Relation between frequency response and impulse response

We have derived

\[ y(n) = \sum_{m=-\infty}^{\infty} h(m) x(n - m) \]

Let \( x(n) = e^{i\omega n} \). So,

\[ y(n) = \sum_{m=-\infty}^{\infty} h(m) e^{i\omega(n-m)} \]

\[ = e^{i\omega n} \sum_{m=-\infty}^{\infty} h(m) e^{-i\omega m} \]
We also know that when \( x(n) = e^{i\omega n} \),

\[
\forall n \in \text{Integers}, \quad y(n) = H(\omega)e^{i\omega n}.
\]

So, we have

\[
\forall \omega \in \text{Reals}, \quad H(\omega) = \sum_{m=-\infty}^{\infty} h(m)e^{-i\omega m}.
\]

This is known as the discrete-time Fourier transform (DTFT). The frequency response of a discrete-time system is the DTFT of the impulse response.
Example.

\[ h(n) = \delta(n) + \delta(n - 1) + 2\delta(n - 2) + 5\delta(n - 3). \]
\[ H(\omega) = 1 + e^{-i\omega} + 2e^{-2i\omega} + 5e^{-5i\omega}. \]
Properties

1. If $h$ is real-valued, $H^*(-\omega) = H(\omega)$. This means that the magnitudes are the same, and thus the gain is the same for a frequency and its negative counterpart.

2. $\forall \omega \in \text{Reals}, H(\omega + 2\pi) = H(\omega)$. 
Continuous-time version

\[ H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}. \]

This is known as the continuous-time Fourier transform, but we will simply call it the Fourier transform (FT). The frequency response \( H \) of a continuous-time LTI system is the FT of the impulse response, \( h \).
Example.

\[ h(t) = 1.5, \ 0 \leq t \leq 1, \ \text{and zero otherwise.} \]
\[ H(\omega) = \int_0^1 1.5 e^{-i\omega t} \]
\[ = -\frac{1.5}{\omega} e^{-i\omega t} \bigg|_0^1 \]
\[ = \frac{1.5}{i\omega} - \frac{1.5}{i\omega} e^{-i\omega} \]
Property

\[ H^*(-\omega) = H(\omega) \]

This has the same implication as for the discrete-time version, that the magnitudes are the same, and thus the gain does not depend on the sign of the frequency.
A system is *causal* if the output value at a particular time depends only on the input values at that time or before.

\[ y(n) = \sum_{m=-\infty}^{\infty} h(m)x(n - m) \]

We must have \( h(m) = 0 \), for \( m < 0 \). Why?
When is causality important? not important?
Finite Impulse Response filters

Suppose that \( h(n) = 0 \) if \( n < 0 \) and \( h(n) = 0 \) if \( n \geq L \). Any system whose impulse response satisfies this property is a finite impulse response (FIR) filter. These are easy to work with, as the expression for the output reduces to

\[
y(n) = \sum_{m=0}^{L-1} x(n - m)h(m)
\]

This is easy to implement.
No equivalent notion for continuous-time, as integration over a finite interval not any easier than integration over an infinite interval.
Example 9.12

Suppose that

\[ y(n) = \frac{1}{L} \sum_{m=0}^{L-1} x(n - m) \]

This is LTI, and has impulse response

\[ h(n) = \begin{cases} 
1/L & 0 \leq n < L \\
0 & \text{otherwise}
\end{cases} \]

The frequency response is the FT of the impulse response:

\[ H(\omega) = \sum_{n=-\infty}^{\infty} h(m) e^{-i\omega m} = \frac{1}{L} \sum_{m=0}^{L-1} e^{-i\omega m} \]
Fact:

$$\sum_{m=0}^{L-1} a^m = \frac{1 - a^L}{1 - a}$$

So,

$$H(\omega) = \frac{1}{L} \left( \frac{1 - e^{-i\omega L}}{1 - e^{-i\omega}} \right)$$
L=4;
omega=[-pi:pi/250:pi];
H=(1/L)*((1-exp(-i*omega*L))./(1-exp(-i*omega)));
plot(omega,abs(H));
① only shown for $-\pi$ to $\pi$
② lowpass structure (zero frequency passed unchanged)
③ some higher frequencies eliminated, but overall uneven
Filter design

• Using a moving-average filter is too ad-hoc. What we would really like to do is specify the filter behaviour, then have that filter designed automatically.

• This is a deep field, we cannot do it any justice here, but you should be able to explore it on your own if you so desire.
fir1(8,0.5) designs an FIR filter using the “window” method. The output of this command is: $h(0) = 0$, $h(1) = -0.0227$, $h(2) = 0$, $h(3) = 0.2740$, $h(4) = 0.4974$, $h(5) = 0.2740$, $h(6) = 0$, $h(7) = -0.0227$, $h(8) = 0$. The result is much smoother.
Filter design

May wish to specify:

1. passband
2. stopband
3. ripple constraints
4. etc.

Several algorithms try to satisfy these constraints as much as possible.
Example 9.14

\[ h=\text{remez}(63, [0, 0.125, 0.25, 1]; [1, 1, 0, 0]) \]; where 63 is the highest order term in the filter, 0 and 0.125 specify the passband, 0.25 and 1 specify the stopband and [1,1,0,0] specifies that the passband has gain 1 and the stopband has gain 0.