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CERTAIN VALUES OF COMPLETENESS AND SATURATEDNESS OF A UNIFORM IDEAL RULE OUT CERTAIN SIZES OF THE UNDERLYING INDEX SET

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ABSTRACT. Using the method of non-well-founded generic ultrapowers, we shall prove a generalization of a theorem of Taylor that certain values of completeness and saturatedness of a uniform ideal rule out certain sizes of the underlying index set.

1. **Introduction**. "There is no κ^+-complete κ^+-saturated ideal over κ^+, κ an uncountable cardinal" is the straightforward generalization of the classical result of Ulam (see [2] or [6]) "there is no nontrivial σ-additive measure on $\aleph_1$", proved by so-called Ulam matrices. The method of well-founded generic ultrapowers was first used by Solovay (see [4]) to prove that if "there exists a κ-complete κ-saturated ideal over κ", κ must be a large cardinal (badly Mahlo). Later they were extensively studied by Jech and Prikry (see [3]) in connection with precipitous ideals.

The method of non-well-founded generic ultrapowers was first used by Silver (see [5]).

Kunen observed (private communication) that using the method of well-founded generic ultrapower one can show that there is no $\aleph_1$-complete $\aleph_2$-saturated uniform ideal over a cardinal κ if $\aleph_\omega < \kappa < \aleph_{\omega_1}$.

Taylor (private communication) proved a generalization of this, namely "there is no $\aleph_\alpha$-saturated λ^+-complete uniform ideal over a cardinal κ if $\aleph_\lambda < \kappa < \aleph_{\lambda}+$ and $\alpha < \lambda$ and λ is an infinite cardinal", using some combinatorial results of Jech and Prikry. His proof is purely combinatorial.

Inspired by Kunen's observation and using a technical insight into generic ultrapowers developed in [3], we shall prove a generalization of Taylor's theorem with a significantly shorter proof.

2. **Definitions**. (For details, though for κ-complete ideals over κ rather than λ-complete ideals over κ, $\lambda \leq \kappa$, see [3]).

Let I be an ideal over a set S.

Then $I^+ = \{X \subset S: X \notin I\}$.

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$W \subset I^+$ is *I-disjoint* if $(\forall X, Y \in W)(X \cap Y \in I)$.

Let λ be a cardinal, I is λ-*saturated* if no $X \in [I^+]^\lambda$ is I-disjoint.

I is λ-*complete* if for any $\xi < \lambda$ and any $\{X_\alpha: \alpha \in \xi\} \subset I$, $U\{X_\alpha: \alpha \in \xi\} \in I$.

Let M be a transitive model of ZFC. Let $\kappa \in \mathrm{Ord}^M$ and $\lambda \in \mathrm{Card}^M$. $G \subset M$ is a *non-principal M-λ-complete M-ultrafilter over* κ if

(1) $(\forall x \in G)(\forall y \in P(\kappa) \cap M)(y \supset x \Rightarrow y \in G)$;

(2) $(\forall x \in P(\kappa) \cap M)(x \in G$ or $\kappa - x \in G)$;

(3) $(\forall X \in [M]^{<\lambda} \cap M)(X \subset G \Rightarrow \cap X \in G)$;

(4) $\cap G = \phi$.

If $f, g \in {}^\kappa M \cap M$, then

$$f \in^* g \quad \text{iff} \quad \{\alpha \in \kappa: f(\alpha) \in g(\alpha)\} \in G$$
$$f =^* g \quad \text{iff} \quad \{\alpha \in \kappa: f(\alpha) = g(\alpha)\} \in G.$$

For every $f \in {}^\kappa M \cap M$ let us choose (in V) a representative $[f]$ from the class $\{g \in {}^\kappa M \cap M: g =^* f\}$, and form (in V) *generalized ultrapower* $\mathrm{Ult}(M, G) = \{[f]: f \in {}^\kappa M \cap M\}$.

Let $ext([f]) = \{[g] \in {}^\kappa M \cap M: [g] \in^* [f]\}$.

For every $x \in M$ define $c_x \in {}^\kappa M \cap M$ by $c_x(\alpha) = x$ for all $\alpha \in \kappa$. Then as usual j defined by $j(x) = [c_x]$ is an elementary embedding of M into $\mathrm{Ult}(M, G)$ (it is often called *canonical embedding*) and (Loś theorem) $\mathrm{Ult}(M, G) \models \phi([f_0], \ldots, [f_n])$ iff $\{\alpha \in \kappa: M \models \phi(f_0(\alpha), \ldots, f_n(\alpha))\} \in G$, for every formula $\phi(x_0, \ldots, x_n)$ and every sequence $\langle [f_0], \ldots, [f_n] \rangle \in \mathrm{Ult}(M, G)$. In the case that $\in^*$ is well-founded on the whole class $\mathrm{Ult}(M, G)$, we identify $\mathrm{Ult}(M, G)$ with its transitive collapse.

3. Preliminaries.

LEMMA 1. *Let $M \subset V$ be a transitive model of ZFC. Let $G \in V$ be a non-principal M-λ-complete M-ultrafilter over κ, $\aleph_1^M \leqslant \lambda \leqslant \kappa$ cardinals in M. Let $j: M \to \mathrm{Ult}(M, G)$ be the canonical embedding. Then*

(1) $|\alpha| \leqslant |\mathrm{ext}(j(\alpha))|$ *for all* $\alpha \in \mathrm{Ord}^M$;

(2) $|\alpha| = |\mathrm{ext}(j(\alpha))|$ (*since* $\mathrm{ext}(j(\alpha)) = \{[c_\beta]: \beta \in \alpha\}$) *for all* $\alpha \in \lambda$;

(3) $|\mathrm{ext}(j(\aleph_\alpha^M))| \leqslant \aleph_\alpha^V$ *for all* $\alpha \in \lambda$;

(4) $\{[c_\beta]: \beta \in \lambda\}$ *is an initial segment of* $\mathrm{Ord}^{\mathrm{Ult}(M,G)}$;

(5) *if G is* uniform, *i.e.* $(\forall x \in G)(|x|^M = \kappa)$, *then* $|(\kappa^+)^M| \leqslant |\mathrm{ext}(j(\kappa))|$.

(Note: the cardinalities are computed in V.)

PROOF. (1)−(4) follow from 2.2.2, 2.2.4, 2.2.5 and 2.3.1 in [3], when generalized from M-κ-complete M-ultrafilters over κ to M-λ-complete M-ultrafilters over κ, $\lambda \leqslant \kappa$.

(5) Choose, in M, a family $F \subset {}^\kappa\kappa$ of size κ^+ of almost disjoint functions (such family always exists, see e.g. [2]). Since G is uniform, $f \neq g \in F \Rightarrow [f] \neq [g]$ as $\{\gamma \in \kappa: f(\gamma) = g(\gamma)\} \supset \beta$ for some $\beta \in \kappa$ and hence $\{\gamma \in \kappa: f(\gamma) \neq g(\gamma)\} \in G$. So $|\mathrm{ext}(j(\kappa))| \geqslant |F|$. □

NOTE. Let M be a transitive model of ZFC. Let, in M, I be an ideal over S. Let $X \subset^* Y$ mean $X - Y \in I$. One can view the poset $\langle I^+, \subset^* \rangle$ as a forcing notion. Then, if G is $\langle I^+, \subset^* \rangle$-generic over M, we shall say that G is *I-generic over M*.

LEMMA 2. *Let M be a transitive model of ZFC. Let I be, in M, a λ-complete (uniform) ideal over a cardinal κ so that $\aleph_1^M \leq \lambda \leq \kappa$. Let G be I-generic over M. Then G is a non-principal M-λ-complete (uniform) M-ultrafilter over κ.*

PROOF. Easy. Left to the interested reader, or see [3]. $\square$

4. **Main result**.

THEOREM 3. *Let λ be an uncountable cardinal, $\alpha < \lambda$ and $\mu = \omega_0 \cdot \alpha$. Let $\aleph_\mu < \kappa < \aleph_\lambda$. Then there is no λ-complete $\aleph_\alpha$-saturated uniform ideal over κ.*

PROOF. Assume that there are an M, a transitive model of ZFC, and I, a λ-complete $\aleph_\alpha^M$-saturated uniform ideal over κ in M, and that $\alpha < \lambda$, λ is an uncountable cardinal and $\aleph_\mu^M < \kappa < \aleph_\lambda^M$ and $\mu = \omega_0 \cdot \alpha$. Let G be I-generic over M.

Since I is $\aleph_\alpha^M$-saturated, $\aleph_\alpha^M$ is a cardinal in $M[G]$. Let $\aleph_\alpha^M = \aleph_\delta^{M[G]}$ for some $\delta \leq \alpha$. Let $\xi = \alpha - \delta$. Then $\alpha + \omega_0 \cdot \xi = \delta + \omega_0 \cdot \xi \leq \omega_0 \cdot \alpha = \mu$. Thus $\aleph_\gamma^M = \aleph_\gamma^{M[G]}$ for all $\gamma \geq \mu$. Let $\kappa = \aleph_\beta^M$ for some β. Then $\mu < \beta < \lambda$. By Lemma 1 (5) and (3) (since $\beta < \lambda$),

$$\aleph_{\beta+1}^{M[G]} = \aleph_{\beta+1}^M = |\aleph_{\beta+1}^M| \leq |\mathrm{ext}(j(\aleph_\beta^M))| \leq \aleph_\beta^{M[G]},$$

a contradiction. $\square$

COROLLARY 4. *Taylor's theorem.*

PROOF. Let I be a ξ^+-complete $\aleph_\alpha$-saturated uniform ideal over a cardinal κ, $\alpha < \xi$, ξ an infinite cardinal and $\aleph_\xi < \kappa < \aleph_{\xi^+}$. Let $\lambda = \xi^+$. Let $\mu = \omega_0 \cdot \alpha$. Then $\aleph_\mu < \kappa < \aleph_\lambda$, λ is an uncountable cardinal and I is λ-complete, $\aleph_\alpha$-saturated and uniform, which contradicts Theorem 3. $\square$

NOTE. (1) $\kappa \leq \aleph_\lambda$ is the best upper bound, for Foreman and Magidor (private communication) constructed a model with an $\aleph_1$-complete $\aleph_2$-saturated ideal over $\aleph_{\omega_1+1}$.

(2) Theorem 3 gives a better lower estimate for κ than Taylor's theorem, and if ξ is weakly inaccessible, then Theorem 3 shows the non-existence of ξ^+-complete $\aleph_\xi$-saturated uniform ideals over κ, $\aleph_{\omega_0 \cdot \xi} < \kappa < \aleph_{\xi^+}$, while Taylor's theorem deals only with $\aleph_\alpha$-saturated ideals for $\alpha < \xi$.

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