LARGE SETS OF MUTUALLY ALMOST DISJOINT STEINER TRIPLE SYSTEMS NOT FROM STEINER QUADRUPLE SYSTEMS

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Abstract. We construct what we believe to be the first examples of large sets of v MAD STS(v) which are not obtained from Steiner quadruple systems.

1. Introduction

A *Steiner triple system* of order v (briefly STS(v)) is a pair $(V, \mathcal{B})$ where V is a v-set, and $\mathcal{B}$ is a collection of 3-subsets of V called *triples* such that each 2-subset of V is contained in exactly one triple. It is well known that an STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6).

A family $(V, \mathcal{B}_1), (V, \mathcal{B}_2), ..., (V, \mathcal{B}_q)$ of q Steiner triple systems of order v, all on the same set V, is a *large set* of STS(v) if every 3-subset of V is contained in at least one STS of the family. Two STS(v), $(V, \mathcal{B}_1), (V, \mathcal{B}_2)$ are *disjoint* if $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ and *almost disjoint* if $|\mathcal{B}_1 \cap \mathcal{B}_2| = 1$. The classic papers of Lu [L1],[L2] and Teirlinck [T] established the existence of large sets of mutually disjoint (MD) STS(v) for all admissible $v \neq 7$. Such large sets necessarily contain $v - 2$ systems.

In [LR], Lindner and Rosa considered large sets of mutually almost disjoint (MAD) STS(v). They showed that for admissible $v \geq 15$ such large sets must contain $v - 1$ (later shown to be impossible, cf. [GR]), v, or $v + 1$ systems, and proved the existence for every $v \equiv 1$ or 3 (mod 6), of a large set of v MAD STS(v). For this end, they employed the existence of *Steiner* *quadruple systems.* The definition of a Steiner quadruple system of order v (SQS(v)) parallels that of an STS(v): an SQS(v) is a pair $(V, \mathcal{B})$ where V is a v-set and $\mathcal{B}$ is a collection of 4-subsets of V called *quadruples* such that each 3-subset of V is contained in exactly one quadruple. In a celebrated paper [H], Hanani established that an SQS(v) exists if and only if $v \equiv 2$ or 4 (mod 6).

Given an SQS(v), $(V, \mathcal{B})$, and an element $x \in V$, consider the pair $(V_x, \mathcal{B}_x)$ where $V_x = V \setminus \{x\}$, and $\mathcal{B}_x = \{\{a,b,c\} : \{a,b,c,x\} \in \mathcal{B}\}$. Clearly, $(V_x, \mathcal{B}_x)$ is an STS($v-1$) called the *derived* triple system through x of the SQS(v). It is now very easy to give a description of the construction used in [LR] to prove the existence of a large set of v MAD STS(v). Take an SQS($v+1$) with $V = \{0, 1, 2, ..., v\}$, and let $\mathcal{B}_{xy}$ be the set of all triples of the derived triple system through x of the SQS($v+1$) in which then y is replaced with x. It is then easily seen that $(V, \mathcal{B}_{10}), (V, \mathcal{B}_{20}), \ldots, (V, \mathcal{B}_{v0})$ is a large set of v MAD STS(v) [LR] (cf. also [GR]).

Let us call any large set of v MAD STS(v) obtained from an SQS($v+$ 1) via the construction described above an *SQS-delivered* large set. Two fundamental questions posed already over 20 years ago in [LR] asked (i) do there exist large sets of $v+1$ MAD STS(v), and (ii) do there exist large sets of v MAD STS(v) which are *not* SQS-delivered. In [GR] we answered the first of these in the affirmative; in this paper we answer the second question in the affirmative, too.

2. Another approach to construct large sets of v MAD STS(v)

In what follows we consider STS(v), $(V, \mathcal{B})$ where $V = Z_v = \{0, 1, \ldots, v-1\}$. As usual, the *distance* $d(x,y)$ between two elements $x, y \in Z_v$ is given by $d(x,y) = min(|x-y|, v-|x-y|)$. Then for every triple $B = \{x,y,z\} \in \mathcal{B}$, $x < y < z$, we can associate a *cyclically ordered difference triple* $D = D(\{x,y,z\}) = \langle d(x,y), d(y,z), d(z,x) \rangle$.

Let $O_1, O_2, ..., O_q$ be the orbits of triples under the action of Z_v. It is well known that then $q = \lceil \frac{1}{6}(v-1)(v-2) \rceil$, and all orbits are of length v except when $v \equiv 3$ (mod 6) in which case there is exactly one short orbit

of length $v/3$. The *rotation distance* $r(B_1, B_2)$ between two triples B_1, B_2 belonging to the same orbit (i.e. with the same difference triple D) is defined as $r(B_1, B_2) = \min\ \{i, j : B_1 + i = B_2, B_2 + j = B_1\}$.
Given a set of triples $S, S \subset O_i$, let $r(S)$ be the multiset
$r(S) = \{r(B_i, B_j) : B_i, B_j \in S, B_i \neq B_j\}$. Thus $r(S) = \binom{|S|}{2}$; in particular, $r(S) = \emptyset$ if $S = 1$.

For the set $\mathcal{B}$ of triples of our STS(v), write

$$\mathcal{B} = \bigcup_{i=1}^{q} \mathcal{B}_i$$

where $\mathcal{B}_i \subset O_i$, and let $p_i = |\mathcal{B}_i|$; p_i is simply the number of triples of $\mathcal{B}$ that belong to the orbit O_i. An STS(v), $(Z_v, \mathcal{B})$ is called Z_v*-extensive* (or simply *extensive*) if $p_i \geq 1$ for each $i = 1, \ldots, q$ (i.e $\mathcal{B}$ contains at least one representative of each orbit O_i).

Theorem 1. *Let $v \equiv 1 \pmod 6$, and suppose there exists an STS(v), $(Z_v, \mathcal{B})$ with the following properties:*

(i) *$(Z_v, \mathcal{B})$ is extensive;*
(ii) *if ${O_i}^*, i = 1, \ldots, l$, are the orbits of triples of $\binom{V}{3}$ under Z_v for which $p_i > 1$ then*

$$\bigcup_{j=1}^{l} r(\mathcal{B}_j^*) = \{1, 2, \ldots, \frac{1}{2}(v-1)\}.$$

Then $(Z_v, \mathcal{B} + k)$, $k \in Z_v$ is a large set of v MAD STS(v).

(Note that here $\{1, 2, \ldots, \frac{1}{2}(v-1)\}$ is a set, not a multiset.)

Proof. That the set $\{(Z_v, \mathcal{B}_0), (Z_v, \mathcal{B}_1), \ldots, (Z_v, \mathcal{B}_{v-1})\}$ is a *large set* of STS(v) follows from our assumption (i). Consider now two STSs from our set, say, $(Z_v, \mathcal{B}_x)$ and $(Z_v, \mathcal{B}_y)$. Let $d(x, y) = w$ where, say (w.l.o.g), $y = x + w$. Since $w \in \{1, 2, \ldots, \frac{1}{2}(v-1)\}$, there exists exactly one index j, say, $j = z$, such that $w \in r(\mathcal{B}_z^*)$. This means that if, say, B' and B'' are the two triples of $\mathcal{B}_z^*$ with distance $d(B', B'') = w$, and, say, $B'' = B' + w$, then $B', B'' \in \mathcal{B}_x$ and $B' + w, B'' + w \in \mathcal{B}_y$. But $B' + w = B''$, thus the triple B'' is common to both $\mathcal{B}_x$ and $\mathcal{B}_y$. At the same time, due to the uniqueness of z, $\mathcal{B}_x$ and $\mathcal{B}_y$ have no other triple in common. This completes the proof. □

The next theorem applies to the case $v \equiv 3 \pmod 6$ and is really only a slight modification of Theorem 1. Its proof is similar to the proof of Theorem 1 and hence is omitted.

Theorem 2. *Let $v \equiv 3 \pmod 6$, and suppose there exists an STS(v), $(Z_v, \mathcal{B})$ with the following properties:*

(i) *$(Z_v, \mathcal{B})$ is extensive;*
(ii) *if O_q is the short orbit of triples under Z_v then $p_q = 1$;*
(iii) *if $O_i^*, i = 1, \ldots, l$ are the orbits of triples of $\binom{V}{3}$ under Z_v for which $p_i > 1$ then*

$$\bigcup_{j=1}^{l} r(\mathcal{B}_j^*) = \{1, 2, \ldots, \frac{1}{2}(v-1)\} \setminus \{\frac{1}{3}v\}$$

.

Then $(Z_v, \mathcal{B} + k), k \in Z_v$ is a large set of v MAD STS(v).

Finally in this section we prove a result about the structure of an SQS-delivered large set of v MAD STS(v) which enables us to deduce that the example for $v = 15$ in the next section is not SQS-delivered. An easy counting argument (cf. [GR]) shows that if $(V, \mathcal{B}_0), (V, \mathcal{B}_1), \ldots, (V, \mathcal{B}_{v-1})$ is any large set of v MAD STS(v) then $\binom{V}{3} = \mathcal{V}^1 \cup \mathcal{V}^3$ where $\mathcal{V}^1$ [resp. $\mathcal{V}^3$] is the set of triples that belong to exactly one [resp. three] of the sets $\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_{v-1}$. Moreover, $(V, \mathcal{V}^3)$ is itself an STS(v) (cf. [LR]). Now if the large set is SQS-delivered from an SQS($v+1$),$(\{0, 1, 2, \ldots, v-1, \infty\}, \mathcal{B})$), where w.l.o.g. ∞ plays the rôle of the special element (cf. Introduction), then $\mathcal{V}^3$ is the derived STS(v) through ∞. Further, since $\mathcal{B}_x$ is obtained from $\mathcal{B}$ by deleting the element x from all quadruples which contain it and replacing ∞ with x, it follows that all triples $B \in \mathcal{V}^3 \cap \mathcal{B}_x$ must contain the element x, and no triple $B' \in \mathcal{V}^1 \cap \mathcal{B}_x$ can contain x. Thus we have proved the following theorem.

Theorem 3. *In any STS(v) belonging to an SQS-delivered large set of v MAD STS(v), $(V, \mathcal{B}_0), (V, \mathcal{B}_1), \ldots, (V, \mathcal{B}_{v-1})$, for each $x \in \{0, 1, \ldots, v-1\}$ those triples that belong to $\mathcal{V}^3 \cap \mathcal{B}_x$ contain a common element.*

We also have the following obvious corollary.

Corollary 4. *If $(Z_v, \mathcal{B} + k), k \in Z_v$ is a set of v MAD STS(v) obtained from Theorem 1 or Theorem 2, and the triples in $\mathcal{V}^3 \cap \mathcal{B}$ do not have a common element then the set is not SQS-delivered.*

3. Large sets of v MAD STS(v) from Theorems 1 and 2 for small orders v

v=7: It is easily checked that $(Z_7, \mathcal{B})$ where $\mathcal{B} = \{\{0,1,2\}, \{0,3,5\}, \{0,4,6\}, \{1,3,4\}, \{1,5,6\}, \{2,3,6\}, \{2,4,5\}\}$ is an STS(7) that satisfies the conditions of Theorem 1, i.e. is extensive, and $r(\{\{0,4,6\},\{1,3,4\},\{2,4,5\}\}) = \{1,2,3\}$; the orbit with difference triple $\langle 2,1,3\rangle$ is the only one having more than one representative in the STS(7). It is also immediately apparent (the element 4 occurs in all three triples) that this STS does not satisfy the conditions of Corollary 4. In fact the resulting large set of 7 MAD STS(7) is SQS-delivered. Other examples for v=7 are essentially equivalent to this one.

v = 9: An exhaustive check reveals that, somewhat disappointingly, there exists no Z_9-extensive STS(9)! This exhaustive examination can be actually carried out by hand, for example, by inspecting the 120 distinct STS(9) that contain a fixed triple, such as $\{0,1,2\}$.

v=13: Here checking by hand is no longer feasible. We carried out an exhaustive computer backtrack search for an STS(13) satisfying the conditions of Theorem 1. And while there is a huge number of Z_{13}-extensive STS(13), there is no STS(13) satisfying the conditions of Theorem 1.

v=15: Here at last our computer backtrack search (by far not exhaustive) was successful. Below is one of several hundred STS(15) found which satisfy the conditions of Theorem 2. It is easily verified that it satisfies Corollary 4 as well, and thus the resulting large set of 15 MAD STS(15) is not SQS-delivered. For the sake of brevity, all set-notation brackets and commas are omitted.

Triples:
0 1 2 (difference triple 1 1 2) 2 3 5 (difference triple 1 2 3)
1 3 4 (difference triple 2 1 3) 4 5 8 (difference triple 1 3 4)
7 10 11, 0 11 14, 10 13 14 (difference triple 3 1 4)
7 8 12 (difference triple 1 4 5) 2 6 7 (difference triple 4 1 5)
3 12 13 (difference triple 1 5 6) 3 8 9 (difference triple 5 1 6)
1 9 10 (difference triple 1 6 7) 5 6 14 (difference triple 6 1 7)
4 11 12 (difference triple 1 7 7) 5 7 9 (difference triple 2 2 4)

2 12 14 (difference triple 2 3 5)   8 11 13 (difference triple 3 2 5)
4 6 10 (difference triple 2 4 6)   2 4 13 (difference triple 4 2 6)
0 5 13 (difference triple 2 5 7)   5 10 12 (difference triple 5 2 7)
2 9 11 (difference triple 2 6 7)
0 6 8, 1 8 14, 2 8 10 (difference triple 6 2 7)
0 9 12 (difference triple 3 3 6)   6 9 13 (difference triple 3 4 7)
0 4 7 (difference triple 4 3 7)   3 6 11 (difference triple 3 5 7)
0 3 10 (difference triple 5 3 7)   1 7 13 (difference triple 3 6 6)
3 7 14 (difference triple 4 4 7)   1 6 12 (difference triple 4 5 6)
1 5 11 (difference triple 5 4 6)   4 9 14 (difference triple 5 5 5)

There are only two orbits, namely those with difference triples $\langle 3,1,4\rangle$ and $\langle 6,2,7\rangle$ which have more than one representative among the triples of our STS. The rotation distances between the 3 triples of the first of these are 1,3,4, and of the second are 2,6,7. The system itself is isomorphic to #33 of the standard listing [MPR].

4. Large sets of v MAD STS(v) for order 19.

For $v \geq 19$, the search space is too large for a backtrack to be feasible. Another method to construct an STS satisfying the conditions of Theorem 1 or Theorem 2 (according as $v \equiv 1$ or $v \equiv 3 \pmod 6$) is needed.

Let $v = 6s + 1$. Then the number of triples in an STS(v) is $s(6s+1)$ and the number of orbits of triples under the action of Z_v is $s(6s-1)$, all of length v. Using Theorem 1, in order to construct a large set of v MAD STS(v) we need to assemble an STS(v) containing precisely one representative of $6s^2 - 2s$ orbits and three representatives of s orbits. Our method is to introduce a multiplier m of order s and make the assumption that $B \in \mathcal{B}$ implies $mB \in \mathcal{B}$. This has the additional implication that the STS(v),(V,$\mathcal{V}^3$) will not only be cyclic but also stabilized by the multiplier m. The details of the method are best illustrated by the particular cases given in this and the following sections.

For $v = 19$, the 51 cyclic orbits are partitioned by the multiplier 7 of order 3 into 15 classes of three orbits and 6 stabilized orbits generated respectively by the triples $\{0,1,8\},\{0,2,16\},\{0,4,13\},\{0,1,12\},\{0,2,5\}$,

$\{0,4,10\}$. We choose as the set $\mathcal{V}^3$ of triples the orbits generated by $\{0,1,8\},\{0,4,13\},\{0,2,5\}$ and as the triples in $\mathcal{V}^3 \cap \mathcal{B}$ those containing the element 0. We then seek to complete the construction of an STS(19) by adjoining one representative of each the remaining 48 orbits with the additional condition that the inclusion of any triple implies that the triple obtained by multiplying by 7 is also included. Of course, any such system constructed will not satisfy the condition of Corollary 4 and may be SQS-delivered. In fact, an exhaustive computer search produced precisely three solutions of which one is SQS-delivered. The triples which complete each of these three solutions are listed below together with details of the determination of whether SQS-delivered and proof of nonisomorphism of the large sets constructed. Again all set-notation brackets and commas are omitted.

Solution 19/1.

3 4 5, 2 9 16, 14 6 17, 15 16 18, 10 17 12, 13 5 8, 5 6 9, 16 4 6,
17 9 4, 6 7 11, 4 11 1, 9 1 7, 8 9 14, 18 6 3, 12 4 2, 13 14 1,
15 3 7, 10 2 11, 9 10 18, 6 13 12, 4 15 8, 7 8 17, 11 18 5, 1 12 16,
16 17 11, 17 5 1, 5 16 7, 17 18 13, 5 12 15, 16 8 10, 14 15 11, 3 10 1,
2 13 7, 1 2 18, 7 14 12, 11 3 8, 9 11 13, 6 1 15, 4 7 10, 15 17 2,
10 5 14, 13 16 3, 6 8 2, 4 18 14, 9 12 3, 10 13 15, 2 3 14, 8 12 18.

The resulting large set is not SQS-delivered for the following reason. Suppose otherwise, i.e. that the large set is SQS-delivered from an SQS(v+1), $(\{0,1,\ldots,v-1,\infty\},\mathcal{B})$ where ∞ is the special element. Then $\{3,4,5\} \in \mathcal{B}_0$ implies $\{0,3,4,5\} \in \mathcal{B}$ implies $\{0,4,5\} \in \mathcal{B}_3$ implies $\{16,1,2\} \in \mathcal{B}_0$ by the cyclic nature of generating the large set. But $\{16,1,2\} \notin \mathcal{B}_0$.

Solution 19/2.

8 9 10, 18 6 13, 12 4 15, 5 6 8, 16 4 18, 17 9 12, 12 13 16, 8 15 17,
18 10 5, 3 4 8, 2 9 18, 14 6 12, 7 8 13, 11 18 15, 1 12 10, 10 11 17,
13 1 5, 15 7 16, 16 17 6, 17 5 4, 5 16 9, 14 15 5, 3 10 16, 2 13 17,
1 2 15, 7 14 10, 11 3 13, 13 14 9, 15 3 6, 10 2 4, 17 18 4, 5 12 3,
16 8 2, 6 7 4, 4 11 9, 9 1 6, 18 1 3, 12 7 2, 8 11 14, 5 7 11,
16 11 1, 17 1 7, 7 9 3, 11 6 2, 1 4 14, 10 13 15, 2 3 14, 8 12 18.

Again the resulting large set is not SQS-delivered by a similar argument to that given under Solution 1 using the triple $\{8, 9, 10\}$ or, indeed, many other triples.

For the sake of completeness, the third solution, which using the same argument can be shown to be SQS-delivered, is also given.

Solution 19/3.

4 5 6, 9 16 4, 6 17 9, 9 10 12, 6 13 8, 4 15 18, 12 13 16, 8 15 17,
18 10 5, 3 4 8, 2 9 18, 14 6 12, 8 9 14, 18 6 3, 12 4 2, 15 16 3,
10 17 2, 13 5 14, 6 7 15, 4 11 10, 9 1 13, 16 17 7, 17 5 11, 5 16 1,
1 2 15, 7 14 10, 11 3 13, 17 18 13, 5 12 15, 16 8 10, 14 15 11, 3 10 1,
2 13 7, 7 8 5, 11 18 16, 1 12 17, 7 9 11, 11 6 1, 1 4 7, 3 5 9,
2 16 6, 14 17 4, 18 1 14, 12 7 3, 8 11 2, 10 13 15, 2 3 14, 8 12 18.

The system formed by the set $\mathcal{V}^3$ of triples in all of the above three solutions is the well-known Netto system (A4 in the standard listing [MPR]). In Solutions 1 and 3 we discovered that the systems themselves which comprise the large set are also the Netto systems. Nevertheless, the large sets are nonisomorphic since one is SQS-delivered whereas the other is not. The systems which make up the large set of Solution 2 are not the Netto systems, so all three solutions are pairwise nonisomorphic.

There are two further possibilities for the set $\mathcal{V}^3$ of triples. We may use either the orbits generated by $\{0, 1, 8\}$, $\{0, 2, 5\}$, $\{0, 4, 10\}$ which is system A3 in [MPR] or the orbits generated by $\{0, 1, 4\}$, $\{0, 7, 9\}$, $\{0, 6, 11\}$ which is one of the 15 classes of three orbits and is system A2 in [MPR]. We conducted exhaustive computer searches in both these cases, again using as the triples in $\mathcal{V}^3 \cap \mathcal{B}$ those containg the element 0. To our surprise, we found no solutions in the latter case but the former yielded three further solutions whose systems, and hence large sets, are pairwise nonisomorphic and which are not SQS-delivered. The triples which complete these systems are given below.

Solution 19/4.

8 9 10, 18 6 13, 12 4 15, 17 18 1, 5 12 7, 16 8 11, 13 14 17, 15 3 5,
10 2 16, 5 6 10, 16 4 13, 17 9 15, 7 8 13, 11 18 15, 1 12 10, 10 11 17,
13 1 5, 15 7 16, 12 13 2, 8 15 14, 18 10 3, 4 5 14, 9 16 3, 6 17 2,

1 2 15, 7 14 10, 11 3 13, 16 17 12, 17 5 8, 5 16 18, 6 7 3, 4 11 2,
9 1 14, 3 4 1, 2 9 7, 14 6 11, 4 6 8, 9 4 18, 6 9 12, 12 14 18,
8 3 12, 18 2 8, 9 11 5, 6 1 16, 4 7 17, 10 13 15, 2 3 14, 1 7 11.

Solution 19/5.

15 16 17, 10 17 5, 13 5 16, 3 4 6, 2 9 4, 14 6 9, 6 7 10, 4 11 13,
9 1 15, 12 13 17, 8 15 5, 18 10 16, 7 8 13, 11 18 15, 1 12 10, 13 14 1,
15 3 7, 10 2 11, 14 15 4, 3 10 9, 2 13 6, 8 9 18, 18 6 12, 12 4 8,
4 5 18, 9 16 12, 6 17 8, 5 6 1, 16 4 7, 17 9 11, 17 18 14, 5 12 3,
16 8 2, 1 2 18, 7 14 12, 11 3 8, 5 7 9, 16 11 6, 17 1 4, 8 10 14,
18 13 3, 12 15 2, 1 3 16, 7 2 17, 11 14 5, 10 13 15, 2 3 14, 1 7 11.

Solution 19/6.

16 17 18, 17 5 12, 5 16 8, 14 15 17, 3 10 5, 2 13 16, 3 4 7, 2 9 11,
14 6 1, 15 16 1, 10 17 7, 13 5 11, 10 11 16, 13 1 17, 15 7 5, 6 7 13,
4 11 15, 9 1 10, 8 9 17, 18 6 5, 12 4 16, 13 14 4, 15 3 9, 10 2 6,
7 8 2, 11 18 14, 1 12 3, 12 13 8, 8 15 18, 18 10 12, 4 5 1, 9 16 7,
6 17 11, 1 2 18, 7 14 12, 11 3 8, 4 6 8, 9 4 18, 6 9 12, 8 10 14,
18 13 3, 12 15 2, 2 4 17, 14 9 5, 3 6 16, 10 13 15, 2 3 14, 1 7 11.

5. Further large sets of v MAD STS(v).

For $v = 25$, the 92 cyclic orbits are partitioned by the multiplier 7 of order 4 into 20 classes of four orbits and 6 classes of two orbits. But neither any of the classes of four orbits nor any pair of classes of two orbits form an STS(25), and so the method is not applicable.

However, for $v = 31$, we are again in luck. The multiplier 2 of order 5 partitions the 145 cyclic orbits of triples into 29 classes of five orbits. At this value of v we re-encounter the combinatorial explosion and again are not able to make an exhaustive computer backtrack. But below are listed the triples of two nonisomorphic STS(31) satisfying Theorem 1, one of each of the two possibilities for the system $\mathcal{V}^3$ formed by the triples occuring three times in the large set. As with the examples given in the previous section it is easy to show that neither system is SQS-delivered.

Solution 31/1.

$\mathcal{V}^3$ is the set of triples generated $\{0,1,6\}$ under the mappings $f : i \to i{+}1$ $(mod\ 31)$ and $g : i \to 2i$ $(mod\ 31)$. $\mathcal{V}^3 \cap \mathcal{B}$ is the set of triples containing 0. The system is completed by the following triples and their images under repeated application of the mapping g.

1 2 3, 1 4 7, 3 8 13, 5 6 8, 6 7 10, 4 5 9, 12 13 19, 26 27 4,
18 19 28, 23 24 3, 10 11 22, 21 22 3, 8 9 22, 20 21 6, 14 15 1, 24 25 12,
28 29 17, 19 20 9, 7 8 29, 22 23 15, 13 14 7, 27 28 22, 20 23 29, 23 26 2,
24 27 5, 17 20 30, 2 5 22, 29 1 19.

Solution 31/2.

$\mathcal{V}^3$ is the set of triples generated by $\{0,1,12\}$ under the mappings f and g as in Solution 1. $\mathcal{V}^3 \cap \mathcal{B}$ is again the set of triples containing 0. The system is completed by the following triples and their images under repeated application of g.

1 2 3, 1 4 7, 1 6 11, 5 6 8, 6 7 10, 4 5 9, 27 28 2, 29 30 5,
10 11 19, 14 15 24, 18 19 29, 23 24 5, 15 16 29, 11 12 28, 21 22 8, 8 9 27,
20 21 9, 7 8 28, 12 13 3, 28 29 21, 25 26 19, 26 27 21, 2 5 11, 9 12 19,
3 6 15, 12 15 25, 21 24 10, 8 11 29.

6. Order 15 revisited.

The method outlined in Section 4 can also be applied to the case where $v = 6s + 3$. Then the number of triples in an STS(v) is $(3s+1)(2s+1)$, and there are, under the action of Z_v, $3s(2s+1)$ orbits of triples of length v, plus the short orbit of length $v/3$. Using Theorem 2, in order to construct a large set of v MAD STS(v) we need to assemble an STS(v) containing one representative each of $2s(3s+1)$ orbits of length v and the short orbit, and three representatives of the remaining s orbits. Again we use a multiplier of order s. Of the three values of v (15,21,27) within range of this method only for $v = 15$ do we have a positive result. For $v = 27$ we need a multiplier of order 4, but $\phi(27) = 18$ so no such multiplier exists. For $v = 21$, in spite of an extensive though not exhaustive computer search no solution was found. We doubt if one exists by this method. On the other hand, the results for $v = 15$ are interesting.

Under the multiplier 4 of order 2 the 31 cyclic orbits are partitioned into 12 classes of two orbits and 7 stabilized orbits, including the short one, generated respectively by the triples $\{0,1,4\}$, $\{0,2,8\}$, $\{0,1,12\}$, $\{0,2,9\}$, $\{0,3,6\}$, $\{0,6,12\}$, $\{0,5,10\}$. There are two possibilities for the set $\mathcal{V}^3$ of triples; either the orbits generated by $\{0,1,4\}$, $\{0,2,8\}$, $\{0,5,10\}$ which is the projective STS(15) (#1 in [MPR]) or the orbits generated by $\{0,1,4\}$, $\{0,2,9\}$, $\{0,5,10\}$ which is the anti-Pasch STS(15) (also called the Netto system; #80 in [MPR]). As before, we will choose as the triples in $\mathcal{V}^3 \cap \mathcal{B}$ those containing the element 0. In the former case we obtained 9 solutions among which there are two pairs of isomorphic ones. Of the seven pairwise nonisomorphic solutions, five are not SQS-delivered, but two are. We list these below identifying which of the 80 STS(15) in the standard listing of [MPR] is obtained and whether the large set is SQS-delivered or not.

Solution 15/1. (System #1; *not* SQS-delivered)
5 6 7, 5 9 13, 3 4 6, 12 1 9, 2 3 7, 8 12 13, 13 14 4, 7 11 1,
1 2 10, 4 8 10, 8 9 3, 2 6 12, 10 12 14, 10 3 11, 11 13 2, 14 7 8,
9 11 4, 6 14 1, 1 3 13, 4 12 7, 2 4 5, 8 1 5, 6 8 11, 9 2 14,
9 10 6, 3 5 12, 7 10 13, 14 5 11.

Solution 15/2. (System #5; *not* SQS-delivered)
2 3 4, 8 12 1, 8 9 11, 2 6 14, 4 5 9, 1 5 6, 6 7 12, 9 13 3,
1 2 10, 4 8 10, 13 14 8, 7 11 2, 10 12 14, 10 3 11, 1 3 7, 4 12 13,
14 1 9, 11 4 6, 6 8 3, 9 2 12, 5 7 8, 5 13 2, 11 13 1, 14 7 4,
9 10 6, 3 5 12, 7 10 13, 14 5 11.

Solution 15/3. (System #31; *not* SQS-delivered)
2 3 4, 8 12 1, 8 9 11, 2 6 14, 4 5 9, 1 5 6, 6 7 12, 9 13 3,
10 11 4, 10 14 1, 13 14 8, 7 11 2, 4 6 8, 1 9 2, 1 3 7, 4 12 13,
8 10 3, 2 10 12, 12 14 9, 3 11 6, 5 7 8, 5 13 2, 11 13 1, 14 7 4,
9 10 6, 3 5 12, 7 10 13, 14 5 11.

Solution 15/4. (System #73; *not* SQS-delivered)
1 2 3, 4 8 12, 5 6 8, 5 9 2, 12 13 2, 3 7 8, 6 7 12, 9 13 3,
4 5 13, 1 5 7, 13 14 8, 7 11 2, 2 4 6, 8 1 9, 10 12 1, 10 3 4,
9 11 4, 6 14 1, 12 14 9, 3 11 6, 8 10 11, 2 10 14, 11 13 1, 14 7 4,
9 10 6, 3 5 12, 7 10 13, 14 5 11.

Solution 15/5. (System #78; *not* SQS-delivered)
4 5 6, 1 5 9, 8 9 11, 2 6 14, 3 4 8, 12 1 2, 6 7 12, 9 13 3,
10 11 4, 10 14 1, 13 14 8, 7 11 2, 8 10 12, 2 10 3, 1 3 7, 4 12 13,
6 8 1, 9 2 4, 12 14 9, 3 11 6, 5 7 8, 5 13 2, 11 13 1, 14 7 4,
9 10 6, 3 5 12, 7 10 13, 14 5 11.

Solution 15/6. (System #1; SQS-delivered)
5 6 7, 5 9 13, 8 9 11, 2 6 14, 7 8 12, 13 2 3, 3 4 9, 12 1 6,
1 2 10, 4 8 10, 13 14 8, 7 11 2, 10 12 14, 10 3 11, 1 3 7, 4 12 13,
14 1 9, 11 4 6, 6 8 3, 9 2 12, 2 4 5, 8 1 5, 11 13 1, 14 7 4,
9 10 6, 3 5 12, 7 10 13, 14 5 11.

Solution 15/7. (System #78; SQS-delivered)
4 5 6, 1 5 9, 8 9 11, 2 6 14, 6 7 11, 9 13 14, 3 4 9, 12 1 6,
10 11 4, 10 14 1, 1 2 11, 4 8 14, 8 10 12, 2 10 3, 1 3 7, 4 12 13,
12 14 7, 3 11 13, 6 8 3, 9 2 12, 5 7 8, 5 13 2, 2 4 7, 8 1 13,
9 10 6, 3 5 12, 7 10 13, 14 5 11.

We found it interesting, to say the least, that the two systems which occur in the large sets generated by this method which are SQS-delivered, i.e. #1 and #78 also occur in non SQS-delivered solutions. In the case where the set $\mathcal{V}^3$ is the anti-Pasch STS(15) there are no solutions. When we checked the several hundred STS(15) which were found by the computer backtrack search from Section 3 we discovered that in all of them the set $\mathcal{V}^3$ also is the projective STS(15). In other words, we have no solution where $\mathcal{V}^3$ is the anti-Pasch STS(15). Clearly, there is much still to be investigated in this area.

Acknowledgement

Research of the first and the third author was supported by NSERC of Canada Grants No. OGP0025112 (FF) and OGP007268 (AR). The second author wishes to thank McMaster University for hospitality whilst this paper was written.

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