The Knaster-Tarski Fixed Point Theorem for Complete Partial Orders

Stephen Forrest

October 31, 2005

Complete Lattices

Let $(X, \leq)$ be a be partially ordered set, and let $S \subseteq X$.

Then $\bigvee S$ ("the *meet* of S") denotes the least upper bound of S with respect to $\leq$, if it exists. Similarly, $\bigwedge S$ ("the *join* of S") denotes the greatest lower bound of S with respect to $\leq$.

We say that X is a *complete lattice* if, for every $S \subseteq X$, then both $\bigvee S$ and $\bigwedge S$ exist in X.

Example 1: Take $(\mathbb{N} \cup \{0\}, \preceq)$, where $a \preceq b$ if a divides b. This is a complete lattice, where $a \wedge b = \text{gcd}(a, b)$, $a \vee b = \text{lcm}(a, b)$, $\bot = 1$, and $\top = 0$.

Example 2: Take $(\mathcal{P}(X), \subseteq)$ for any set X. This is a complete lattice, where $\bigvee S = \bigcup S$, $\bigwedge S = \bigcap S$, $\bot = \emptyset$, and $\top = X$.

Definitions

Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be partially ordered sets.

A function $\varphi : X \to Y$ is *monotone* or *order-preserving* if $x_1 \leq_X x_2$ implies $f(x_1) \leq_Y f(x_2)$.

A point $x \in X$ is a *fixed point* of a function $\varphi : X \to Y$ if $\varphi(x) = x$. Denote the set of fixed points of φ by fix(φ).

Knaster-Tarski Fixed-Point Theorem for Complete Lattices

Let L be a complete lattice and $\Psi : L \to L$ be monotone. Then

$$\bigvee\{x \in L \mid x \leq \Psi(x)\} \in \mathsf{fix}(\Psi)$$

Proof: Let $H = \{x \in L \mid x \leq \Psi(x)\}$ and $\alpha = \bigvee H$. For all $x \in H$ we have $x \leq \alpha$, so then $x \leq \Psi(x) \leq \Psi(\alpha)$ by monotonicity. Therefore $\Psi(\alpha)$ is an upper bound for H, hence $\alpha \leq \Psi(\alpha)$.

Then, by monotonicity, we have $\Psi(\alpha) \leq \Psi(\Psi(\alpha))$, so $\Psi(\alpha) \in H$, and therefore $\Psi(\alpha) \leq \alpha$. Hence by antisymmetry, $\alpha = \Psi(\alpha)$.

Knaster-Tarski (cont'd)

The Knaster-Tarski theorem can be generalized to state that fix(Ψ) is itself a complete lattice, hence a smallest fixed-point of Ψ can be chosen. We will state a further generalization to complete partial orders. First, some more definitions:

Let $(X, \leq)$ be a partially ordered set. Let $S \subseteq X$ with $S \neq \emptyset$.

S is a *directed subset* if whenever $x, y \in S$ there exists $z \in S$ such that $x \leq z$ and $y \leq z$.

Example: Take $(\mathbb{N} \cup \{0\}, \preceq)$ as before, i.e. $a \preceq b$ if a divides b. Then $\{2, 3\}$ is not a directed subset of X, but $\{2, 3, 6\}$ is.

Complete Partially Ordered Sets

Let $(X, \leq)$ be a partially ordered set. We say that X is a *complete partially ordered set* (CPO) if:

- X has a bottom element $\perp$.

- $\bigvee D$ exists for each directed subset D of X.

Any complete lattice is a CPO, as is any finite partially ordered set with a least element ($\perp$).

Example: Take the set of all partial functions defined on some set S, where $f \leq g$ if $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $f(x) = g(x)$ for all $x \in \mathrm{dom}(f)$. This is a CPO, with the least element being the function defined nowhere.

Knaster-Tarski Fixed-Point Theorem for Complete Partial Orders

Let X be a CPO and $\Psi : P \to P$ be an order-preserving map. Then fix(Ψ) is a CPO.

In particular, this means that we can choose a smallest element from fix(Ψ).

Consequences

The Knaster-Tarski theorem has many applications and consequences.

In mathematics, it provides a short proof of the Schröder-Bernstein Theorem. In computer science, it is used heavily in the field of denotational semantics and abstract interpretation, where the existence of fixed points can be exploited to guarantee well-defined semantics for a recursive algorithm.

Fixed Points and Program Semantics

The semantics of recursive functions are difficult to specify: to use a function name before it is defined. The factorial function can be specified as the unique fixed point of the function

$$F = \lambda f \;.\; \lambda x \;.\; \text{if } x = 0 \text{ then } 1 \text{ else } x * f(x-1)$$

It is clear that **factorial** *is* a fixed point, since

$$F\ \textbf{factorial} = \lambda x \;.\; \text{if } x = 0 \text{ then } 1 \text{ else } x * \textbf{factorial}(x-1) = \textbf{factoria}$$

The factorial can be represented in lambda-calculus as $Y F$, with F as above and Y given by

$$Y = \lambda G \;.\; (\lambda g \;.\; G(g\ g))(\lambda g \;.\; G(g\ g))$$

Y is the Y combinator discovered by Haskell Curry. It finds a fixed point of its argument if it exists.

The Schröder-Bernstein Theorem

Let $f : A \to B$ and $g : B \to A$ be injections. Then there exists a bijection h between A and B.

Proof: As stated earlier $\mathcal{P}(A)$ is a complete lattice. For any $S \subseteq A$, define $\varphi : \mathcal{P}(A) \to \mathcal{P}(A)$ by $\varphi(S) = A \setminus g(B \setminus f(S))$. Since f, g are injective, then φ is monotone. Hence φ has a fixed point $C \subseteq A$, therefore $A \setminus C = g(B \setminus f(C))$.

Then we can define a bijection $h : A \to B$ simply by setting $h(x) = f(x)$ if $x \in C$, and $h(x) = g^{-1}(x)$ if $x \in A \setminus C$.