Verifying a Border Array in Linear Time

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Abstract

A border of a string \( x \) is a proper (but possibly empty) prefix of \( x \) that is also a suffix of \( x \). The border array \( \beta = \beta[1..n] \) of a string \( x = x[1..n] \) is an array of nonnegative integers in which each element \( \beta[i] \), \( 1 \leq i \leq n \), is the length of the longest border of \( x[1..i] \). In this paper we first present a simple linear-time algorithm to determine whether or not a given array \( y = y[1..n] \) of integers is a border array of some string on an alphabet of unbounded size and then a slightly more complex linear-time algorithm for an alphabet of any given (bounded) size \( \alpha \). We then consider the problem of generating all possible distinct border arrays of given length \( n \) on a bounded or unbounded alphabet, and doing so in time proportional to the number of arrays generated. A previously published algorithm that claims to solve this problem in constant time per array generated is shown to be incorrect, and new algorithms are proposed. We conclude with an equally efficient on-line algorithm for this problem.

1 Introduction

The classical method for computing the border array \( \beta = \beta[1..n] \) of a given string \( x = x[1..n] \) is the so-called “failure function” algorithm [AHU74], that executes in \( O(n) \) time. A recent paper [MSM99] introduces the idea of \( b \)-equivalent — that is, strings with the same border array — and shows how to construct, on a standard alphabet, \( b \)-canonical strings that are the unique representatives of each \( b \)-equivalent class. The paper then describes an algorithm to generate all possible border arrays of length \( n \) together with their corresponding \( b \)-canonical strings in time proportional to the number of arrays generated. If \( b_n \) denotes the

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number of distinct border arrays of length \( n \) that can exist when the alphabet is unbounded, then the sequence

\[
B = \{ b_1, b_2, \ldots \} \\
= \{ 1, 2, 4, 9, 20, 47, 110, 263, 630, 1525, \ldots \}
\]

is shown to be a new integer sequence [SP95].

In this paper we extend the results of [MSM99] in two ways:

(1) We describe an \( O(n) \)-time algorithm that determines whether or not a given array \( y[1..n] \) of integers is a border array of some string (on a bounded or unbounded alphabet)

(2) We show that the [MSM99] algorithm to generate all possible border arrays is actually incorrect, in the sense that it requires more than constant time per string generated. We then describe a time- and space-optimal algorithm that generates all border arrays of length at most \( n \) (on a bounded or unbounded alphabet) without the need to store the underlying \( b \)-canonical strings. These arrays constitute a new infinite class of integer sequences.

Unlike the algorithm described in [MSM99], this algorithm generates a trie in a depth-first fashion and so is not on-line — that is, the set of border arrays for \( n+1 \) cannot be efficiently derived from the set of border arrays for \( n \). We conclude by describing another algorithm that is on-line and achieves constant time per string generated.

2 Identifying Valid Border Arrays For Unbounded Alphabets

This paper deals with arrays \( y = y[1..n] \) of nonnegative integers. For these arrays it will be convenient to make use of the notation \( y'[i] = y[i] \) for every \( i \in 1..n \), while

\[
y'[i] = y[y'^{-1}[i]]
\]

for every \( j > 1 \) such that \( y'^{-1}[i] \in 1..n \). It follows from the definition of border that for a border array \( \beta \), \( 0 \leq \beta[i] < i \) for every \( i \), so that the sequence \( i, \beta[i], \beta^2[i], \ldots \) is monotone decreasing to zero, hence finite. We state a well-known result [AHU74]:

Lemma 2.1 For some integer \( n \geq 1 \), let \( x = x[1..n] \) denote a string with border array \( \beta \). Let \( k \) be the integer such that \( \beta^k[n] = 0 \). Then

(a) for every integer \( j \in 1..k \), \( x[1..\beta^j[n]] \) is a border of \( x[1..n] \);

(b) for any choice of letter \( \lambda \), every border of \( x[1..n+1] = x[1..n]\lambda \) has a length that is an element of the following set:

\[
S^n = \{ S^0_n, S^1_n, \ldots, S^k_n \} \\
= \{ 0, \beta[n]+1, \beta[n]+1, \ldots, \beta^k[n]+1 \}. \quad \square
\]
The set \( S^n \) defined in Lemma 2.1(b) is called the *admissible set* of the border array \( \beta[1..n] \), and each of its elements \( S_j^n \), \( j = 0, 1, \ldots, k \) is called an *admissible extension* of \( \beta \). Thus the lemma tells us that the only possible border arrays \( \beta[1..n+1] = \beta[1..n|m] \) are those for which \( m \) is an admissible extension. To see that the converse is not true — that is, that not all admissible extensions give rise to border arrays — consider the following example \( (n = 11) \):

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\beta &=& 0 & 0 & 1 & 1 & 2 & 3 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

Here the border array \( \beta \) corresponds to a string \( x = abaabaabaaba \), for example. In fact, it is easy to see that, up to an isomorphism on the alphabet, this string is the *only* one that corresponds to \( \beta \). From Lemma 2.1(b) we see that the admissible extensions \( m \) of \( \beta \) are

\[
m = \begin{cases} 
0 & \\
\beta[11]+1 = 7 \\
\end{cases}
\]

Of these five admissible extensions, only three \((m = 0, 7, 4)\) can actually be used to extend \( \beta \) to a border array 001123234567; these extensions correspond to appending the letters \( c, b, a \), respectively, to \( x \). (Of course, in place of \( c \), any letter other than \( a \) or \( b \) could be used.) The extensions \( m = 1, 2 \) do not yield valid border arrays because, even though they give the lengths of borders of \( x[1..12] = x[1..11]a \) and \( x[1..11]b \), respectively, they do not give the lengths of the *longest* borders.

In order to characterize those values of \( m \) that can be used to extend a border array \( \beta[1..n] \) to a border array \( \beta[1..n|m] \), we make use of the following definition:

A nonzero admissible extension \( m \) of a border array \( \beta \) is said to be *invalid* if and only if there exists an admissible extension \( m' \) of \( \beta \) such that \( m = \beta[m'] \). Any other admissible extension of \( \beta \) is said to be a *valid* extension.

It follows from this definition that for \( n \geq 1 \) the admissible extensions

\[
S^n_0 = 0, \quad S^n_1 = \beta[n]+1 
\]

are always valid. We shall see in Theorem 2.2 that every valid extension determines a distinct border array; thus \( b_n \geq 2^{n-1} \), as in fact we have seen in the sequence \( B \) whose first ten terms were given in the Introduction.

Observe that this definition is useful only for an unbounded alphabet. For example, the border array

\[
\beta[1..15] = 001012301234567
\]
corresponds to a string
\[ x = abacabaabacaba \]
and has five valid extensions \( m = 0, 8, 4, 2, 1 \) that result from appending the
letters \( c, d, e, b, a \), respectively, to \( x \). However, if the alphabet size were limited
to \( \alpha = 4 \), we would presumably not wish to regard \( m = 0 \) as “valid”. The
following theorem provides a justification for our use of this term.

**Theorem 2.2** For every \( n \geq 1 \), an integer array \( y = y[1..n] \) is a border array if
and only if \( y[1] = 0 \) and each \( y[i] \) is a valid extension of \( y[1..i-1] \), \( i = 2, 3, \ldots, n \).

**Proof** The result is trivially true for \( n = 1 \), and so we may suppose \( n \geq 2 \).

To prove necessity, suppose that for some \( i \in 1..n-1 \),

\[ y[1..i] \text{ and } y[1..i+1] = y[1..i]m \]

are both border arrays, and let \( x = x[1..i+1] \) denote a string with border array
\( y[1..i+1] \). By Lemma 2.1(b), \( m \) must be an admissible extension of \( y[1..i] \). We
suppose however that \( m \) is invalid and derive a contradiction.

Since \( m \) is invalid, there exists an admissible extension \( m' > m \) of \( y[1..i] \)
such that \( y[m'] = m \). Then \( m' = y'[i] + 1 \) for some integer \( r \geq 1 \), and the
following statements are true:

1. \( x[1..m] = x[i-m+2..i+1] \) since \( y[i+1] = m \);
2. \( x[1..m] = x[m'-m+1..m'] \) since \( y[m'] = m \);
3. \( x[1..m'-1] = x[i-m'+2..i] \) since \( m'-1 = y'[i] \).

From (1) and (2) we conclude that

\[ x[m'] = x[m] = x[i+1], \]

so that (3) can be extended to

\[ x[1..m'] = x[i-m'+2..i+1]. \]

Thus \( x[1..i+1] \) has a border of length \( m' > m \), contradicting the assumption
that \( y[1..i+1] = y[1..i]m \) is a border array. We conclude that \( m \) must be valid,
as required.

To prove sufficiency, let \( y = y[1..n] \) be an array such that \( y[1] = 0 \) and each
\( y[i] \) is a valid extension of \( y[1..i-1] \), \( i = 2, 3, \ldots, n \). We show by induction that
\( y \) is a border array of some string.

Since \( y[1] = 0 \), the result holds for \( n = 1 \). Suppose then that for \( n \geq 2 \) and
some \( i \in 2..n \), \( y = y[1..i-1] \) is a border array of some string \( x[1..i-1] \). We show
that therefore \( y[1..i] \) must be a border array.

Let \( m = y[i] \). By hypothesis \( m \) is a valid extension of \( y[1..i-1] \) and so by
Lemma 2.1(b) two cases arise:
m = 0. In this case \( y[1..i] \) is a border array of a string \( x[1..i-1] \), where the letter \( \lambda \) is chosen to be distinct from every previous letter in \( x[1..i-1] \).

\( m \geq 0 \). Here \( m = y'[i-1]+1 \) for some integer \( p \geq 1 \), so that by the inductive hypothesis \( x[1..m-1] \) is a border of \( x[1..i-1] \). Then we can choose \( x[i] = x[m] \), so that \( x[1..m] \) is a border of \( x[1..i] \) — we want to show that it is the longest border.

If \( x[1..m] \) is not the longest border of \( x[1..i] \), there must exist a longer border \( x[1..m'] \) such that \( m = y[m'] \). By Lemma 2.1(b), \( m' = y'[i-1]+1 \) for some positive integer \( r < p \). But then by definition \( m \) is invalid, contrary to the original assumption that each \( y[i] \) is a valid extension of \( y[1..i-1] \). We conclude that \( y[1..i] \) is the border array of the string \( x[1..i] = x[1..i-1]x[m] \), as required.

\( \square \)

This theorem makes clear that an extension \( m = y[i] \) of a border array \( y = y[1..i-1] \) yields a border array \( y[1..i] \) if and only if

1. \( m \) is an admissible extension of \( y \);
2. there exists no admissible extension \( m' > m \) of \( y \) such that \( y[m'] = m \).

The algorithm that determines whether or not a given array is a border array simply evaluates these two conditions in a straightforward manner for every position \( i \in 2..n \). Thus the outline of the main algorithm can be expressed as follows:

```plaintext

- For \( y[1..n] \), \( n \geq 1 \), return either \( n+1 \)
- or the first position \( i \in 1..n \)
- such that \( y[i] \) is invalid.
if \( y[1] \neq 0 \) then return 1
- repeatedly call the function valid to check each
- value \( y[i] \) until the whole array \( y \) is processed
i \( \leftarrow 2 \)
while \( i \leq n \) and valid(i, y[1..i]) do
  i \( \leftarrow i + 1 \)
return i
```

The Boolean function \( \text{valid} \) returns \( \text{TRUE} \) if and only if conditions (1) and (2) are satisfied by \( m = y[i] \), as shown in Figure 1. Since the algorithm processes \( y \) position-by-position from left to right, terminating whenever an invalid position is found, we may assume that for every \( i \geq 2 \), \( y[1..i-1] \) is a valid border array. Observe that the algorithm described here makes no reference to any corresponding string \( x \), but bases its determination of validity entirely on the properties of the given array \( y \).

Thus, based on Theorem 2.2 and this discussion, we may conclude that our algorithm is correct. To see that it executes in \( O(n) \) time, we need to show that
function valid(i, y[1..i])
— Given that y[1..i−1] is a border array,
— return TRUE iff y[i] is valid.
— First determine whether y[i] is admissible.  
if y[i] = 0 then return TRUE  
else  
  b ← y[i − 1]  
  while b > 0 and y[i] ≠ b + 1 do  
    b ← y[b]  
  if y[i] ≠ b + 1 then return FALSE  
else  
  Now determine whether y[i] = b + 1 satisfies condition (2).  
  b' ← y[i − 1]  
  while b' > b and b + 1 ≠ y[b' + 1] do  
    b' ← y[b']  
  return (b' ≤ b)

Figure 1: The Boolean Function valid

the total number of operations performed in the while loops of function valid is  
O(n). But this follows as in the failure function algorithm [AHU74] [KMP77].  
Specifically, consider the worst case in which valid always returns TRUE. From  
one call of valid to the next, the value of b = y[i−1] is increased by at most 1.  
However, each execution of (1.1) decreases b by at least 1. Therefore, (1.1) is  
executed at most n − 1 times in total. Also note that for each specific i, (1.2)  
is executed at most the same number of times as (1.1) because of the condition  
b' > b.

The same analysis applies to the case where FALSE is returned except that  
the algorithm may terminate earlier, thus executing fewer steps. We conclude  
that our algorithm executes in linear time. This establishes the second main  
result of this section:

**Theorem 2.3** The algorithm presented in this section correctly determines in  
time O(n) whether or not a given integer array y[1..n] is a border array. □

To conclude this section, we remark that a version of Theorem 2.2 appears  
as Theorem 3.2 in [MSM99]; however, the result as it is given there is much less  
clear and its proof depends on an elaborate theory of b-canonical strings that  
we have avoided here with a proof that is elementary.
3 Identifying Valid Border Arrays For Bounded Alphabets

As shown in the previous section, the definition of a valid extension of a border array $y[1..n]$ is not appropriate for dealing with strings over an alphabet of a fixed finite size $\alpha$. The example there shows that the problem is how to determine when 0 is valid in relation to the size of the alphabet. For the rest of this section we assume that $\alpha \geq 2$ is the fixed finite alphabet size. We capture the revised notion of validity in the following definition:

A nonzero admissible extension $m$ of a border array $\beta = \beta[1..n]$ is said to be an $\alpha$-valid extension of $\beta$ if and only if it is a valid extension of $\beta$ (as defined in the previous section). $0$ is said to be an $\alpha$-valid extension of $\beta$ if and only if for the set

$$M_n = \{m : m > 0, m \text{ an } \alpha - \text{valid extension of } \beta\},$$

$$|M_n| < \alpha.$$

Lemma 3.1 below shows that $|M_n|$ is precisely the number of letters (of any alphabet of size $\alpha$) used in all possible extensions of all previous borders in any string of which $\beta$ is a border array.

Let us illustrate using the example of the previous section, where 8,4,2, and 1 are all valid (hence $\alpha$-valid) extensions of the border array $\beta[1..15] = 001012301234567$. The string $x = abacabadabacaba$ is a string of which $\beta$ is a border array, and so we see that the alphabet \{a, b, c, d\} must have size at least 4. Hence if $\alpha = 4$, 0 is not $\alpha$-valid, while if $\alpha \geq 5$, it is $\alpha$-valid (appending $e$ to $x$ gives 0).

**Lemma 3.1** Let $n \geq 2$ be an integer, and let $y[1..n]$ be a border array. Let $m_1$ and $m_2 \neq m_1$ be two distinct $\alpha$-valid extensions of $y[1..n]$. Then for any string $x[1..n]$ such that $y[1..n]$ is its border array, $x[m_1] \neq x[m_2]$.

**Proof** By contradiction. We assume that there exists a string $x$ such that $x[m_1] = x[m_2]$, where $m_1 = y'[n]+1$ and $m_2 = y'[n]+1$ for some $1 \leq i < j \leq k$ (k being the smallest integer such that $y'[n] = 0$). Moreover assume that, for given $j$, the integer $i$ is the maximum value satisfying this condition.

Since $x[y'[n]+1] = x[y'[n]+1]$ and $i < j$, $y'[n]+1$ must be an $\alpha$-valid extension of $y'[n]+1$. Since $y'[n]+1$ is $\alpha$-valid, $y'[n]+1 \neq y'[n]+1$, and so $y[y'[n]+1] > y'[n]+1$. Furthermore, $y[y'[n]+1] = y'[n]+1$ for some $r$, $i < r < k$. Thus


Since $y'[n]+1 > y'[n]+1$, $1 \leq i < r < j \leq k$. But this contradicts the maximality of $i$. $\square$

The following theorem is an obvious modification of Theorem 2.2 for bounded alphabets.
Theorem 3.2 For every $n \geq 1$, and for any $\alpha \geq 2$, an integer array $y = y[1..n]$ is a border array of a string over an alphabet of size $\alpha$ if and only if $y[1] = 0$ and each $y[i]$ is an $\alpha$-valid extension of $y[1..i-1]$, $i = 2, \ldots, n$.

Proof If $y[i] > 0$, the proof follows the same argument as in the proof of Theorem 2.2, so we refer the reader there. Below, we cover the case when $y[i] = 0$.

Necessity: let $y[1..i]$ be a border array and $y[i] = 0$. By contradiction assume that $0$ is not an $\alpha$-valid extension of $y[1..i-1]$; that is, that $|M_{i-1}| = \alpha$. Take any string $x[1..i]$ of which $y[1..i]$ is a border array. Then for some $\alpha$-valid $m > 0$, $x[m] = x[i]$. It follows that $y[i] \geq m > 0$, a contradiction.

Sufficiency: let $y[1..i-1]$ be a border array and suppose that $|M_{i-1}| < \alpha$. Take any string $x[1..i-1]$ of which $y[1..i-1]$ is a border array. Suppose $\lambda \neq x[m]$ for any $m \in M_{i-1}$ (The fact that such a $\lambda$ exists follows from Lemma 3.1.) We will show that $y[1..i-1]0$ is a border array of $x[1..i-1]$. By contradiction assume that the border array $y[1..i]$ of $x[1..i-1]$ is such that $y[i] \neq 0$. But then by this theorem for nonzero values, $y[i] = m$ for some $\alpha$-valid $m$, hence $\lambda = x[i] = x[m]$, a contradiction. □

This result makes it clear that for $y[i] \neq 0$, the algorithm of Section 2 can be used as it stands to determine whether or not $y[i]$ is $\alpha$-valid. However, whenever $y[i] = 0$ and $i > 1$, the definition of $\alpha$-valid requires additional processing to determine whether or not the number of valid nonzero extensions of $y[1..i-1]$ equals $\alpha$. This processing can be viewed as a modification to function valid, that affects only the case $y[i] = 0$. The code is presented in Figure 2. A Boolean array $V[1..n]$ indicates whether or not each admissible extension $m$ is valid ($V[m] = \text{TRUE}$) or not valid ($V[m] = \text{FALSE}$). We initialize $V[1..n]$ to $\text{FALSE}$ before the first invocation of valid.

The new code is straightforward and falls into four parts:

* determine the number $k$ of nonzero admissible extensions $m$ and set each corresponding $V[m]$ to $\text{TRUE}$;
* identify invalid admissible extensions and thus compute $v$, the number of nonzero valid admissible extensions;
* reset to $\text{FALSE}$ the components of $V$ that were changed in the first part;
* return $\text{TRUE}$ if and only if $v < \alpha$.

We claim that the revised algorithm correctly determines whether or not a given array $y$ is a border array of some string on an alphabet of size $\alpha$.

To analyze the complexity of the revised algorithm, we need to show that the extra processing involves when $y[i] = 0$ still requires only $O(n)$ steps. To this end, fix such an $i$ and let $s$ be the number of times (2.1) is executed. Then $s \leq y[i-1] + 1$ since $b$ is set to $y[i-1]$ on the first pass and is reduced by at least 1 on each subsequent pass. Note that $s$ distinct components of $V$ are set to TRUE.
if $y[i] = 0$ then
  $k \leftarrow 0; \ b \leftarrow i - 1$
  while $b \neq 0$ do
    $k \leftarrow k + 1; \ b \leftarrow y[b]; \ V[b + 1] \leftarrow \text{TRUE}$  
  $v \leftarrow k; \ b \leftarrow i - 1$
  while $b \neq 0$ do
    $b \leftarrow y[b]$
    $b' \leftarrow b + 1$
    if $V[b']$ then
      while $y[b'] > 0$ do
        $v \leftarrow v - 1; \ b' \leftarrow y[b']$; $V[b'] \leftarrow \text{FALSE}$
      $b \leftarrow i - 1$
    while $b \neq 0$ do
      $b \leftarrow y[b]$; $V[b + 1] \leftarrow \text{FALSE}$
  return $(v < \alpha)$

else — (The rest of function valid follows.)

Figure 2: Determining whether $y[i] = 0$ is $\alpha$-valid or not

Now the second while loop is controlled by the same parameters as the first. In particular, (2.2) is executed $s$ times. Further, each execution of (2.3) sets to FALSE a component of $V$ that was previously TRUE. Thus (2.3) executes at most $s$ times in total – more than once on some passes and not at all on others. The third while loop (which resets $V$) is also executed $s$ times.

Now, as in Section 2, $b = y[i - 1]$ increases by at most 1 on each call of valid. However, $b$ decreases by $y[i - 1]$ when $y[i] = 0$. Thus the sum of all these $y[i - 1]$ cannot exceed $n - 1$ and hence the sum of the corresponding values of $s$ cannot exceed $2(n - 2)$. It follows that overall, $O(n)$ time is sufficient to handle the cases in which $y[i] = 0$, in spite of the nested while loops. Since the remainder of the processing is unchanged, we conclude:

Theorem 3.3 The algorithm presented in this section correctly determines in time $O(n)$ whether or not a given integer array $y[1..n]$ is a border array on an alphabet of specified size $\alpha$. □

4 Computing All Border Arrays of Length At Most $n$

In [MSM99] all the border arrays of length at most $n$ are generated by growing a trie $T_n$ of height $n$ in which every simple path of length $k \leq n$ from the root spells out both a unique border array $\beta = \beta[1..k]$ and the $b$-canonical string $x[1..k]$ corresponding to $\beta$. Thus each node of $T_n$ may be thought of as being
Figure 3: Trie $T_3$ — All Border Arrays of Length $k \leq 3$

labelled with an integer pair $(i, \beta)$, where $i$ denotes the $i^{th}$ smallest letter $\lambda_i$ in an ordered standard alphabet, and $\beta$ is the value of a corresponding entry in the border array $B$. For $n = 3$, $T_n$ appears as shown in Figure 3, representing strings

$$\lambda_1 \lambda_1 \lambda_1, \lambda_1 \lambda_1 \lambda_2, \lambda_1 \lambda_2 \lambda_1, \lambda_1 \lambda_2 \lambda_2$$

with corresponding border arrays

012, 010, 001, 000.

The algorithm described in [MSM99] uses the canonical string $x[1..k]$ spelled out by the path from the root to the current node $N$ as a means of determining the children of $N$. Effectively, standard letters $\lambda_1, \lambda_2, \ldots$ are appended one-by-one to $x[1..k]$ yielding new strings $x[1..k] \lambda_i$, $i = 1, 2, \ldots$; for each new string formed, the corresponding $(k + 1)^{st}$ border array element is computed. The process terminates when a standard letter, say $\lambda_r$, is appended for which the corresponding border array value is zero (see Lemma 3.4 in [MSM99]). Thus for each new node of $T_n$, one step in the failure function calculation is performed, requiring amortized constant time as discussed in Section 2. Hence the claim that $T_n$ is constructed in time proportional to the number of nodes; that is, proportional to the number of border arrays (and corresponding strings) generated.

But the algorithm described in [MSM99] generates $T_n$ in a breadth-first or on-line manner: $T_{k+1}$ is actually computed from the leaf nodes of $T_k$ for every $k \in 1..n-1$. Since for every node $N$ both the corresponding $x[1..k]$ and $B[1..k]$ need to be available for the failure function calculation, $\Theta(k)$ time will be required to traverse the path from the root to node $N$ in order to compute them. Thus the time required to compute each child of node $N$ in a breadth-first algorithm is not constant, but rather $\Theta(k/r)$, where $r$ is the number of children of $N$.

The obvious correction to the [MSM99] algorithm is to build $T_n$ in a depth-first manner that uses two working-storage arrays $x = x[1..n]$ and $B[1..n]$ to store the path from the root of $T_n$ to the current node $N$. Then for each node
the current values $\alpha[1..k]$ and $\beta[1..k]$ are known and can be used to compute the children of $N$: each extension $\alpha[1..k] \lambda_i, i = 1, 2, \ldots, r$, can be formed so that corresponding extensions of $\beta[1..k]$ can be computed using a single constant-time step of the failure function algorithm. A depth-first recursion that computes all the children of each $N$ before computing any of $N$’s siblings will then lead to the result claimed in [MSM99]: $T_n$ will be constructed in $\Theta(b_n)$ time using $\Theta(b_n)$ space.

The depth-first approach also enables us to solve efficiently a problem raised in [MSM99]: the computation of a trie $T'_n$ whose node labels consist only of border array values $\beta$, omitting the elements of the underlying canonical string $\alpha$. This can easily be accomplished by maintaining the working storage array $\alpha[1..\eta]$ but not storing the current letter in the current node $N$: the algorithm will execute recursively in exactly the same way.

Note that it is straightforward to modify each of these depth-first algorithms to generate a trie for a given bounded alphabet $A$ of size $\alpha$: it is necessary only to replace the number $\eta$ of children computed at each node by $\min\{r, \alpha\}$. We can now state formally the main result of this section:

**Theorem 4.1** For any given positive integer $\eta$, the two algorithms outlined in this section compute all possible border arrays of length at most $\eta$ on either a bounded or unbounded alphabet in time $\Theta(b_n)$ and space $\Theta(b_n)$, where $b_n$ is the number of arrays generated. □

We remark that the depth-first algorithms described above have the disadvantage that they provide no means of efficiently computing $T_{n+1}$ (respectively, $T'_{n+1}$) from $T_n$ (respectively, $T'_n$). In the next section, we provide an on-line (breadth-first) algorithm that performs the same computation with equal efficiency.

5 An On-line Algorithm for Computing All Border Arrays of Length At Most $n$

We wish to construct the trie $T_n$ in a breadth-first fashion using only constant time to compute the border array entry of each node. The failure function algorithm satisfies this requirement, but it needs the corresponding arrays $\alpha$ and $\beta$ to be available. The elements of the two arrays are already stored in the ancestor nodes of the tree. We will modify the failure function algorithm to use this information instead of the two arrays.

The failure function algorithm accesses the arrays $\alpha$ and $\beta$ in the following two ways:

(a) $\beta[b]$, which returns the length of the longest border of the position $b$; and

(b) $\alpha[b+1]$, which returns the letter at the next position of $b$.

Each node in our trie $T_n$ will represent a border array/b-canonical string pair. Then the above operation (a) is equivalent to finding the node corresponding
to the longest border of the string represented by the current node. Operation (b) is equivalent to finding one of the children of the current node. We refine the tree structure of $T_n$ to simulate these two operations.

First we introduce an extra node $E$ which corresponds to the 0-position or empty string/border. A pointer in $E$ points to the “real” root $R$ of $T_n$: $(\lambda_1, 0, 1)$, which means that the current letter is $\lambda_1$, with empty border, and the length of the string is 1.

Except for $E$, each node $N$ in $T_n$ represents a border array $\beta[1..j]$ and its $b$-canonical string $x[1..j]$. We have the following data in each node $N$:

- $\lambda$ the letter $x[j]$;
- $\beta$ the border array entry $\beta[j]$;
- $j$ the length of the corresponding border array/b-canonical string
- $pr$ a pointer to the parent: the node of $\beta[1..j-1]/x[1..j-1]$;
- $br$ a pointer to the border: the node of $\beta[1..\beta[j]]/x[1..\beta[j]]$;
- $ne$ a pointer to the next node of the same level as $N$;
- $cp$ a list of pointers to the children of $N$;
- $ci$ a pointer to the child whose descendant is currently being processed.

The pointers $ci$ are maintained to form a path from $E$ to the node $N$ to which a child $N'$ is being added. Note that $j$ is just the level in the tree where $E.j = 0$. Also, we define $E.ci = R$ and $R.br = E$. These values do not change.

We now describe the construction of $T_n$. Beginning with $E$, each new level is an ordered sequence of nodes. Successive children of a given node are constructed by appending a different letter (using the ordering of the alphabet) to the string represented by the parent until an empty border results. The algorithm for computing the border is given below.

When adding a new node, it is trivial to determine the values of $\lambda$, $j$, $pr$, $ne$ and $cp$. Also, $\beta$ is just the level pointed to by $br$. We now show how to compute $br$ and $ci$:

$br$: When adding a child $N'$ to the current node $N$, we proceed as follows:

$$
N_b \leftarrow N.br; N_{b+1} \leftarrow N_b.ci$$

**while** $N_b \neq E$ **and** $N'.\lambda \neq N_{b+1}.\lambda$ **do**

$$
N_b \leftarrow N_b.br; N_{b+1} \leftarrow N_b.ci
$$

**if** $N'.\lambda = N_{b+1}.\lambda$ **then**

$$
N'.br \leftarrow N_{b+1}
$$

**else**

$$
N'.br \leftarrow E
$$
First we use $N.br$ to find the border node $N_b$ of $N$, then use $N_b.ci$ to find
the next node $N_{b,i}$ along the path from $N_b$ to $N$. Now we compare
the letters of $N'$ and $N_{b,i}$. We continue to do so until we find a match or
reach the empty border. Finally we set the $br$ and $\beta$ according to whether
the letters match or not.

ci: When a first child of $N$ is added, we can set $N.ci$ to this child. This value
does not have to be changed until we start adding grandchildren for $N$.
When finished adding children to a node $N$, we use $Nлеч$ to find the next node.
Now we need to update the $ci$ of ancestor nodes of $N$. For example,
suppose that a node $N_p$ has two children $N_1$ and $N_2$. When we are adding
children to $N_1$, $N_p.ci = N_1$. Now we want to add children to $N_2$. We need
to update $ci$ so that $N_p = N_2$. Each time we finish adding children to a
node $N$, we call the procedure update $ci(N.pr)$. It is possible that $N$
is the last child of its parent $N_p$. Then we may need to update the $ci$ of
the parent nodes recursively. In the following, $next$ denotes the pointer field
in the linked list $cp$, and $cp.first$ returns the first node of $cp$:

\[
\text{update}_ci(N.pr) \\
\text{procedure update}_ci(N_p) \\
\quad \text{if} \ ci.next \neq \text{NULL} \ \text{then} \\
\quad \quad ci \leftarrow ci.next \\
\quad \text{else} \\
\quad \quad ci \leftarrow cp.first \\
\quad \quad \text{if} \ N_p \neq R \ \text{then} \\
\quad \quad \quad \text{update}_ci(N_p.pr)
\]

We update $ci$ to point to the next child, if there is any. Otherwise, we set
$ci$ to point to the first child, and update $ci$ of the parent node.

According to the above description, we have an on-line algorithm that com-
putes all the border arrays/b-canonical strings of length $n$. Except for $cp$, all
data in each node require constant space. The total number of nodes in $cp$ is
equal to the total number of child nodes in the tree, which is $|T_n|−1$. So the
algorithm still needs space proportional to the number of arrays generated.

The code for $br$ is basically the same as the failure function algorithm. Thus
it takes amortized constant time for each node. When we generate $T_{n+1}$ from
$T_n$, the number of times we update $ci$ is the same as the number of nodes in $cp$,
which is $|T_n|−1$. Thus we get the following conclusion about the complexity of
the algorithm:

**Theorem 5.1** For every positive integer $n$, the algorithm outlined in this sec-
tion computes all border arrays/b-canonical strings of length $n$ in $\Theta(b_n)$ time
and represents them in $\Theta(b_n)$ space. □
An easy modification can be made to the algorithm for a bounded alphabet of size $\alpha \geq 2$. Theorem 5.1 also holds in this case.

References


http://www.research.att.com/~njas/sequences/

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