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COMPLETION OF FACTOR ALGEBRAS OF IDEALS

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ABSTRACT. Let $\Im$ be a κ-complete ideal over κ. The structure of the completion of the Boolean algebra $\wp(\kappa)/\Im$ is investigated with respect to properties of the ideal $\Im$ and the cardinal κ. It is shown that under certain conditions $\mathrm{Comp}(\wp(\kappa)/\Im)$ is isomorphic to a collapse algebra.

1. Introduction. In [**BV**] it is proven that $\mathrm{Comp}(\wp(\kappa)/\Im_\kappa)$ and $\mathrm{Col}(\omega, \kappa^+)$ are isomorphic if $2^\kappa = \kappa^+$ and κ is a regular uncountable cardinal (and where $\Im_\kappa$ is the ideal of all subsets of κ of size $< \kappa$). In this paper we extend this result to:

THEOREM 1. *Let κ be a regular uncountable cardinal such that $2^\kappa = \kappa^+$. Let $\Im$ be a κ-complete nowhere precipitous ideal over κ. Then* $\mathrm{Comp}(\wp(\kappa)/\Im)$ *and* $\mathrm{Col}(\omega, \kappa^+)$ *are isomorphic.*

COROLLARY 2. *Assume $V = L$. For every regular uncountable cardinal κ and every κ-complete ideal $\Im$ over κ,* $\mathrm{Comp}(\wp(\kappa)/\Im)$ *and* $\mathrm{Col}(\omega, \kappa^+)$ *are isomorphic.*

THEOREM 3. *Let κ be a singular cardinal. Assume $2^\kappa = \kappa^+$ and $2^{cf(\kappa)} = cf(\kappa)^+$.*
(1) *If $cf(\kappa) = \omega$, then* $\mathrm{Comp}(\wp(\kappa)/\Im)$ *and* $\mathrm{Col}(\omega_1, \kappa^+)$ *are isomorphic.*
(2) *If $cf(\kappa) > \omega$, then* $\mathrm{Comp}(\wp(\kappa)/\Im_\kappa)$ *and* $\mathrm{Col}(\omega, \kappa^+)$ *are isomorphic.*

REMARK. "If $V = L$, then $\mathrm{Comp}(\wp(\omega_1)/\Im)$ and $\mathrm{Col}(\omega, \omega_2)$ are isomorphic" is proven independently using a different method in [**BTW**].

REMARK. Let $\lambda = \mu^+ \leq \kappa$. Let $\Im$ be a λ-complete and not λ^+-complete ideal over κ. Then $\wp(\kappa)/\Im$ as a forcing notion collapses either λ^+ to λ, or λ to μ (see [**F**]). If $\lambda = \kappa$, the latter happens (see [**BTW**]).

2. Notation and definitions. Lowercase Greek letters are reserved for ordinals. Let κ, τ, μ be cardinals.

Let X, Y be sets. $\wp(X)$ is the set of all subsets of X. ${}^XY$ is the set of all functions from X into Y,

$$[X]^{<\lambda} = \{Y \subseteq X : |Y| < \lambda\}, \quad [X]^\lambda = \{Y \subseteq X : |Y| = \lambda\},$$
$$\kappa^{<\lambda} = \sum\{\kappa^\gamma : \gamma < \lambda \ \&\ \gamma \text{ is a cardinal}\}, \quad {}^{<\lambda}\kappa = \bigcup\{{}^\alpha\kappa : \alpha \in \lambda\}.$$

$\mathrm{Col}(\lambda, \kappa)$ is the Boolean completion of the set of all functions f with $\mathrm{dom}(f) \in [\lambda]^{<\lambda}$ and $\mathrm{rng}(f) \subseteq \kappa$ ordered by the inverse inclusion. (In generic extensions obtained via $\mathrm{Col}(\lambda, \kappa)$, the cardinal λ is preserved and the cardinal κ becomes an ordinal of size λ.)

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If $X, Y \subset \wp(\kappa)$, then $X \subseteq\subseteq Y$ iff $(\forall y \in Y)(\exists x \in X)(x \subseteq y)$; we say that X *refines* Y, or that X is a *refinement* of Y.

If B is a Boolean algebra, then 1_B is its greatest element, while 0_B is its least element. $B^+ = B - \{0_B\}$. If $b \in B$, then $B|b = \{c \in B : c \leq b\}$. Comp$(B)$ denotes the Boolean completion of B (it can be defined as the algebra of regular open sets of the Stone space of B and is unique up to isomorphism).

Let $b \in B^+$. $P \subseteq B^+$ is a *partition of* b iff P is a maximal disjoint subfamily of elements of $(B|b)^+$. If P_1, P_2 are partitions of b, then

$$P_1 \leq\leq P_2 \quad \text{iff } (\forall c \in P_1)(\exists d \in P_2)(c \leq d);$$

we say P_1 *refines* P_2, or that P_1 is a *refinement of* P_2. $\langle P_\alpha : \alpha < \lambda \rangle$ is a *descending sequence of partitions of* b iff $P_\alpha \leq\leq P_\beta$ whenever $\beta < \alpha < \lambda$.

$\Im$ is a λ-*complete* ideal over κ iff $\Im \subseteq \wp(\kappa)$, $\Im \neq \wp(\kappa)$, $\Im$ is closed under unions of size $< \lambda$ and under the subset operation, and $\Im$ contains all singletons of κ. $\Im^+ = \wp(\kappa) - \Im$. $\Im^* = \{X \subseteq \kappa : (\kappa - X) \in \Im\}$, which is the dual filter to $\Im$.

For an $A \in \Im^+$ define $\Im|A = \{X \subseteq \kappa : (X \cap A) \in \Im\}$ (which also is a λ-complete ideal over κ and $\Im \subseteq \Im|A$).

$\Im_\kappa = \{X \subseteq \kappa : |X| < \kappa\}$ is the *Fréchet ideal* over κ.

Let $X, Y \in \Im$. Then $X \subseteq^* Y$ iff $(X - Y) \in \Im$. $P \subseteq \wp(\kappa)$ is $\Im$-*disjoint* iff $(\forall X \neq Y \in P)(X \cap Y \in \Im)$. $P \subseteq \wp(\kappa)$ is an *almost disjoint family* iff P is $\Im_\kappa$-disjoint.

Let $S \in \Im^+$, then $P \subseteq \wp(S) \cap \Im^+$ is an $\Im$-*partition* of S iff P is $\Im$-disjoint and maximal. $\langle P_\alpha : \alpha \in \lambda \rangle$ is a *descending sequence of* $\Im$-*partitions* of S iff for every $\alpha \in \lambda$, P_α is an $\Im$-partition of S and $P_\alpha \subseteq\subseteq P_\beta$ whenever $\alpha < \beta < \lambda$. Let $\langle P_n : n < \omega \rangle$ be a descending sequence of $\Im$-partitions of S. Then a sequence $\langle X_n : n < \omega \rangle$ is a *path through* $\langle P_n : n < \omega \rangle$ iff for every $n \in \omega$, $X_n \in P_n$ and also $X_{n+1} \subseteq X_n$.

$\Im$ is a *precipitous ideal* iff for every $S \in \Im^+$ and for every descending sequence $\langle P_n : n \in \omega \rangle$ of $\Im$-partitions of S, there is a path through with a nonempty intersection. $\Im$ is *nowhere precipitous* iff for every $A \in \Im^+$, $\Im|A$ is not precipitous.

Let $X \subseteq \kappa$. Define

$$X/\Im = \{Y \subseteq \kappa : (X - Y) \cup (Y - X) \in \Im\}, \quad \wp(\kappa)/\Im = \{X/\Im : X \subseteq \kappa\},$$
$$X/\Im \wedge Y/\Im = (X \cap Y)/\Im, \quad X/|\Im \vee Y/\Im = (X \cup Y)/\Im, \quad -X/\Im = (\kappa - X)/\Im.$$

Thus $\wp(\kappa)/\Im$, $\wedge$, and $\vee$ form a λ-complete Boolean algebra (for $\Im$ is λ-complete) with $\emptyset/\Im = \Im$ being its smallest element and $\kappa/\Im$ being its greatest element. For any $\xi < \lambda$,

$$\sum\{X_\alpha/\Im : \alpha < \xi\} = \left(\bigcup\{X_\alpha : \alpha < \xi\}\right)/\Im$$

and

$$\prod\{X_\alpha : \alpha < \xi\} = \left(\bigcup\{X_\alpha : \alpha < \xi\}\right)/\Im.$$

Let $\kappa \geq 2$ and $\lambda \geq \omega$. A Boolean algebra B is (λ, μ, κ)-*distributive* iff for every sequence $\langle P_\alpha : \alpha \in \lambda \rangle$ of partitions of 1_B of size $\leq \mu$, here is Q, a partition of 1_B, so that $(\forall \alpha \in \lambda)(\forall q \in Q)(|\{p \in P_\alpha : p \wedge q \neq 0_B\}| < \kappa)$. B is $(\lambda, \cdot, \kappa)$-*distributive* iff the above holds without the restriction on the size of P_α's. B is $(\lambda, \cdot, \kappa)$-*nowhere distributive* iff for every $b \in B^+$, $B|b$ is not $(\lambda, \cdot, \kappa)$-distributive.

Let P be a set partially ordered by $\leq$. $D \subseteq P$ is *dense* in P iff $(\forall p \in P)(\exists d \in D)(d \leq p)$. P is *λ-closed* iff for any descending sequence $\langle p_\alpha \colon \alpha < \beta \rangle$ of length $\beta < \lambda$ of elements of P, there is a $p \in P$ such that $(\forall \alpha < \beta)(p \leq p_\alpha)$. $d(P) = \min\{|D| \colon D \text{ is dense in } P\}$.

3. Preliminaries.

LEMMA 4. *Let λ be an infinite cardinal and κ be a cardinal ≥ 2. A Boolean algebra B is $(\lambda, \cdot, \kappa)$-nowhere distributive iff there is a sequence $\langle P_\alpha \colon \alpha < \lambda \rangle$ of partitions of 1_B such that $(\forall b \in B^+)(\exists \alpha \in \lambda)(|\{p \in P_\alpha \colon p \wedge b \neq 0_B\}| \geq \kappa)$.*

PROOF. See Lemma 1.11 in [**BSV**]. $\square$

LEMMA 5. *Let λ be an infinite cardinal and κ be a cardinal ≥ 2. Let B be a complete $(\lambda, \cdot, \kappa)$-nowhere distributive Boolean algebra containing a λ-closed dense subset. Let $d(B) = \kappa^{<\lambda}$. Then B is isomorphic to $\mathrm{Col}(\lambda, \kappa)$.*

PROOF. See Theorem 1.15 and Corollary 1.16 in [**BSV**]. $\square$

LEMMA 6. *Let κ be an uncountable cardinal. Let $\Im$ be a countably complete ideal over κ. $\Im$ is nowhere precipitous iff there is a descending sequence of $\Im$-partitions of κ with no path through whose intersection is nonempty.*

PROOF. Assume that $\Im$ is nowhere precipitous (the opposite direction is obvious). Note that if $A \in \Im^+$, then $(\Im|A)^+ \subseteq \Im^+$ and so if $C \in (\Im|A)^+$ and W is an $(\Im|A)$-partition of C, $C \in \Im^+$ and W is an $\Im$-partition of C as well. Since $\Im$ is nowhere precipitous, $\Im|A$ is not precipitous for any $A \in \Im^+$. I.e., there are $C \in (\Im|A)^+$ and a descending sequence of $(\Im|A)$-partitions with no path through whose intersection is nonempty. Thus there is $C \in \Im^+$ so that $C \subseteq A$ and

(*) there is a descending sequence $\langle W(C,n) \colon n < \omega \rangle$ of $\Im$-partitions of C with no path through whose intersection is nonempty.

We have just shown that the set of all $C \in \Im^+$ for which (*) holds is dense in $\langle \Im^+, \subseteq \rangle$. Therefore there is an $\Im$-disjoint partition F of κ so that (*) holds for every $C \in F$. Now for every $n < \omega$ define $W_n = \bigcup\{W(C,n) \colon C \in F\}$. Since F is an $\Im$-disjoint family, all W_n's are $\Im$-partitions of κ and form a descending sequence of partitions. If $\langle X_n \colon n < \omega \rangle$ is a path through $\langle W_n \colon n < \omega \rangle$, then $X_0 \in W(C, 0)$ for some $C \in F$ and since $X_n \subseteq X_0 \subseteq C$, $\langle X_n \colon n < \omega \rangle$ is a path through $\langle W(C,n) \colon n < \omega \rangle$, and hence its intersection is empty. $\square$

LEMMA 7. *Let κ, λ be uncountable cardinals. Let $\Im$ be a λ-complete nowhere precipitous ideal over κ. Then $\wp(\kappa)/\Im$ is $(\omega, \cdot, \lambda^+)$-nowhere distributive.*

PROOF. By Lemma 6 there is a descending sequence $\langle W_n \colon N < \omega \rangle$ of $\Im$-partitions of κ with no path through whose intersection is nonempty. We shall prove that

$$(\forall X \in \Im^+)(\exists n < \omega)(|\{Y \in W_n \colon X/\Im \wedge Y/\Im \neq \emptyset/\Im\}| \geq \lambda^+).$$

Then, by Lemma 4, $\wp(\kappa)/\Im$ is $(\omega, \cdot, \lambda^+)$-nowhere distributive. By way of contradiction assume $(\exists X \in \Im^+)(\forall n < \omega)(|\{Y \in W_n \colon X/\Im \wedge Y/\Im\}| < \lambda^+)$. Since W_n is an $\Im$-partition of κ, $Q_n = \{Y/\Im \colon Y \in W_n \; \& \; X/\Im \wedge Y/\Im \neq \emptyset/\Im\}$ is a partition of $X/\Im$ of size $\leq \lambda$ (in the Boolean algebra $\wp(\kappa)/\Im$). For each $Y/\Im \in Q_n$

choose a representative and enumerate them, i.e. $Q_n = \{Y^n_\alpha/\Im\colon \alpha < \lambda_n\}$, where $\lambda_n = |Q_n| \le \lambda$. Let $\hat{Q}_n = \{Y^n_\alpha\colon \alpha < \lambda_n\}$. Then $\hat{Q}_n$ is an $\Im$-partition of X of size λ_n. Define $Z^n_0 = Y^n_0$ and $Z^n_\alpha = Y^n_\alpha - \bigcup\{Y^n_\beta\colon \beta < \alpha\}$ for all $\alpha < \lambda_n$. Since $\Im$ is λ-complete and $\lambda_n \le \lambda$, $\{Z^n_\alpha\colon \alpha < \lambda_n\}$ is a disjoint $\Im$-partition of X refining $\hat{Q}_n$. Let $Z_n = X - \bigcup\{Z^n_\alpha\colon \alpha < \lambda_n\}$. Then every $Z_n \in \Im$ and so $X - \bigcup\{Z_n\colon n < \omega\} \ne \emptyset$, for $\Im$ is countable complete. Pick any $\rho \in X - \bigcup\{Z_n\colon n < \omega\}$. For every $n < \omega$, $\rho \in X$ and $\rho \notin Z_n$, hence there must be some $\alpha_n < \lambda_n$ so that $\rho \in Z^n_{\alpha_n} \subseteq Y^n_{\alpha_n}$. Since $\langle W_n\colon n < \omega\rangle$ is a descending sequence of $\Im$-partitions of κ, $\langle Y^n_{\alpha_n}\colon n < \omega\rangle$ is a (descending) path through $\langle W_n\colon n < \omega\rangle$ which has a nonempty intersection (for it contains at least ρ), a contradiction. □

LEMMA 8. *Let κ, λ be cardinals, $\kappa \ge 2$ and $\lambda \ge \omega$. Let B be a $(\lambda, \cdot, \kappa)$-nowhere distributive Boolean algebra. Then*
(1) *B is atomless and thus ω-closed,*
(2) *if $D \subseteq B$ is dense in B, then $|D| \ge \kappa$,*
(3) $\mathrm{Comp}(B)$ *is also $(\lambda, \cdot, \kappa)$-nowhere distributive.*

PROOF. Easy, left to the interested reader. □

LEMMA 9. *Let κ, λ be uncountable cardinals. Let $\Im$ be a λ-complete nowhere precipitous ideal over κ so that $\wp(\kappa)/\Im$ has a dense set of size λ^+. Then* $\mathrm{Comp}(\wp(\kappa)/\Im)$ *and* $\mathrm{Col}(\omega, \lambda^+)$ *are isomorphic.*

PROOF. Since $\Im$ is λ-complete and nowhere precipitous, by Lemma 6, $\wp(\kappa)/\Im$ is $(\omega, \cdot, \lambda^+)$-nowhere distributive. By Lemma 8, $\mathrm{Comp}(\wp(\kappa)/\Im)$ is also $(\omega, \cdot, \lambda^+)$-nowhere distributive and $d(\mathrm{Comp}(\wp(\kappa)/\Im)) \ge \lambda^+$. Since $\wp(\kappa)/\Im$ has a dense set of size λ^+, so does $\mathrm{Comp}(\wp(\kappa)/\Im)$. Hence $d(\mathrm{Comp}(\wp(\kappa)/\Im)) = \lambda^+ = (\lambda^+)^{<\omega}$. By Lemma 8, $\wp(\kappa)/\Im$ is ω-closed. Thus all requirements of Lemma 5 are satisfied and so $\mathrm{Comp}(\wp(\kappa)/\Im)$ and $\mathrm{Col}(\omega, \lambda^+)$ are isomorphic. □

These were preliminaries needed to deal with ideals over regular cardinals. Let us now turn our attention to singular cardinals. Unfortunately, we can only describe the structure of $\mathrm{Comp}(\wp(\kappa)/\Im_\kappa)$.

Let $\omega \le \lambda = cf(\kappa) < \kappa$. (Recall that λ is regular, $\wp(\lambda)/\Im_\lambda$ can be regularly imbedded into $\wp(\kappa)/\Im_\kappa$, and $\Im_\kappa$ is λ-complete.)

As usual, for $X, Y \in [\lambda]^\lambda$ define $X \subseteq^* Y$ iff $(X - Y) \in \Im_\lambda$, i.e. iff $|X - Y| < \lambda$.

LEMMA 10. *Let $\lambda \ge \omega$ be regular. For every $X \in [\lambda]^\lambda$ define by induction $h_X(\alpha) = \min(\lambda - \bigcup\{h_X(\beta)\colon \beta < \alpha\})$ for all $\alpha \in X$. Then*
(1) *h_X is strictly increasing,*
(2) *$(\forall \alpha \in X)(h_X(\alpha) \le \alpha)$,*
(3) *if $Y \in [\lambda]^\lambda$ and $Y \subseteq^* X$, then*

$$(\exists \beta \in X \cap Y)(\forall \alpha \in X \cap Y)(\alpha \ge \beta \to h_X(\alpha) \ge h_Y(\alpha)),$$

(4) *if $s\colon X \to \lambda$ is strictly increasing, then $(\forall \alpha \in X)(h_X(\alpha) \le s(\alpha))$.*

PROOF. Standard, left to the reader. (Note that h_X is the inverse of the enumeration function for X.) □

LEMMA 11. *Let $\omega \le \lambda = cf(\kappa) < \kappa$. Assume that there is a sequence $\langle P_\alpha\colon \alpha \in \mu\rangle$ of $\Im_\lambda$-partitions of λ so that $(\forall X \in [\lambda]^\lambda)(\exists \alpha \in \mu)(\exists Y \in P_\alpha)(Y \subseteq^* X)$. Then $\wp(\kappa)/\Im_\kappa$ is $(\mu, \cdot, \kappa^+)$-nowhere distributive.*

PROOF. Let $\kappa = \bigcup\{Q_\alpha : \alpha \in \lambda\}$ where $\langle Q_\alpha : \alpha \in \lambda\rangle$ is a disjoint family of subsets of κ so that $\langle |Q_\alpha| : \alpha \in \lambda\rangle$ is a strictly increasing sequence of regular cardinals and $\kappa = \sum\{|Q_\alpha| : \alpha \in \lambda\}$. Let $\kappa_\alpha = |Q_\alpha|$ for every $\alpha \in \lambda$. For any $X \in [\lambda]^\lambda$ define $\hat{X} = \bigcup\{Q_\alpha : \alpha \in X\}$. Let $h_X : X \to \lambda$ be as in Lemma 10. Define

$$S_X = \{D \in [\kappa]^\kappa : (\forall \alpha \in X)(|D \cap Q_\alpha| \leq |Q_{h_X(\alpha)}|) \& (\forall \alpha \in (\lambda - X))(D \cap Q_\alpha = \varnothing)\}.$$

Claim 1. For any $X \in [\lambda]^\lambda$, S_X is dense in $([\hat{X}]^\kappa, \subseteq)$.

If $D \in S_X$, then $D = \bigcup\{D \cap Q_\alpha : \alpha \in X\} \subseteq \bigcup\{Q_\alpha : \alpha \in X\} = \hat{X}$, so $D \in [\hat{X}]^\kappa$. Let $Y \in [\hat{X}]^\kappa$. Then

$$(\forall \gamma \in \lambda)(\forall \beta \in X)(\exists \alpha \in X)(\alpha > \beta \ \& \ |Y \cap Q_\alpha| \geq |Q_\gamma|).$$

If not, then $(\exists \gamma \in \lambda)(\exists \beta \in X)(\forall \alpha \in X)(\alpha > \beta \to |Y \cap Q_\alpha| < |Q_\gamma|)$ and so $|Y| < |Q_\beta| + |G_\gamma| < \kappa$, a contradiction. Thus we can define by induction a strictly increasing function $f : \lambda \to X$ so that $|Y \cap Q_{f(\gamma)}| \geq |Q_\gamma|$ for all $\gamma \in \lambda$ and $\gamma \leq h_X(f(\gamma)) \leq f(\gamma)$. For every $\alpha \in X$ define D_α by

(i) if $\alpha = f(\gamma)$ for some $\gamma \in \lambda$, then let D_α be a subset of $(Y \cap Q_{f(\gamma)})$ of size $\geq |Q_\gamma|$ but $\leq |Q_{h_X(f(\gamma))}|$. Notice that this is possible as $\gamma \leq h_X(f(\gamma))$ and so $|Q_\gamma| \leq |Q_{h_X(f(\gamma))}|$,

(ii) otherwise $D_\alpha = \varnothing$.

Let $D = \bigcup\{D_\alpha : \alpha \in X\}$. Then $D \subseteq Y$, and $|D| \geq \sum\{|Q_\gamma| : \gamma \in \operatorname{rng}(f)\} = \kappa$ as $\operatorname{rng}(f) = \lambda$, and also $|D \cap Q_\alpha| \leq |Q_{h_X(\alpha)}|$ for all $\alpha \in X$. Hence $D \in S_X$ and so the claim is proven.

By Claim 1, for any $X \in [\lambda]^\lambda$ there is an almost disjoint partition of $\hat{X}$ consisting entirely of elements of S_X. Choose one and denote it R_X. Define

$$W_\alpha = \bigcup\{R_X : X \in P_\alpha\}$$

for every $\alpha \in \mu$.

Claim 2. Each W_α is an almost disjoint partition of κ.

Clearly, for each $Z \in W_\alpha$, $|Z| = \kappa$. Now let $Z_1, Z_2 \in W_\alpha$. If $Z_1, Z_2 \in R_X$ for some $X \in P_\alpha$, then $|Z_1 \cap Z_2| < \kappa$. So assume that $Z_1 \in R_{X_1}, Z_2 \in R_{X_2}$, and $X_1 \neq X_2 \in P_\alpha$. Assume by way of contradiction that $|Z_1 \cap Z_2| = \kappa$. If $\beta \in Z_1 \cap Z_2$, then $\beta \in \hat{X}_1 \cap \hat{X}_2$, so $\beta \in Q_\gamma$ for some $\gamma \in X_1 \cap X_2$. Thus there is a function $f : (Z_1 \cap Z_2) \to (X_1 \cap X_2)$ so that $\beta \in Q_{f(\beta)}$. Since $|X_1 \cap X_2| < \lambda$, and hence $|\operatorname{rng}(f)| < \lambda$, there are $\gamma \in X_1 \cap X_2$ and $Z_3 \in [Z_1 \cap Z_2]^\kappa$ such that $f(\beta) = \gamma$ for all $\beta \in Z_3$. Then $Z_3 \subseteq Q_\gamma$, so $|Q_\gamma| = \kappa$, a contradiction. Hence $|Z_1 \cap Z_2| < \kappa$. The claim is proven.

Claim 3. $\{W_\alpha : \alpha \in \mu\}$ witnesses that $\wp(\kappa)/\mathfrak{I}_\kappa$ is $(\mu, \cdot, \kappa^+)$-nowhere distributive.

Let $C \in [\kappa]^\kappa$. Then $(\forall \gamma \in \lambda)(\forall \beta \in \lambda)(\exists \alpha \in \lambda)(\alpha > \beta \ \& \ |C \cap Q_\alpha| > |Q_\gamma|)$. If not, then $(\exists \gamma \in \lambda)(\exists \beta \in \lambda)(\forall \alpha \in \lambda)(\alpha > \beta \to |C \cap Q_\alpha| \leq |Q_\gamma|)$. Then $|C| \leq |Q_\beta| + |Q_\gamma| < \kappa$, a contradiction. Thus there is a strictly increasing sequence $\langle \alpha_\gamma : \gamma \in \lambda\rangle \subseteq \lambda$ so that $|C \cap Q_{\alpha_\gamma}| > |Q_\gamma|$. Let $X = \{\alpha_\gamma : \gamma \in \lambda\}$. So $X \in [\lambda]^\lambda$. Define $s(\alpha_\gamma) = \gamma$ for all $\gamma \in \lambda$. Thus $s : X \to \lambda$ is strictly increasing and therefore $h_X(\alpha) \leq s(\alpha)$ for all $\alpha \in X$. By the assumption of this lemma, $(\exists \delta \in \mu)(\exists Y \in P_\delta)(Y \subseteq^* X)$. Then there is $\overline{\alpha} \in X \cap Y$ so that $h_Y(\alpha) \leq h_X(\alpha)$ for all $\alpha \in (X \cap Y) - \overline{\alpha}$. Recall that R_Y is an almost disjoint partition of $\hat{Y}$ consisting of elements of S_Y, and also $R_Y \subseteq W_\delta$. If $Z \in R_Y$, then $Z \in [\kappa]^\kappa$, $(\forall \alpha \in Y)(|Z \cap Q_\alpha| \leq |Q_{h_Y(\alpha)}|)$ and $(\forall \alpha \in \lambda - Y)(Z \cap Q_\alpha = \emptyset)$.

(*) There are at least κ^+ elements of R_Y which have an intersection of size κ with C.

If not, let $\{Z_\rho \colon \rho \in \kappa\}$ be a list of elements of R_Y which contains all $Z \in R_Y$ so that $|Z \cap C| = \kappa$ (if need be, with repetitions). Let $D = C \cap \hat{X} \cap \hat{Y}$. Then

$$(**) \qquad (\forall \beta \in \lambda)(\exists \gamma \in \lambda)\Big(\gamma > \beta \ \& \ \Big|\Big(D - \bigcup\{Z_\rho \colon \rho \in \kappa_\beta\}\Big) \cap Q_{\alpha_\gamma}\Big| > |Q_\gamma|\Big).$$

Let $\beta \in \lambda$. Since $Y \subseteq^* X, |X \cap Y| = \lambda$. There is $\gamma \in \lambda$ so that $\gamma > \beta$ and $\alpha_\gamma \in (X \cap Y) - \overline{\alpha}$. If $\rho < \kappa_\beta$, then $|Z_\rho \cap Q_{\alpha_\gamma}| \leq |Q_{h_Y(\alpha_\gamma)}| \leq |Q_{h_X(\alpha_\gamma)}| \leq |Q_{s(\alpha_\gamma)}| = |Q_\gamma|$. Thus

$$\Big|\bigcup\{Z_\rho \colon \rho < \kappa_\beta\} \cap Q_{\alpha_\gamma}\Big| \leq \kappa_\beta \times |Q_\gamma| = |Q_\gamma|$$

as $\gamma > \beta$ and $|Q_\gamma| = \kappa_\gamma$.

Since $\alpha_\gamma \in X \cap Y, Q_{\alpha_\gamma} \subseteq \hat{X} \cap \hat{Y}$. $|D \cap Q_{\alpha_\gamma}| = |(C \cap \hat{X} \cap \hat{Y}) \cap Q_{\alpha_\gamma}| = |C \cap Q_{\alpha_\gamma}| > |Q_\gamma|$. Thus

$$\begin{aligned}&\Big|\Big(D - \bigcup\{Z_\rho \colon \rho \in \kappa_\beta\}\Big) \cap Q_{\alpha_\gamma}\Big| \\ &\quad = \Big|(D \cap Q_{\alpha_\gamma}) - \Big(\bigcup\{Z_\rho \colon \rho < \kappa_\beta\} \cap Q_{\alpha_\gamma}\Big)\Big| > |Q_\gamma|.\end{aligned}$$

Therefore (**) holds.

By (**) there is a strictly increasing sequence $\langle \gamma_\beta \colon \beta < \lambda\rangle \subseteq \lambda$ so that $|(D - \bigcup\{Z_\rho \colon \rho \in \kappa_\beta\}) \cap Q_{\alpha_{\gamma\beta}}| > |Q_{\gamma\beta}|$. For each $\beta \in \lambda$ let A_β be a subset of $(D - \bigcup\{Z_\rho \colon \rho \in \kappa_\beta\}) \cap Q_{\alpha_{\gamma\beta}}$ of size $|Q_{\gamma\beta}|$. Let $A = \bigcup\{A_\beta \colon \beta \in \lambda\}$. Then $A \subseteq D$, so $A \subseteq \hat{Y}$ and $A \subseteq C$. $|A| = \sum\{|A_\beta| \colon \beta \in \lambda\} = \sum\{|Q_{\gamma\beta}| \colon \beta \in \lambda\} = \sum\{\kappa_{\gamma\beta} \colon \beta \in \lambda\} = \kappa$. So $A \in |\hat{Y}|^\kappa$. Since R_Y is an almost disjoint partition of $\hat{Y}$, for some $Z \in R_Y$, $|Z \cap A| = \kappa$. Then also $|Z \cap C| = \kappa$. Therefore $Z = Z_{\rho_0}$ for some $\rho_0 \in \kappa$. Let $\beta \in \lambda$ so that $\rho_0 < \kappa_\beta$ and $Z \cap A_\beta \neq \emptyset$. Let $x \in Z \cap A_\beta$. Then $x \notin \bigcup\{Z_\rho \colon \rho < \kappa_\beta\}$, so $x \notin Z_{\rho_0} = Z$, a contradiction. Hence (*) holds.

Thus Claim 3 is proven.

This completes the proof of the lemma. □

LEMMA 12. *Let $\lambda \geq \omega_1$ be regular. Let $B = \mathrm{Col}(\omega, \lambda^+)$. There is a descending sequence $\langle P_n \colon n < \omega\rangle$ of partitions of 1_B such that $(\forall b \in B^+)(\exists n < \omega)(\exists c \in P_n)(c < b)$ and so that $\bigcup\{P_n \colon n < \omega\}$ is dense in B.*

PROOF [SKETCH]. Let $C = {}^{<\omega}(\lambda^+)$. For $f, g \in C$ let $f \preceq g$ iff $g \subseteq f$. Then $B = \mathrm{Comp}((C, \preceq))$. For every $n < \omega$, let $P_n = {}^n(\lambda^+)$. Now it is easy to check that $\langle P_n \colon n < \omega\rangle$ is a required sequence. □

LEMMA 13. *Let $2^\omega = \omega_1$. There is a sequence $\langle P_\alpha \colon \alpha \in \omega_1\rangle$ of $\Im_\omega$-partitions of ω such that $(\forall X \in [\omega]^\omega)(\exists \alpha \in \omega_1)(\exists Y \in P_\alpha)(Y \subseteq X)$.*

PROOF. Easy, left to the reader (or follows from Base Matrix Theorem in [**BPS**] (see [**BVo**])).

LEMMA 14. *Let $\omega = cf(\kappa) < \kappa$. Then $\wp(\kappa)/\Im_\kappa$ is ω_1-closed.*

PROOF. Let $\langle \kappa_n \colon n < \omega\rangle$ be an increasing sequence of regular cardinals cofinal in κ. $\wp(\kappa)/\Im_\kappa$ is atomless, for if $X \notin \Im_\kappa$, then $|X| = \kappa$ and so there is $Y \subseteq X$ so that $|Y| = \kappa$ and $|X - Y| = \kappa$, hence $\varnothing/\Im_\kappa < Y/\Im_\kappa < X/\Im_\kappa$ and so $X/\Im_\kappa$ is not an atom. Thus $\wp(\kappa)/\Im_\kappa$ is ω-closed. Let $\langle X_n/\Im_\kappa \colon n < \omega\rangle$ be a strictly decreasing sequence of elements of $\wp(\kappa)/\Im_\kappa$. Let $Y_n = \bigcap\{X_m \colon m \leq n\}$ for all $n < \omega$. Since $X_n = Y_n \cup \bigcup\{X_n - X_m \colon m < n\}$ and $\bigcup\{X_n - X_m \colon m < n\} \in \Im_\kappa$, $Y_n/\Im_\kappa = X_n/\Im_\kappa$ for every $n < \omega$. Thus $\langle Y_n/\Im_\kappa \colon n < \omega\}$ is a strictly decreasing sequence of elements of $\wp(\kappa)/\Im_\kappa$, and therefore $|Y_{n+1} - Y_n| = \kappa$ for all $n < \omega$. For each $n < \omega$ choose some $A_n \subset (Y_n - Y_{n+1})$ of size κ_n. Let $A = \bigcup\{A_n \colon n < \omega\}$. Then $|A| = \sum\{\kappa_n \colon n < \omega\} = \kappa$, $A \subseteq Y_0$, and for every $n \geq 1$, $A - Y_n = \bigcup\{A_i \colon i < n\}$, hence $A \subseteq^* Y_n$ and so $A \subseteq^* X_n$ for all $n < \omega$. Thus $\varnothing/\Im_\kappa < A/\Im_\kappa \leq X_n/\Im_\kappa$ for all $n < \omega$. □

4. Main results.

PROOF OF THEOREM 1. By Lemma 7, $\wp(\kappa)/\Im$ is $(\omega, \cdot, \kappa^+)$-nowhere distributive and so by Lemma 8 $d(\wp(\kappa)/\Im) \geq \kappa^+$. Since $|\wp(\kappa)/\Im| \leq |\wp(\kappa)| = 2^\kappa = \kappa^+$, $d(\wp(\kappa/\Im)) = \kappa^+$ and so $\wp(\kappa)/\Im$ has a dense set of size κ^+. Thus, by Lemma 9, $\mathrm{Comp}(\wp(\kappa)/\Im)$ and $\mathrm{Col}(\omega, \kappa^+)$ are isomorphic. □

PROOF OF COROLLARY 2. It is known that under $V = L$, there are no measurable cardinals, and hence no countably complete precipitous ideals (see e.g. [**J**]) and that the G.C.H. holds. Therefore for any κ-complete ideal $\Im$ over κ, all requirements of Theorem 1 are satisfied, and so $\mathrm{Comp}(\wp(\kappa)/\Im)$ and $\mathrm{Col}(\omega, \kappa^+)$ are isomorphic. □

NOTE. Why cannot Theorem 1 be applied to singular cardinals? For if $\Im$ is a κ-complete ideal over a singular cardinal κ, then $\kappa \in \Im$ and it is a contradiction. Is it necessary that $\Im$ be κ-complete? As long as we can find a dense set of size λ^+ in $\wp(\kappa)/\Im$, we need just λ-completeness (see Lemma 9). Thus we need κ-completeness only to get "close" to the "natural" estimate of the size of $\wp(\kappa)/\Im$.

PROOF OF THEOREM 3. Let κ be a singular cardinal. Assume $2^\kappa = \kappa^+$ and $2^{cf(\kappa)} = cf(\kappa)^+$.

(1) Assume that $cf(\kappa) = \omega$.

By Lemmas 13 and 11, $\wp(\kappa)/\Im_\kappa$ is $(\omega_1, \cdot, \kappa^+)$-nowhere distributive. By Lemma 8, $\mathrm{Comp}(\wp(\kappa)/\Im_\kappa)$ is $(\omega_1, \cdot, \kappa^+)$-nowhere distributive. By Lemma 14, $\wp(\kappa)/\Im_\kappa$ is ω_1-closed. By Lemma 8, $d(\mathrm{Comp}(\wp(\kappa)/\Im_\kappa)) \geq \kappa^+$. Since $|\wp(\kappa)/\Im_\kappa| \leq 2^\kappa = \kappa^+, d(\mathrm{Comp}(\wp(\kappa)/\Im_\kappa)) = \kappa^+$. Since $2^\kappa = \kappa^+$ and $cf(\kappa) = \omega, (\kappa^+)^{<\omega_1} = \kappa^+$. Therefore, by Lemma 5, $\mathrm{Comp}(\wp(\kappa)/\Im_\kappa)$ and $\mathrm{Col}(\omega_1, \kappa^+)$ are isomorphic.

(2) Assume that $\lambda = cf(\kappa) > \omega$.

By Theorem 1, $\mathrm{Comp}(\wp(\lambda)/\Im_\lambda)$ is isomorphic to $\mathrm{Col}(\omega, \lambda^+)$, for $\Im_\lambda$ is nowhere precipitous (see [**J**] or [**JP**]), λ-complete and $2^\lambda = \lambda^+$. Thus, by Lemma 12, there is a descending sequence $\langle P_n : n < \omega\rangle$ of $\Im_\lambda$-partitions of λ so that $(\forall X \in [\lambda]^\lambda)(\exists n < \omega)(\exists Y \in P_n)(Y \subseteq^* X)$. Hence, by Lemma 11, $\wp(\kappa)/\Im_\kappa$ is $(\omega, \cdot, \kappa^+)$-nowhere distributive. By Lemma 8, $\mathrm{Comp}(\wp(\kappa)/\Im_\kappa)$ is $(\omega, \cdot, \kappa^+)$-nowhere distributive, $d(\mathrm{Comp}(\wp(\kappa)/\Im_\kappa)) \geq \kappa^+$, and $\wp(\kappa)/\Im_\kappa$ is ω-closed. Since $|\wp(\kappa)/\Im_\kappa| \leq 2^\kappa = \kappa^+$, $d(\mathrm{Comp}(\wp(\kappa)/\Im_\kappa)) = \kappa^+$. Clearly $(\kappa^+)^{<\omega} = \kappa^+$ and so, by Lemma 5, $\mathrm{Comp}(\wp(\kappa)/\Im_\kappa)$ and $\mathrm{Col}(\omega, \kappa^+)$ are isomorphic. □

REFERENCES

[**BPS**] B. Balcar, J. Pelant and P. Simon, *The space of ultrafilters on N covered by nowhere dense sets*, Fund. Math. **110** (1980), 11–24.

[**BSV**] B. Balcar, P. Simon and P. Vojtáš, *Refinement properties and extensions of filters in Boolean algebras*, Trans. Amer. Math. Soc. **267** (1981), 265–283.

[**BTW**] J. E. Baumgartner, A. D. Taylor and S. Wagon, *Ideals on uncountable cardinals*, Logic Colloquium 77, Proc. Conf. Wroclaw, 1977.

[**BV**] B. Balcar and P. Vopěnka, *On systems of almost disjoint sets*, Bull. Acad. Polon. Sci. Sér. Sci. Tech. **20** (1972).

[**BVo**] B. Balcar and P. Vojtáš, *Almost disjoint refinement of families of subsets of N*, Proc. Amer. Math. Soc. **79** (1980), 465–470.

[**F**] F. Franek, *Some results about saturated ideals and about isomorphisms of κ-trees*, Ph.D. Dissertation, Dept. of Math., University of Toronto, 1983.

[**J**] T. Jech, *Set theory*, Academic Press, 1978.

[**JP**] T. Jech and K. Prikry, *Ideals over uncountable sets: applications of almost disjoint functions and generic ultrapowers*, Mem. Amer. Math. Soc., No. 214 (1979).

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