A COMPLETION CONJECTURE FOR KIRKMAN TRIPLE SYSTEMS

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1. Introduction and a completion conjecture

A *Steiner triple system* of order v (STS(v)) is a pair $(V, \mathcal{B})$ where V is a v-set of *elements* and $\mathcal{B}$ is a collection of 3-subsets of V called *triples* such that every 2-subset of V is contained in exactly one triple. It is well known that an STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6).

A *parallel class* in an STS(v) $(V, \mathcal{B})$ is a subcollection of $\mathcal{B}$ which partitions the set of elements V. A partition $\mathcal{R}$ of $\mathcal{B}$ into parallel classes is called a *resolution*, and an STS which admits a resolution is said to be *resolvable*. For an STS(v) to possess a parallel class, v must be divisible by 3, and thus $v \equiv 3$ (mod 6) is a necessary condition for the existence of a resolvable STS(v).

A *Kirkman triple system* of order v (KTS(v)) is a triple $(V, \mathcal{B}, \mathcal{R})$ where $(V, \mathcal{B})$ is an STS(v) (called the *underlying* STS of the KTS), and $\mathcal{R}$ is its resolution. In other words, a KTS(v) is an STS(v) together with a *specific* resolution $\mathcal{R}$. It is possible for an STS to be the underlying STS of several distinct, indeed, even of nonisomorphic KTSs. [The smallest v for which this occurs is $v = 15$.]

The existence problem for resolvable STSs (or, which is the same, the existence problem for KTSs) was posed by Kirkman in 1847 [K]. Some 120 years later, Ray-Chaudhuri and Wilson [RW] proved that a KTS(v) exists for all $v \equiv 3$ (mod 6).

Consider two distinct parallel classes $\mathcal{R}_1, \mathcal{R}_2$ in a KTS(v). The *block-intersection graph* $G(\mathcal{R}_1, \mathcal{R}_2)$ has as its vertices all triples of $\mathcal{R}_1 \cup \mathcal{R}_2$, with two vertices adjacent if the corresponding two triples intersect. Clearly, $G(\mathcal{R}_1, \mathcal{R}_2)$ is a cubic bipartite graph with $\frac{2}{3}v$ vertices, and v edges.

Let $\mathcal{C}_1, ..., \mathcal{C}_m$ be a list of all nonisomorphic cubic bipartite graphs with $\frac{2}{3}v$ vertices. Then we can associate with any KTS(v) $(V, \mathcal{B}, \mathcal{R})$, where $\mathcal{R} = \{\mathcal{R}_1, ..., \mathcal{R}_{(v-1)/2}\}$, the *type matrix* $\mathcal{B} = (b_{ij})$ (where $b_{ii} = 0$, and for $i \neq j, b_{ij} = k$ if $G(\mathcal{R}_i, \mathcal{R}_j) \simeq \mathcal{C}_k$), and the *type vector* $T = (t_1, ..., t_m)$ where $t_k = |\{b_{ij} : i < j, b_{ij} = k\}|$. These have been used in [R] to classify the 7 nonisomorphic KTS(15) (see also [FMR]), and formally proposed in [PP2] as invariants to distinguish KTSs.

Conversely, one may ask whether any cubic bipartite graph with $\frac{2}{3}v$ vertices

The third author has conjectured that this is always so. An equivalent formulation of this is the following completion conjecture.

Conjecture. Let $v \equiv 3 \pmod 6$, $v > 3$, and let $\mathcal{R}_1, \mathcal{R}_2$ be two disjoint parallel classes of triples on a v-set V. Then there exists a KTS(v) containing $\mathcal{R}_1, \mathcal{R}_2$ as parallel classes.

We have verified this conjecture for $v \leq 21$. There exists a unique bipartite cubic graph on 6 vertices, $K_{3,3}$, and, trivially, the block-intersection graph of any two parallel classes in the unique STS(9) is isomorphic to $K_{3,3}$. There exist two nonisomorphic bipartite cubic graphs on 10 vertices (cf. No.14 and No.17 on the list of [BF], with automorphism group of order 48 and 20, respectively; see Fig.1).

No.1 No.2

Fig.1 The bipartite cubic graphs on 10 vertices

Four different type vectors are associated with the 7 nonisomorphic KTS(15) (cf. [R]).

Table 1

Type vectors Graph No. 1	Type vectors Graph No. 2	Number of associated KTS
21	0	2
9	12	2
3	18	2
0	21	1

Further, there are 13 nonisomorphic connected (and a unique disconnected) bipartite cubic graphs on 14 vertices [BF]; cf. also [PP2]. Among all known KTS(21) (see Section 2 below), there is a unique KTS(21) discovered by Mathon [M] whose type vector has 13 nonzero components corresponding to the 13 connected cubic bipartite graphs with 14 vertices. This single KTS(21), together with any of the KTS(21) whose type vector has nonzero component corresponding to the unique

disconnected cubic bipartite graph on 14 vertices (there are 27 KTS(21) of the latter type among the known ones) shows that the conjecture above holds for v=21 (even though many other sets of two or more known KTS(21) provide the same verification).

Verification of the conjecture for v=27 - the next logical step -, while still feasible, would likely require a more considerable computational effort: according to [BF], the number of nonisomorphic cubic graphs on 18 vertices is 149 while according to [MR2], there are at least 909 noisomorphic KTS(27). Even though verifying it for v=27 would lend further support to the conjecture, it is not likely to throw any light on the conjecture itself which, frankly, appears at this point quite hopeless.

It is perhaps this feeling which leads one to consider some of the more promising questions suggested by examining the type vector (and the type matrix). Let us call a KTS(v) *type-uniform* of class k if t_k is the unique nonzero component of its type-vector T.

The unique KTS(9) is trivially type-uniform. Table 1 shows that three out of the 7 KTS(15) are type-uniform; two of them are type-uniform of class 1, and one of class 2. It is easily seen that the KTS(3^n) corresponding to the affine resolution of AG($n, 3$), the n-dimensional affine space over GF(3), is type-uniform.

Does there exist a type-uniform KTS(v) for all $v \equiv 3 \pmod 6$? No type-uniform KTS(21) is known at present. Among the known KTS(21), there are three which come "closest" to being type-uniform: these are the systems A.1a, A.2c and A.4d of [MPR]; each of these has only two nonzero components in their respective type vectors. Moreover, the KTS(21) No. A.2c has $t_{12} = 42, t_{13} = 3$ while A.4d has $t_7 = 3, t_{12} = 42$ (and all other t_i's are equal 0). Here the numbering of the connected cubic bipartite graphs on 14 vertices is as in [BF], and we label the unique disconnected cubic graph on 14 vertices $\mathcal{C}_{14}$.

We are able to show that there exists no type-uniform KTS(21) of class 14.

Indeed, suppose that a parallel class $\mathcal{R}_1$ of a KTS(21) is

$\mathcal{R}_1$: AAA BBB CCC DDD EEE FFF GGG

(where we simply suppressed indices 1,2,3). Since $G(\mathcal{R}_1, \mathcal{R}_2)$ is to be disconnected, we may assume without loss of generality the second parallel class $\mathcal{R}_2$ to be $\mathcal{R}_2$: ABC ABC ABC DEF DEG DFG EFG. Then there are essentially 4 possibilities for the third parallel class $\mathcal{R}_3$ (keeping again in mind that $G(\mathcal{R}_1, \mathcal{R}_3)$ is to be disconnected):

(a) ABC ABC ABC DEF DEG DFG EFG
(b) ABD ABD ABD CEF CEG CFG EFG
(c) ADE ADE ADE BCF BCG BFG CFG
(d) DEF DEF DEF ABC ABC ACG BCG

But $G(\mathcal{R}_2, \mathcal{R}_3)$ must also be disconnected. With $\mathcal{R}_1$ and $\mathcal{R}_2$ as above, it is easily seen that possibilities (b) and (d) both lead to connected graphs $G(\mathcal{R}_2, \mathcal{R}_3)$. Thus $\mathcal{R}_3$ can be of type (a) or (c) only. On the other hand, there can be in a KTS(21) at most 3 parallel classes of type (a), and, without loss of generality, at most 6 parallel classes of type (c), say, 3 as given above, and 3 of the form

AFG AFG AFG BCD BCE BDE CDE.

But such a system would then contain a subsystem KTS(9) which is impossible in a KTS(21).

At the other end of the spectrum, one could require the T-vector of a KTS(v) to contain a maximum possible number of nonzero components. Let us call such a KTS *type-heterogeneous*. For v=15, somewhat trivially, the four KTS(15) that are not type-uniform, are type-heterogeneous (as there are only two nonisomorphic cubic bipartite graphs on 10 vertices). It is numerically possible for a KTS(21) to have in its T-vector all 14 components positive. No such KTS(21) is known at present, however. For $v \geq 27$, the maximum possible number of nonzero components is easily seen to be $\binom{(v-1)/2}{2}$. Do there exist type-heterogeneous KTS(v) for any orders $v \equiv 3 \pmod 6$, $v \geq 21$?

More generally, we can ask whether there exists, for any $k, 1 \leq k \leq \binom{(v-1)/2}{2}$, a KTS having exactly k ($\leq k, or \geq k$, respectively) nonzero components in its T-vector. One observation we can make is that if the automorphism group of a KTS(v) acts transitively on its parallel classes then at most $\frac{1}{2}(v-1)$ components of its T-vector can be nonzero. But whether a KTS(v) transitive on its parallel classes exists for every $v \equiv 3 \pmod 6$ is another open question... even though such KTSs are known to exist whenever $\frac{1}{3}v$ or $\frac{1}{2}(v-1)$ is a prime-power $\equiv 1 \pmod 6$ (cf., e.g., [RC]), as well as for many other orders.

2. The type vector and type matrix

As mentioned already, it was suggested earlier (cf., e.g., [PP]) to use the type vector and type matrix as invariants for KTSs. Their sensitivity appears to be surprisingly high. In the process of examining the sensitivity of the type vector as an invariant, we have first collected all data on the KTS(21) constructed to date. These include:

(i) 33 nonisomorphic KTS(21) for each of which its underlying STS(21) has an automorphism consisting of three disjoint cycles of length 7 [MPR];

(ii) 5 nonisomorphic KTS(21) whose underlying STS(21) has an automorphism of order 7 with 7 fixed points and not included in (i), [T];

(iii) 48 nonisomorphic KTS(21) whose underlying STS(21) is 4-rotational [MR1];

(iv) 19 nonisomorphic constructed by Petrenjuk by means of H-transformations (cf. [PP3]) of systems in (i) and (ii) [PP1];

(v) 99 nonisomorphic KTS(21) constructed by Mathon from the STS in (i) by replacing subsystems of order 7 (their number is three, one or zero) in all possible ways, and then finding their resolutions, or by assuming an automorphism of order 3 consisting of 7 disjoint cycles of length 3, [M].

The only possible overlaps among the 205 systems of the above 5 types could appear between those of type (ii) and (iv), (ii) and (v), and (iv) and (v). There turned out to be no overlap between (ii) and (iv) but there were 5 KTS(21) appearing in both (ii) and (v), and 7 KTS(21) in both (iv) and (v). Thus the number of known pairwise nonisomorphic KTS(21) is 192. Among the KTS(21) of type (v) constructed by Mathon, one finds the first known examples of KTSs having no nontrivial automorphism.

The type vector distinguishes the systems almost perfectly. There is only one pair of nonisomorphic KTS(21) *not* distinguished by type vectors. This pair, however, was distinguished by the type matrices. At the same time, calculating the type vector (or type matrix) proved extremely fast. Determining the type vector would normally involve testing graphs for isomorphism. But instead of using a general graph isomorphism algorithm, we simply made use of the fact that the number of 4-cycles and 6-cycles through the vertices of cubic bipartite graphs on 14 vertices distinguishes these graphs completely. Thus computing the number of 4-cycles and 6-cycles in these graphs, and, consequently, computing the type vector (and type matrix) proved extremely easy and fast.

A list of 192 known KTS(21) together with their type vectors is available on request from the authors through e-mail. Requests should be directed to *franya@maccs.dcss.mcmaster.ca* or to *rosa@mcmail.cis.mcmaster.ca* .

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