The Knaster-Tarski Fixed Point Theorem for Complete Partial Orders

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Complete Lattices

Let \((X, \leq)\) be a be partially ordered set, and let \(S \subseteq X\).

Then \(\bigvee S\) ("the meet of \(S\" ) denotes the least upper bound of \(S\) with respect to \(\leq\), if it exists. Similarly, \(\bigwedge S\) ("the join of \(S\") denotes the greatest lower bound of \(S\) with respect to \(\leq\).

We say that \(X\) is a complete lattice if, for every \(S \subseteq X\), then both \(\bigvee S\) and \(\bigwedge S\) exist in \(X\).

**Example 1:** Take \((\mathbb{N} \cup \{0\}, \preceq)\), where \(a \preceq b\) if \(a\) divides \(b\). This is a complete lattice, where \(a \wedge b = \gcd(a, b)\), \(a \vee b = \lcm(a, b)\), \(\bot = 1\), and \(\top = 0\).

**Example 2:** Take \((\mathcal{P}(X), \subseteq)\) for any set \(X\). This is a
complete lattice, where \( \lor S = \bigcup S \), \( \land S = \bigcap S \), \( \bot = \emptyset \), and \( \top = X \).
Definitions

Let \((X, \leq_X)\) and \((Y, \leq_Y)\) be partially ordered sets.

A function \(\varphi : X \to Y\) is monotone or order-preserving if \(x_1 \leq_X x_2\) implies \(f(x_1) \leq_Y f(x_2)\).

A point \(x \in X\) is a fixed point of a function \(\varphi : X \to Y\) if \(\varphi(x) = x\). Denote the set of fixed points of \(\varphi\) by \(\text{fix}(\varphi)\).
Knaster-Tarski Fixed-Point Theorem for Complete Lattices

Let $L$ be a complete lattice and $\psi : L \to L$ be monotone. Then

$$\bigvee \{ x \in L \mid x \leq \psi(x) \} \in \text{fix}(\psi)$$

**Proof:** Let $H = \{ x \in L \mid x \leq \psi(x) \}$ and $\alpha = \bigvee H$. For all $x \in H$ we have $x \leq \alpha$, so then $x \leq \psi(x) \leq \psi(\alpha)$ by monotonicity. Therefore $\psi(\alpha)$ is an upper bound for $H$, hence $\alpha \leq \psi(\alpha)$.

Then, by monotonicity, we have $\psi(\alpha) \leq \psi(\psi(\alpha))$, so $\psi(\alpha) \in H$, and therefore $\psi(\alpha) \leq \alpha$. Hence by antisymmetry, $\alpha = \psi(\alpha)$. 

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The Knaster-Tarski theorem can be generalized to state that \( \text{fix}(\Psi) \) is itself a complete lattice, hence a smallest fixed-point of \( \Psi \) can be chosen. We will state a further generalization to complete partial orders. First, some more definitions:

Let \((X, \leq)\) be a partially ordered set. Let \(S \subseteq X\) with \(S \neq \emptyset\).

\(S\) is a directed subset if whenever \(x, y \in S\) there exists \(z \in S\) such that \(x \leq z\) and \(y \leq z\).

**Example:** Take \((\mathbb{N} \cup \{0\}, \preceq)\) as before, i.e. \(a \preceq b\) if \(a\) divides \(b\). Then \(\{2, 3\}\) is not a directed subset of \(X\), but \(\{2, 3, 6\}\) is.
Complete Partially Ordered Sets

Let \((X, \leq)\) be a partially ordered set. We say that \(X\) is a complete partially ordered set (CPO) if:

- \(X\) has a bottom element \(\bot\).
- \(\bigvee D\) exists for each directed subset \(D\) of \(X\).

Any complete lattice is a CPO, as is any finite partially ordered set with a least element \((\bot)\).

Example: Take the set of all partial functions defined on some set \(S\), where \(f \leq g\) if \(\text{dom}(f) \subseteq \text{dom}(g)\) and \(f(x) = g(x)\) for all \(x \in \text{dom}(f)\). This is a CPO, with the least element being the function defined nowhere.
Knaster-Tarski Fixed-Point Theorem for Complete Partial Orders

Let $X$ be a CPO and $\Psi : P \to P$ be an order-preserving map. Then $\text{fix}(\Psi)$ is a CPO.

In particular, this means that we can choose a smallest element from $\text{fix}(\Psi)$.
Consequences

The Knaster-Tarski theorem has many applications and consequences.

In mathematics, it provides a short proof of the Schröder-Bernstein Theorem. In computer science, it is used heavily in the field of denotational semantics and abstract interpretation, where the existence of fixed points can be exploited to guarantee well-defined semantics for a recursive algorithm.
Fixed Points and Program Semantics

The semantics of recursive functions are difficult to specify: to use a function name before it is defined. The factorial function can be specified as the unique fixed point of the function

\[ F = \lambda f . \lambda x . \text{if } x = 0 \text{ then } 1 \text{ else } x \times f(x - 1) \]

It is clear that \texttt{factorial} is a fixed point, since

\[ F \texttt{factorial} = \lambda x . \text{if } x = 0 \text{ then } 1 \text{ else } x \times \texttt{factorial}(x - 1) = \texttt{factorial} \]

The factorial can be represented in lambda-calculus as \( YF \), with \( F \) as above and \( Y \) given by

\[ Y = \lambda G . (\lambda g . G(g
g))(\lambda g . G(g
g)) \]

\( Y \) is the \( Y \) combinator discovered by Haskell Curry. It finds a fixed point of its argument if it exists.
The Schröder-Bernstein Theorem

Let $f : A \to B$ and $g : B \to A$ be injections. Then there exists a bijection $h$ between $A$ and $B$.

**Proof:** As stated earlier $\mathcal{P}(A)$ is a complete lattice. For any $S \subseteq A$, define $\varphi : \mathcal{P}(A) \to \mathcal{P}(A)$ by $\varphi(S) = A \setminus g(B \setminus f(S))$. Since $f, g$ are injective, then $\varphi$ is monotone. Hence $\varphi$ has a fixed point $C \subseteq A$, therefore $A \setminus C = g(B \setminus f(C))$.

Then we can define a bijection $h : A \to B$ simply by setting $h(x) = f(x)$ if $x \in C$, and $h(x) = g^{-1}(x)$ if $x \in A \setminus C$. 