3.4 is optimal. The proof of the uniqueness of the optimal policy is similar to the uniqueness proof in Theorem 3.1 of Andradóttir and Ayhan [6] (the lower bound on α needs to be strict to ensure that the policies in Theorems 3.2 and 3.3 are not optimal and that idling actions are not optimal in states $1, \ldots, B+1$).

Proof of Theorem 4.4

The proof of Theorem 4.4 is similar to the proof of Theorem 3.4. It follows from our assumptions on service rates that $\mu_{11} > 0$, $\mu_{12} > 0$, and either $\mu_{21} > 0$ or $\mu_{22} > 0$.

The set A_s of allowable actions in state s is the same as the one described in the proof of Theorem 3.4, where we again use the results of Kırkızlar, Andradóttir, and Ayhan [16] to eliminate idling actions for the specified range of α. Since the number of possible states and actions are both finite, the existence of an optimal Markovian stationary deterministic policy follows from Theorem 9.1.8 of Puterman [18].

Under our assumptions on the service rates, the policy described in Theorem 4.4 corresponds to a unichain Markov chain, and, hence, we have a weakly communicating Markov decision process. Therefore, we use the policy iteration algorithm for communicating models (see pages 479 and 480 of Puterman [18]) to prove the optimality of the policy described in Theorem 4.4. This time as the initial policy of the policy iteration algorithm, we choose

$$d_0'(s) = \begin{cases} a_{11} & \text{for } s = 0, \\ a_{12} & \text{for } 1 \leq s \leq s^* - 1, \\ a_{21} & \text{for } s^* \leq s \leq B+1, \\ a_{22} & \text{for } s = B+2, \end{cases}$$

corresponding to the policy described in Theorem 4.4. Then

$$r(s, d_0'(s)) = \begin{cases} 0 & \text{for } s = 0, \\ \mu_{22} & \text{for } 1 \leq s \leq s^* - 1, \\ \mu_{12} & \text{for } s^* \leq s \leq B+1, \\ \alpha(\mu_{12}+\mu_{22}) & \text{for } s = B+2, \end{cases}$$

and

$$p(s'|s,d_0'(s)) = \begin{cases} \frac{\alpha(\mu_{11}+\mu_{21})}{q} & \text{for } s=0,\ s'=1, \\ 1-\frac{\alpha(\mu_{11}+\mu_{21})}{q} & \text{for } s=s'=0, \\ \frac{\mu_{22}}{q} & \text{for } 1\le s\le s^*-1,\ s'=s-1, \\ 1-\frac{\mu_{11}+\mu_{22}}{q} & \text{for } 1\le s\le s^*-1,\ s'=s, \\ \frac{\mu_{11}}{q} & \text{for } 1\le s\le s^*-1,\ s'=s+1, \\ \frac{\mu_{12}}{q} & \text{for } s^*\le s\le B+1,\ s'=s-1, \\ 1-\frac{\mu_{21}+\mu_{12}}{q} & \text{for } s^*\le s\le B+1,\ s'=s, \\ \frac{\mu_{21}}{q} & \text{for } s^*\le s\le B+1,\ s'=s+1, \\ \frac{\alpha(\mu_{12}+\mu_{22})}{q} & \text{for } s=B+2,\ s'=B+1, \\ 1-\frac{\alpha(\mu_{12}+\mu_{22})}{q} & \text{for } s=s'=B+2, \end{cases}$$

where q is the uniformization constant. Since the policy $(d_0')^\infty$ (corresponding to the decision rule d_0') is irreducible, we find a scalar g_0' and a vector h_0' solving

$$r_{d_0'} - g_0'e + (P_{d_0'} - I)h_0' = 0, \tag{12}$$

subject to $h_0'(0) = 0$, where e is again a column vector of ones and I is the identity matrix. Then

$$g_0' = \frac{\Theta_1}{\Theta_2},$$

where

$$\begin{aligned} \Theta_1 &= \alpha(\mu_{11}+\mu_{21})\Big(\frac{\sum_{j=0}^{s^*-1}\mu_{11}^j\mu_{22}^{s^*-1-j}}{\mu_{22}^{s^*-1}} + \frac{\mu_{11}^{s^*-1}\mu_{21}\sum_{j=0}^{B+1-s^*}\mu_{21}^j\mu_{12}^{B+1-s^*-j}}{\mu_{12}^{B+2-s^*}\mu_{22}^{s^*-1}}\Big), \\ \Theta_2 &= 1 + \frac{\alpha(\mu_{11}+\mu_{21})\sum_{j=0}^{s^*-2}\mu_{11}^j\mu_{22}^{s^*-2-j}}{\mu_{22}^{s^*-1}} + \frac{\alpha(\mu_{11}+\mu_{21})\mu_{11}^{s^*-1}}{\mu_{12}^{B+2-s^*}\mu_{22}^{s^*-1}}\Big(\sum_{j=0}^{B+1-s^*}\mu_{21}^j\mu_{12}^{B+1-s^*-j} + \frac{\mu_{21}^{B+2-s^*}}{\alpha(\mu_{12}+\mu_{22})}\Big), \end{aligned}$$

$h_0'(0) = 0$,

$$\begin{aligned} h_0'(s) &= \frac{qg_0'}{\alpha(\mu_{11}+\mu_{21})\mu_{11}^{s-1}}\Big[(\alpha(\mu_{11}+\mu_{21}) - \mu_{11} + \mu_{22})\sum_{j=0}^{s-2}(j+1)\mu_{11}^j\mu_{22}^{s-2-j} + s\mu_{11}^{s-1}\Big] \\ &\quad -\frac{q\mu_{22}}{\mu_{11}^{s-1}}\sum_{j=0}^{s-2}(j+1)\mu_{11}^j\mu_{22}^{s-2-j} \end{aligned}$$

for $1 \le s \le s^*$, and

$$\begin{aligned} h_0'(s) &= h_0'(s^*) + \frac{q\mu_{12}}{\mu_{21}^{s-s^*}\mu_{11}^{s^*-1}}\sum_{j=0}^{s-s^*-1}\mu_{21}^j\mu_{12}^{s-s^*-1-j}\Big[\frac{g_0'}{\alpha(\mu_{11}+\mu_{21})}\Big(\sum_{k=0}^{s^*-2}\mu_{11}^k\mu_{22}^{s^*-2-k}(\alpha(\mu_{11}+\mu_{21}) - \mu_{11} + \\ &\qquad \mu_{22}) + \mu_{11}^{s^*-1}\Big) - \mu_{22}\sum_{i=0}^{s^*-2}\mu_{11}^i\mu_{22}^{s^*-2-i}\Big] + \frac{q}{\mu_{21}^{s-s^*}}(g_0' - \mu_{12})\sum_{j=0}^{s-s^*-1}(j+1)\mu_{21}^j\mu_{12}^{s-s^*-j-1} \end{aligned}$$

for $s^*+1 \le s \le B+2$, constitute a solution to equation (12).

For the next step of the policy iteration algorithm, we choose

$$d_1'(s) \in \arg\max_{a\in A_s} \Big\{ r(s,a) + \sum_{j\in S} p(j|s,a) h_0'(j) \Big\}, \quad \forall s \in S,$$

setting $d_1'(s) = d_0'(s)$ if possible. We now show that $d_1'(s) = d_0'(s)$ for all $s \in S$. In particular, for all $s \in S \setminus \{0, B+2\}$ and $a \in A_s \setminus \{d_0'(s)\}$, we will compute the differences

$$\epsilon'(s,a) \;=\; r(s,d_0'(s)) + \sum_{j\in S} p(j|s,d_0'(s)) h_0'(j) - \Big(r(s,a) + \sum_{j\in S} p(j|s,a) h_0'(j) \Big)$$

and show that the differences are non-negative. Note that for $s = 0$ and $s = B+2$, there is nothing to prove because there is only one possible action in these states, namely $d_0'(0) = a_{11}$ and $d_0'(B+2) = a_{22}$.

For $s \in \{1, \ldots, s^*-1\}$, we have that $d_0'(s) = a_{12}$. We will specify $\epsilon'(s,a)$ for actions a_{11}, a_{21}, and a_{22}. Without loss of generality, assume that $s^* > 1$ because otherwise, this set of states is empty and there is nothing to prove. With some algebra, we obtain

$$\epsilon'(s,a_{11}) \;=\; \frac{\alpha(\mu_{11}+\mu_{21})\mu_{11}^{s^*-s-1} \sum_{j=0}^{s-1} \mu_{11}^j \mu_{22}^{s-1-j} \Big[\mu_{21}^{B+2-s^*} \Upsilon_1(\alpha) + \sum_{j=0}^{B+1-s^*} \mu_{21}^j \mu_{12}^{B+1-s^*-j} \Delta_1(\alpha) \Big]}{\Upsilon'},$$

where

$$\begin{aligned}
\Upsilon' \;&=\; \alpha(\mu_{11}+\mu_{21})(\mu_{12}+\mu_{22}) \Big[\mu_{12}^{B+2-s^*} \sum_{j=0}^{s^*-2} \mu_{11}^j \mu_{22}^{s^*-2-j} + \mu_{11}^{s^*-1} \sum_{j=0}^{B+1-s^*} \mu_{12}^j \mu_{21}^{B+1-s^*-j} \Big] \\
&\quad + (\mu_{12}+\mu_{22})\mu_{22}^{s^*-1}\mu_{12}^{B+2-s^*} + (\mu_{11}+\mu_{21})\mu_{21}^{B+2-s^*}\mu_{11}^{s^*-1} \\
&> \; 0
\end{aligned}$$

and $\Upsilon_1(\alpha)$ is defined in the proof of Theorem 3.4. As mentioned in the proof of Theorem 3.4, if $\alpha \le 1$ and $\mu_{11}\mu_{22} \ge \mu_{12}\mu_{21}$, then $\Upsilon_1(\alpha) \ge 0$. On the other hand, if $\mu_{12}\mu_{21} > \mu_{11}\mu_{22}$, then $\mu_{12} > \mu_{22}$ and $\Upsilon_1(\alpha) \le 0$ if and only if $\alpha \ge \frac{2\mu_{11}\mu_{22}+\mu_{11}\mu_{12}+\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}+\mu_{22})} \ge \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})}$. Then we know from part (ii) of Lemma 4.8 that $f_4(i,\alpha) < 0$ for all $i \in \{2, \ldots, B+2\}$. But then $s^* = 1$, which is a contradiction. Similarly, if $\mu_{11}\mu_{22} \ge \mu_{12}\mu_{21}$, we know from the proof of part (i) of Lemma 4.8 that $\Delta_1(\alpha) \ge 0$. On the other hand, if $\mu_{12}\mu_{21} > \mu_{11}\mu_{22}$, then $\Delta_1(\alpha) \le 0$ if and only if $\alpha \ge \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}-\mu_{22})}$. Then we know from part (ii) of Lemma 4.8 that $f_4(i,\alpha) < 0$ for all $i \in \{2, \ldots, B+2\}$. But then $s^* = 1$, which is a contradiction. Thus, $\epsilon'(s,a_{11}) \ge 0$ with an equality only if $\mu_{11}\mu_{22} = \mu_{12}\mu_{21}$.

Similarly,

$$\begin{aligned}
\epsilon'(s,a_{21}) \;&=\; \frac{f_4(s^*,\alpha)}{\Upsilon'} + \\
&\quad \frac{(\mu_{11}\mu_{12}-\mu_{21}\mu_{22})\mu_{22}^{s-1} \sum_{j=0}^{s^*-s-2} \mu_{11}^j \mu_{22}^{s^*-s-2-j} \Big[\mu_{21}^{B+2-s^*} \Upsilon_1(\alpha) + \sum_{j=0}^{B+1-s^*} \mu_{21}^j \mu_{12}^{B+1-s^*-j} \Delta_1(\alpha) \Big]}{\Upsilon'}.
\end{aligned}$$

One can immediately conclude that $\epsilon'(s, a_{21}) \geq 0$ (with an equality only if $\mu_{11}\mu_{22} = \mu_{12}\mu_{21}$ and $f_4(s^*, \alpha) = 0$) since $\Upsilon' > 0$, $f_4(s^*, \alpha) \geq 0$, $\mu_{11} \geq \mu_{21}$, $\mu_{12} \geq \mu_{22}$, $\Upsilon_1(\alpha) \geq 0$, and $\Delta_1(\alpha) \geq 0$ as discussed above.

On the other hand,

$$\begin{aligned}
& \epsilon'(s, a_{22}) \\
= \;& \alpha(\mu_{12} + \mu_{22})\Big[\frac{\mu_{22}^{s-1}(\mu_{12}^{B+2-s^*}\sum_{j=0}^{s^*-s-1}\mu_{11}^{j}\mu_{22}^{s^*-s-1-j}\Upsilon_1(\alpha) + \mu_{11}^{s^*-s}\sum_{j=0}^{B+1-s^*}\mu_{21}^{j}\mu_{12}^{B+1-s^*-j}\Upsilon_2(\alpha))}{\Upsilon'} \\
& + \frac{\mu_{11}^{s^*-s}\sum_{j=0}^{s-2}\mu_{11}^{j}\mu_{22}^{s-2-j}\sum_{k=0}^{B+1-s^*}\mu_{21}^{k}\mu_{12}^{B+1-s^*-k}(-\Delta_2(\alpha))}{\Upsilon'}\Big],
\end{aligned}$$

where

$$\Upsilon_2(\alpha) \;=\; -\alpha(\mu_{11} + \mu_{21})(\mu_{12} + \mu_{22}) + 2\mu_{12}\mu_{21} + \mu_{11}\mu_{12} + \mu_{21}\mu_{22}.$$

Note that $\Upsilon_1(\alpha) \geq 0$ as mentioned above. If $\mu_{12}\mu_{21} \geq \mu_{11}\mu_{22}$ and $\alpha \leq 1$, then $\Upsilon_2(\alpha) \geq 0$. On the other hand, if $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$, then $\Upsilon_2(\alpha) \leq 0$ if and only if $\alpha \geq \frac{2\mu_{12}\mu_{21}+\mu_{11}\mu_{12}+\mu_{21}\mu_{22}}{(\mu_{11}+\mu_{21})(\mu_{12}+\mu_{22})} \geq \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})}$. Then we know from part (i) of Lemma 4.8 that $f_4(i, \alpha) > 0$ for all $i \in \{2, \ldots, B+2\}$. But then $s^* = B + 2$, which implies that $\sum_{j=0}^{B+1-s^*}\mu_{21}^{j}\mu_{12}^{B+1-s^*-j}\Upsilon_2(\alpha) = 0$. Finally, we know from the proof of Lemma 4.8 that if $\mu_{12}\mu_{21} \geq \mu_{11}\mu_{22}$, then $\Delta_2(\alpha) \leq 0$. If $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$, then $\Delta_2(\alpha) \geq 0$ if and only if $\alpha \geq \frac{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})}$ and part (i) of Lemma 4.8 implies that $f_4(i, \alpha) > 0$ for all $i \in \{2, \ldots, B + 2\}$. But then $s^* = B + 2$, and, hence, $\sum_{j=0}^{B+1-s^*}\mu_{21}^{j}\mu_{12}^{B+1-s^*-j}(-\Delta_2(\alpha)) = 0$. Thus, $\epsilon'(s, a_{22}) \geq 0$ with an equality only if $\mu_{11}\mu_{22} = \mu_{12}\mu_{21}$.

Next we consider $s \in \{s^*, \ldots, B + 1\}$, where we have that $d_0'(s) = a_{21}$. We will specify $\epsilon'(s, a)$ for actions a_{11}, a_{12}, and a_{22}. Without loss of generality, assume that $s^* < B+2$ because otherwise, this set of states is empty and there is nothing to prove. With some algebra we obtain

$$\begin{aligned}
\epsilon'(s, a_{11}) \;=\;& \alpha(\mu_{11} + \mu_{21})\Big[\frac{\sum_{j=0}^{s^*-2}\mu_{11}^{j}\mu_{22}^{s^*-2-j}\Delta_1(\alpha)(\mu_{12}^{s-s^*+1}\sum_{k=0}^{B-s}\mu_{21}^{k}\mu_{12}^{B-s-k} + \mu_{12}^{B+1-s^*})}{\Upsilon'} \\
& + \frac{\mu_{12}\mu_{21}^{B+1-s^*}\sum_{j=0}^{s^*-2}\mu_{11}^{j}\mu_{22}^{s^*-2-j}\Upsilon_1(\alpha) + \mu_{11}^{s^*-1}\mu_{21}^{B+1-s}\sum_{i=0}^{s-s^*}\mu_{21}^{i}\mu_{12}^{s-s^*-i}\Upsilon_2(\alpha)}{\Upsilon'}\Big].
\end{aligned}$$

It follows from the proof of part (i) of Lemma 4.8 that if $\mu_{11}\mu_{22} \geq \mu_{12}\mu_{21}$, then $\Delta_1(\alpha) \geq 0$. On the other hand, if $\mu_{12}\mu_{21} > \mu_{11}\mu_{22}$, as discussed above $\sum_{j=0}^{s^*-2}\mu_{11}^{j}\mu_{22}^{s^*-2-j}\Delta_1(\alpha) = 0$ because $s^* = 1$. Similarly, if $\mu_{11}\mu_{22} \geq \mu_{12}\mu_{21}$, then $\Upsilon_1(\alpha) \geq 0$. If $\mu_{12}\mu_{21} > \mu_{11}\mu_{22}$ then $\sum_{j=0}^{s^*-2}\mu_{11}^{j}\mu_{22}^{s^*-2-j}\Upsilon_1(\alpha) = 0$ because $s^* = 1$. Finally, if $\mu_{12}\mu_{21} \geq \mu_{11}\mu_{22}$, then $\Upsilon_2(\alpha) \geq 0$. On the other hand, if $\mu_{12}\mu_{21} < \mu_{11}\mu_{22}$, then $s^* = B + 2$, which is a contradiction. Thus, $\epsilon'(s, a_{11}) \geq 0$ with an equality only if $\mu_{11}\mu_{22} = \mu_{12}\mu_{21}$.

Next we have

$$\begin{aligned}
\epsilon'(s, a_{12}) \;=\;& \frac{-f_4(s^*+1, \alpha)}{\Upsilon'} + \\
& \frac{(\mu_{11}\mu_{12} - \mu_{21}\mu_{22})\mu_{21}^{B+1-s}\sum_{j=0}^{s-s^*-1}\mu_{21}^{j}\mu_{12}^{s-s^*-1-j}\big[\mu_{22}^{s^*-1}\Upsilon_2(\alpha) + \sum_{j=0}^{s^*-2}\mu_{11}^{j}\mu_{22}^{s^*-2-j}(-\Delta_2(\alpha))\big]}{\Upsilon'}.
\end{aligned}$$

From the definition of s^*, $f_4(s^*+1,\alpha) \le 0$ and $\mu_{11}\mu_{12} - \mu_{21}\mu_{22} \ge 0$ from our assumptions on the service rates. If $\mu_{12}\mu_{21} \ge \mu_{11}\mu_{22}$ then $\Upsilon_2(\alpha) \ge 0$. On the other hand, as discussed above, if $\mu_{12}\mu_{21} < \mu_{11}\mu_{22}$, then $s^* = B+2$, which is a contradiction. Moreover, we know from the proof of Lemma 4.8 that if $\mu_{12}\mu_{21} \ge \mu_{11}\mu_{22}$, then $\Delta_2(\alpha) \le 0$. However, if $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$, then as discussed above, $s^* = B+2$, which is a contradiction. Thus, $\epsilon'(s, a_{12}) \ge 0$ with an equality only if $\mu_{11}\mu_{22} = \mu_{21}\mu_{12}$ and $f_4(s^*+1,\alpha) = 0$.

Finally,

$$\epsilon'(s, a_{22}) \quad = \quad \frac{\alpha(\mu_{12}+\mu_{22}) \sum_{j=0}^{B+1-s} \mu_{21}^{j} \mu_{12}^{B+1-s-j} \mu_{12}^{s-s^*} \left[\sum_{j=0}^{s^*-2} \mu_{11}^{j} \mu_{22}^{s^*-2-j} (-\Delta_2(\alpha)) + \mu_{22}^{s^*-1} \Upsilon_2(\alpha) \right]}{\Upsilon'}.$$

Using the arguments in the previous paragraph, one can immediately conclude that $\epsilon'(s, a_{22}) \ge 0$ with an equality only if $\mu_{11}\mu_{22} = \mu_{12}\mu_{21}$. This proves that $d_1'(s) = d_0'(s)$ for all $s \in S$. By Theorem 9.5.1 of Puterman [18] (which says that in a (weakly) communicating model, policy iteration terminates with an optimal policy) this proves that the policy described in Theorem 4.4 is optimal. The proof of the uniqueness of the optimal policy is similar to the uniqueness proof of Theorem 3.4 (the lower bound on α needs to be strict to ensure that the policies in Theorems 4.2 and 4.3 are not optimal and that idling actions are not optimal in states $1, \ldots, B+1$).

Proof of Proposition 4.1

We first show that when $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$, s^* can take at most two adjacent integer values for each $0 \le \alpha \le 1$. Note that our conditions on the service rates (i.e., $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$, $\mu_{11} \ge \mu_{21}$, and $\mu_{12} \ge \mu_{22}$) imply that $\mu_{11}, \mu_{12}, \mu_{22} > 0$ and $\mu_{11} > \mu_{21}$. If $\mu_{21} = 0$, $\mu_{12} = \mu_{22}$, and $\alpha \le \frac{\mu_{12}}{\mu_{12}+\mu_{22}}$, then Remark 4.2 implies that $s^* = B+2$. On the other hand, if $\alpha \le \frac{\mu_{12}}{\mu_{12}+\mu_{22}}$ and either $\mu_{21} > 0$ or $\mu_{12} > \mu_{22}$, it follows from expressions (2), (3), and (4) of Işık, Andradóttir, and Ayhan [14] that $f_1(i)$ is strictly decreasing in $i \in S \setminus \{0\}$, implying that s^* is either uniquely defined or can be chosen from two adjacent integers (depending on whether $f_1(i) = 0$ for some $i \in S \setminus \{0\}$). Moreover, when $\frac{\mu_{12}}{\mu_{12}+\mu_{22}} \le \alpha \le \frac{\mu_{11}}{\mu_{11}+\mu_{21}}$ and $\beta_2(\alpha) \ge 0$, (2) is strictly positive in the proof of Lemma 4.6 (because the second summation is strictly positive since $\mu_{11} > \mu_{21}$) and when $\beta_2(\alpha) < 0$, (3) is strictly negative (note that $\mu_{21} = 0$ implies that $\beta_2(\alpha) \ge 0$). Thus, either $s^* = B+2$ or $f_2(i,\alpha)$ is strictly decreasing in $i \in S \setminus \{0\}$, and, hence, s^* can take at most two adjacent integer values. Finally, we know from part (i) of Lemma 4.8 that $f_4(i,\alpha) > 0$ when $\frac{\mu_{11}\mu_{12} - \mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})} \le \alpha \le 1$, and, hence, $s^* = B+2$. Otherwise, $\beta_2(\alpha) < 0$, implying that the inequalities (8) and (9) (in the proof of part (v) of Lemma 4.8) are strict, and, thus, $f_4(i,\alpha)$ is strictly decreasing in $i \in S \setminus \{0\}$ and s^* can take at most two adjacent integer values. In order to see this, note that (8) and (9) are strictly negative as long as $\mu_{21} > 0$, but if $\mu_{21} = 0$, then part (i) of Lemma 4.8 will apply.

If $\alpha \le \frac{\mu_{12}}{\mu_{12}+\mu_{22}}$, then s^* is a constant function of α since $f_1(i)$ does not depend on α (as long as

s^* is chosen consistently when $f_1(i) = 0$ for some i). Moreover, for all $i \in \{1, \dots, B+2\}$,

$$\begin{aligned}
\frac{df_2(i,\alpha)}{d\alpha} &= \mu_{21}^{B+2-i} \sum_{j=0}^{i-2} \mu_{22}^{j} \mu_{11}^{i-1-j} (\mu_{12} + \mu_{22})(\mu_{11} - \mu_{21}) \\
&\quad + \mu_{22}^{i-1} \sum_{j=0}^{B-i+2} \mu_{21}^{j} \mu_{12}^{B-i+2-j} (\mu_{12} + \mu_{22})(\mu_{11} - \mu_{21}) \\
&> 0.
\end{aligned}$$

Thus, $f_2(i,\alpha)$ is strictly increasing in α, which implies that s^* is non-decreasing in α when $\mu_{11}\mu_{22} > \mu_{12}\mu_{21}$ and $\frac{\mu_{12}}{\mu_{12}+\mu_{22}} \le \alpha \le \frac{\mu_{11}}{\mu_{11}+\mu_{21}}$. Furthermore, with some algebra, we have

$$f_2(i, \frac{\mu_{12}}{\mu_{12} + \mu_{22}}) = f_1(i)$$

for all $i \in \{1, \dots, B+2\}$. Thus, when $\alpha = \frac{\mu_{12}}{\mu_{12}+\mu_{22}}$, $f_1(\cdot)$ and $f_2(\cdot,\cdot)$ yield the same optimal switch points.

Note that from the proof of Lemma 4.8, one can immediately see that

$$\frac{df_4(i,\alpha)}{d\alpha} = (\mu_{11} + \mu_{21})(\mu_{12} + \mu_{22}) \Big[\sum_{j=0}^{B+2-i} \mu_{21}^{j} \mu_{12}^{B+2-i-j} \mu_{22}^{i-2} (\mu_{22} - \mu_{12}) + \sum_{j=0}^{i-2} \mu_{22}^{j} \mu_{11}^{i-2-j} \mu_{21}^{B+2-i} (\mu_{11} - \mu_{21}) \Big]$$

for all $i \in \{2, \dots, B+2\}$. Thus, for a fixed $i \in \{2, \dots, B+2\}$, $f_4(i,\alpha)$ could be either strictly increasing or non-increasing in α. Let $0 \le \alpha \le 1$ and $i \le s^*$, so that $f_4(i,\alpha) \ge 0$. If $f_4(i,\alpha)$ is strictly increasing, then $i \le s^*$ for all $\alpha' \in [\alpha, 1]$. On the other hand, if $f_4(i,\alpha)$ is non-increasing in α, one can conclude that $f_4(i,\alpha') > 0$, and, hence, $i \le s^*$, for all $\alpha \le \alpha' \le \frac{\mu_{11}\mu_{12} - \mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})}$, because otherwise part (i) of Lemma 4.8 would contradict the assumption that $f_4(i,\alpha)$ is non-increasing in α. Finally, when $\alpha \in [\frac{\mu_{11}\mu_{12} - \mu_{21}\mu_{22}}{(\mu_{11}-\mu_{21})(\mu_{12}+\mu_{22})}, 1]$, part (i) of Lemma 4.8 implies that $s^* = B+2 \ge i$. Since α and $i \le s^*$ are arbitrary, we have shown that s^* is non-decreasing in α for $\frac{\mu_{11}}{\mu_{11}+\mu_{21}} \le \alpha \le 1$. Moreover, with some algebra, we have

$$f_2(i, \frac{\mu_{11}}{\mu_{11} + \mu_{21}}) = \frac{\mu_{11}}{\mu_{11} + \mu_{21}} f_4(i, \frac{\mu_{11}}{\mu_{11} + \mu_{21}})$$

for all $i \ge 0$, which implies that when $\alpha = \frac{\mu_{11}}{\mu_{11}+\mu_{21}}$, $f_2(\cdot,\cdot)$ and $f_4(\cdot,\cdot)$ yield the same optimal switch points. In all cases, the continuity of the functions $f_1(\cdot)$, $f_2(\cdot,\cdot)$, and $f_4(\cdot,\cdot)$ in α and the fact that $f_1(\cdot)$, $f_2(\cdot,\cdot)$, and $f_4(\cdot,\cdot)$ are strictly decreasing in i whenever $s^* < B+2$ imply that jumps in s^* are of size one. Thus, the proof is complete. □

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