ON CLIQUES IN SPANNING GRAPHS OF PROJECTIVE STEINER TRIPLE SYSTEMS

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Abstract. We are interested in what sizes of cliques are to be found in any arbitrary spanning graph of a Steiner triple system $\mathcal{S}$. In this paper we investigate spanning graphs of projective Steiner triple systems, proving, not surprisingly, that *for any positive integer k and any sufficiently large projective Steiner triple system $\mathcal{S}$, every spanning graph of $\mathcal{S}$ contains a clique of size k.*

1. Introduction

In this paper, we investigate cliques in spanning graphs of Steiner triple systems. This research was initially motivated by Rödl's observation (private communication) that using methods used to prove *Strong Ramsey Theorems for Steiner Systems* [NR] one can show that *for any positive integer k there is a positive integer l and a finite partial Steiner $(l, 2)$-system so that any of its spanning graphs contains a clique of size k.* We looked for a class of finite Steiner systems that would exhibit a similar property: i.e. the sizes of cliques in arbitrary spanning graphs of the members of the class asymptoticaly growing to infinity as the orders of the systems are growing to infinity. Not so surprisingly, the class of *projective Steiner triple systems* has this kind of property; i.e. *for any finite size k, any spanning graph $\mathcal{G}$ of any sufficiently large projective Steiner triple system $\mathcal{S}$ contains a clique of size k* (for precise formulation, see Theorem).

The result is not that obvious, for spanning graphs of Steiner triple systems have generally relatively few edges (one third of the number of blocks), and so Turán's or similar theorems - see e.g. [LW] - cannot be used; the fact that spanning graphs have any cliques at all comes from the distribution of edges as enforced by the underlying Steiner triple system, rather than by their density.

Though the result may be considered design-theoretical or graph-theoretical, the methods employed in its proof are rather combinatorial. It is not surprising that strong Ramsey-type results are necessary (Ramsey Theorem, Finite Sums Theorem), for the whole problem could be stated as a Ramsey-like one: *for any size k there is a sufficiently large Steiner triple system $(V, \mathcal{B})$ so that for any coloring* *of pairs of V by three colors so that no two pairs from the same block get the same color, there exists a monochromatic clique of any color of size k.* It also may not be surprising that the combinatorial principles needed are infinite - after all we are investigating an asymptotic behaviour of an infinite class of Steiner triple systems.

2. Notions, notation, definitions

The following basic definitions can be found in many texts, see e.g. [A]. A *Steiner triple system* (STS for short) is a Steiner $(3,2)$-system. A *Steiner* (k,l)-*system* $(V,\mathcal{B})$ is given by a set of elements V and a set $\mathcal{B}$ that is a set of subsets of V of size k, called blocks, with the property that any subset of V of size l is a subset fo a unique block (a *partial system* is such that each subset of V of size l either is a subset of a unique block or is not a subset of any block). A graph G with vertex set V is called a *spanning graph* of Steiner system $\mathcal{S}=(V,\mathcal{B})$ if it contains a single edge from each and every block of $\mathcal{S}$.

A *projective STS of order* $2^{n+1}-1$ is the one represented by points (the elements) and lines (the blocks) of a *finite projective space* $PG(n,2)$ (cf [A]). Such an STS is very often denoted as $PG(n,2)$ as well and we shall use that notation. The properties and uniqueness of projective STS's were studied e.g. in [H], [H1].

A commutative group $(\mathcal{A},\cdot)$ is a *Boolean group* if the operation $\cdot$ satisfies the following for any $a,b,c\in\mathcal{A}$: $(a{\cdot}b){\cdot}(a{\cdot}c)=b{\cdot}c$. Given a Boolean group $(\mathcal{A},\cdot)$ with the identity element $1_\mathcal{A}$, we can define a STS by defining its blocks by $\{a,b,a{\cdot}b\}$ for any $a,b\in\mathcal{A}-\{1_\mathcal{A}\}$. It is easy to see that it does, indeed, define a STS. We denote such a system by $\mathcal{S}(\mathcal{A})$. If the size of $\mathcal{A}$ is finite, then $|\mathcal{A}|=2^n$ for some integer $n\geq 2$, and it is well-known that $\mathcal{S}(\mathcal{A})$ is an STS $PG(n-1,2)$ (cf. [SS],[DP],[R]).

Let $\mathcal{A}$ be a *Boolean algebra* with the usual operations of $\vee$ (*joint*), $\wedge$ (*meet*), $-$ (*complement*) and constants $0_\mathcal{A}$ and $1_\mathcal{A}$ (see e.g. [J]). We can define two binary operations on $\mathcal{A}$, $\triangle$ (so-called *symmetric difference*) and ∇ (so-called *Boolean equality*) by: $a\triangle b=(a-b)\vee(b-a)=(a\vee b)-(a\wedge b)$, and $a\nabla b=-(a\triangle b)$. Since $a\triangle a=0_\mathcal{A}$ and $0_\mathcal{A}\triangle a=a$ for any $a\in\mathcal{A}$, $a\neq 0_\mathcal{A}$, it follows that $(\mathcal{A},\triangle)$ is a Boolean group with the identity $0_\mathcal{A}$ and as such it determines a (projective) STS. Similarly, $(\mathcal{A},\nabla)$ is a Boolean group with the identity $1_\mathcal{A}$ and as such it determines a (projective) STS that is isomorphic to the STS determined by $(\mathcal{A},\triangle)$ (the isomorphism just maps any $a\in\mathcal{A}-\{0_\mathcal{A}\}$ to its complement $-a$). For purely technical reasons we shall consider only the STS's determined by the symmetric difference. If $\mathcal{A}_1\subseteq\mathcal{A}$ is closed under $\triangle$ (and so it must contain $0_\mathcal{A}$), then $(\mathcal{A}_1,\triangle)$ is a Boolean group with the identity $0_\mathcal{A}$ and as such it determines a (projective) STS. We denote it as well as $\mathcal{S}(\mathcal{A}_1)$, if it causes no confusion.

If X is a set, $\mathcal{P}(X)$ denotes the *power set of* X, while $\mathcal{P}^+(X)$ denotes the set of all non-empty subsets of X. $\mathcal{P}_{fin}(X)$ denotes the set of all finite subsets of X and $\mathcal{P}^+_{fin}(X)$ denotes the set of all non-empty finite subsets of X. Since $(\mathcal{P}(X),\cup,\cap,-,\emptyset,X)$ is a Boolean algebra (see e.g. [J]), $\mathcal{P}(X)$ with the symmetric difference is a Boolean group and so it determines a STS of order $2^{|X|}-1$ which we denote as $\mathcal{S}(\mathcal{P}(X))$ (in case that X is finite, then it is a projective STS). Similarly for an infinite X, $\mathcal{P}_{fin}(X)$ with the symmetric difference is a Boolean group and so it determines a STS of order $|X|$ which we denote as $\mathcal{S}(\mathcal{P}_{fin}(X))$.

Following the standard notation in set theory, ω denotes the set of non-negative integers, the first infinite ordinal number, as well as the first infinite cardinal number. An integer n is viewed as a set of all smaller integers, and the canonical well-ordering of ordinals $\leq$ coincides with $\in$-relation, i.e. for two ordinals α and β, $\alpha < \beta$ iff $\alpha \in \beta$. If X is a set and $n \geq 1$ an integer, $[X]^n$ denotes the set of all subsets of X of size n, while $[X]^{\leq n}$ denotes the set of all subsets of X of size $\leq n$. $|X|$ denotes the size (cardinality) of the set X.

In the following we shall define several technical terms and notions that will be needed for the proof of Theorem. We refer the reader to [J] for all concepts and notations in set theory used in this paper.

Definition 1. Let $(X,\preccurlyeq)$ be a well-ordered set and let $a, b, c, d \in \mathcal{P}^+_{fin}(X)$. Moreover let a_{max} (b_{min}) be the maximum (minimum) element of a (b) with respect to $\preccurlyeq$. Then $a \prec\prec b$ if $a_{max} \prec b_{min}$. Furthemore $a{:}b = c{:}d$ *with respect to* $\preccurlyeq$ if $a = \{a_0, ..., a_p\}$, $b = \{b_0, ..., b_l\}$, $c = \{c_0, ..., c_p\}$, $d = \{d_0, ..., d_l\}$, and the elements are listed in an ascending order according to $\preccurlyeq$, and for any $i \leq p$, and any $j \leq l$, $a_i \preccurlyeq b_j$ iff $c_i \preccurlyeq d_j$.

Note. In simple terms $a{:}b = c{:}d$ means that the mutual positions (with respect to $\preccurlyeq$) of elements of a and b is the same as that of elements of c and d. If no confusion arises, we may drop the reference to $\preccurlyeq$.

Definition 2. Let λ be a cardinal, m, n positive integers with $\lambda \geq n, m$. Then $R(\lambda, m, n)$ is defined to be the least cardinal κ satisfying $\kappa \to (\lambda)^n_m$, i.e. for any set X of size $\geq \kappa$ and any coloring of $[X]^n$ by m colors, there is a $Y \subseteq X$, $|Y| \geq \lambda$, so that Y is homogeneous for the coloring (which means that $[Y]^n$ is monochromatic).

Note. It follows from the finite Ramsey theorem that $R(k, m, n) \in \omega$ exists for any positive integers m, n, k so that $k \geq m, n$. Moreover, from the infinite Ramsey theorem it follows that $R(\omega, m, n) = \omega$ for any positive integers m, n.

Definition 3. Let $(X,\preccurlyeq)$ be a well-ordered set, $\mathcal{G}$ be a spanning graph of $\mathcal{S}(\mathcal{P}_{fin}(X))$ and $n \geq 2$ be an integer. We say that $\mathcal{P}_{fin}(X)$ is *n-homogenized for $\preccurlyeq$ and $\mathcal{G}$* if for any non-empty $a, b, c, d \in [X]^{\leq n}$ so that $a{:}b = c{:}d$, $\{a, b\}$ is an edge of $\mathcal{G}$ iff $\{c, d\}$ is an edge of $\mathcal{G}$.

Note. If no confusion arises, we may drop the reference to $\preccurlyeq$ and $\mathcal{G}$.

Definition 4. Let α be a cardinal and n a positive integer with $\alpha \geq n \geq 2$. Then $\beta(\alpha, n)$ is defined to be the least cardinal κ with the following property: for any set X of size $\geq \kappa$, and any well-ordering $\preccurlyeq$ of X, and any spanning graph $\mathcal{G}$ of X, there exists a $Y \subseteq X$, $|Y| \geq \alpha$, so that $\mathcal{P}_{fin}(Y)$ is n-homogenized for $\preccurlyeq$ and $\mathcal{G}$.

Note. Lemma 1 below asserts the existence of $\beta(\alpha, n)$ for any $2 \leq n \leq \alpha \leq \omega$.

Definition 5. Let $(X,\preccurlyeq)$ be a well-ordered set and let $Y \subseteq \mathcal{P}_{fin}(X)$. $y \in Y$ is a $\preccurlyeq$*-left-guard* ($\preccurlyeq$*-right-guard*) of Y if for any $z \in Y$, $z \neq y$, $y_{min} \prec z_{min}$ ($z_{max} \prec y_{max}$).

Let $\mathcal{G}$ be a spanning graph of $\mathcal{S}(\mathcal{P}_{fin}(X))$. A clique Y of $\mathcal{G}$ is called a $\preccurlyeq$*-guarded* clique if (i) every set of the clique Y has a size that is divisible by 4, and (ii) Y has a $\preccurlyeq$-right-guard, and (iii) Y has a $\preccurlyeq$-left-guard.

Note. In simple terms, the left-guard of Y (if it exists) is the unique set whose minimum element is the left-most one of all, and similarly the right guard (if it exists) is the unique set whose maximum element is the right-most one. As usual, if no confusion arises, we may drop the reference to $\preccurlyeq$.

Definition 6. Let k and r be two positive integers with $2 \leq k \leq r < \omega$. The $g(k,r)$ denotes the least integer t such that for every finite well-ordered set$(X,\preccurlyeq)$ of size $\geq t$ and every spanning graph $\mathcal{G}$ of $\mathcal{S}(\mathcal{P}_{fin}(X))$, $\mathcal{G}$ contains either a clique of size r or a $\preccurlyeq$-guarded clique of size k.

Note. Lemma 6 below asserts that $g(k,r)$ exist for all possible k's and r's.

For a set X of size n, n a positive integer, $\mathcal{S}(\mathcal{P}(X))$ has 2^n-1 elements and $\binom{2^n-1}{2}\frac{1}{3} = \frac{1}{6}(2^n-1)(2^n-2)$ blocks. Since for each block a spanning graph of $\mathcal{S}(\mathcal{P}(X))$ selects exactly one edge, there are $3^{\frac{1}{6}(2^n-1)(2^n-2)}$ distinct spanning graphs of $\mathcal{S}(\mathcal{P}(X))$. Since we will need to refer to this number (in proof of Lemma 1 below), we define the following notation $s(n)$:

Definition 7. For any positive integer $n \geq 2$, let $s(n) = 3^{\frac{1}{6}(2^n-1)(2^n-2)}$.

Definition 8. Let X be a set, $\mathcal{G}$ be a spanning graph of $\mathcal{S}(\mathcal{P}(X))$ and $Y \subseteq X$. Then $\mathcal{G}{\restriction}\mathcal{S}(\mathcal{P}(Y))$ is a graph defined on elements of $\mathcal{P}^+(Y)$ by $\{x,y\}$ is an edge of $\mathcal{G}{\restriction}\mathcal{S}(\mathcal{P}(Y))$ iff $\{x,y\}$ is an edge of $\mathcal{G}$, for any $x,y \in \mathcal{P}^+(Y)$.

Note. Clearly, as $\mathcal{G}$ is a spanning graph of $\mathcal{S}(\mathcal{P}(X))$, $\mathcal{G}{\restriction}\mathcal{S}(\mathcal{P}(Y))$ is a spanning graph of $\mathcal{S}(\mathcal{P}(Y))$.

3. Results

Lemma 1. *$\beta(\alpha,n)$ exists for any $2 \leq n \leq \alpha \leq \omega$. If $\alpha < \omega$, then $\beta(\alpha,n)$ is an integer, otherwise $\beta(\omega,n) = \omega$.*

Proof. Given n and α. Define $\{\gamma_i : 2 \leq i \leq 2n{+}1\}$ by setting $\gamma_{2n+1} = \alpha$ and $\gamma_i = R(\gamma_{i+1}, i, s(i))$ for any $2 \leq i \leq 2n$. Set $\beta(\alpha,n) = \gamma_2$. It is clear that if $\alpha < \omega$, the whole sequence $\{\gamma_i : 2 \leq i \leq 2n{+}1\}$ consists of integers, while if $\alpha = \omega$, each $\gamma_i = \omega$.
We have to verify that $\beta(\alpha,n)$ has the required property. Let $(X,\preccurlyeq)$ be a well-ordered set so that $|X| \geq \beta(\alpha,n)$. Let $\mathcal{G}$ be a spanning graph of $\mathcal{S}(\mathcal{P}_{fin}(X))$.
Set Y_1 to a subset of X of size γ_2. By induction define $Y_1, ..., Y_{2n}$ and $C_2, ..., C_{2n}$ so that

(i) for any $1 \leq i \leq 2n$, $|Y_i| = \gamma_{i+1}$.

(ii) for any $1 \le i < j \le 2n$, $Y_j \subseteq Y_i$.
(iii) for any $2 \le i \le 2n$, C_i is a coloring of $[Y_{i-1}]^i$ by $s(i)$ colors defined so that each $x \in [Y_{i-1}]^i$ is assigned as its color the graph $\mathcal{G}\restriction\mathcal{S}(\mathcal{P}_{fin}(x))$.
(iv) for any $2 \le i \le 2n$, Y_i is homogeneous for the coloring C_i.

Set $Y = Y_{2n}$. Then $|Y| = \gamma_{2n+1} = \alpha$ and Y is homogeneous for any coloring $C_2, ..., C_{2n}$. Let $a, b, c, d \in [Y]^{\le n}$ so that $a{:}b = c{:}d$. Then $|a \cup b| = |c \cup d| = i \le 2n$. Since Y is homogeneous for C_i, $\mathcal{G}\restriction\mathcal{S}(\mathcal{P}(a \cup b))$ is the same "color" as $\mathcal{G}\restriction\mathcal{S}(\mathcal{P}(c \cup d))$, and so $\{a, b\}$ is an edge of $\mathcal{G}$ iff $\{c, d\}$ is an edge of $\mathcal{G}$. □

Lemma 2. *$g(2, r)$ exists for any $2 \le r < \omega$.*

Proof. Fix $r \ge 2$. Set $g(2, r) = \beta(2r{+}2, 4)$. We verify that $g(2, r)$ satisfies the requirements.

Let X be a set of size $g(2, r)$. Let $\preccurlyeq$ be a well-ordering of X. Let $\mathcal{G}$ be a spanning graph of $\mathcal{S}(\mathcal{P}(X))$.

Assume that $\mathcal{G}$ does not contain a clique of size r. Without loss of generality we may assume that $X = g(2, r)$ and that $\preccurlyeq$ is $\le$. Since $|X| = \beta(2r{+}2, 4)$, there is a $Y \subseteq X$, $|Y| \ge 2r{+}2$, so that $\mathcal{P}(Y)$ is 4-homogenized for $\le$ and $\mathcal{G}$. Without loss of generality we may assume that $Y = \{0, ..., 2r{+}1\}$.

Consider a triple $\{\{0,1,2,3\}, \{2,3,4,5\}, \{0,1,4,5\}\}$, that is a block of $\mathcal{S}(\mathcal{P}(Y))$.

If $\{\{0,1,2,3\}, \{0,1,4,5\}\}$ is an edge of $\mathcal{G}\restriction\mathcal{S}(\mathcal{P}(Y))$, then so is $\{\{0,1,2l,2l{+}1\}, \{0,1,2p,2p{+}1\}\}$ for any $1 \le l < p \le r$, as $\mathcal{P}(Y)$ is 4-homogenized for $\le$ and $\mathcal{G}$ and $\{0,1,2,3\}{:}\{0,1,4,5\} = \{0,1,2l,2l{+}1\}{:}\{0,1,2p,2p{+}1\}$. Therefore $\{\{0,1,2l,2l{+}1\} : 1 \le l \le r\}$ is a clique of size r, a contradiction.

If $\{\{0,1,4,5\}, \{2,3,4,5\}\}$ is an edge of $\mathcal{G}\restriction\mathcal{S}(\mathcal{P}(Y))$, then so is $\{\{2l,2l{+}1,2r,2r{+}1\}, \{2p,2p{+}1,2r,2r{+}1\}\}$ for any $0 \le l < p < r$, as $\mathcal{P}(Y)$ is 4-homogenized for $\le$ and $\mathcal{G}$ and
$\{0,1,4,5\}{:}\{2,3,4,5\} = \{2l,2l{+}1,2r,2r{+}1\}{:}\{2p,2p{+}1,2r,2r{+}1\}$.
Therefore $\{\{2l,2l{+}1,2r,2r{+}1\} : 0 \le l < r\}$ is a clique of size r, a contradiction.

Therefore $\{\{0,1,2,3\}, \{2,3,4,5\}\}$ is an edge of $\mathcal{G}\restriction\mathcal{S}(\mathcal{P}(Y))$, i.e. a clique of size 2. Every set in the clique has a size that is divisible by 4, $\{0,1,2,3\}$ is its left-guard and $\{2,3,4,5\}$ is its right-guard. Thus it is a guarded clique of size 2. □

Note. In fact, in a similar manner, the existence of $g(3, r)$ for any $3 \le r$ can be proven directly. The proof is more complicated, though. Since all we need is to have a starting value for the induction carried in the proof of Lemma 6, we presented here only the simpler proposition.

For the following theorem conjectured in 1970 by Graham and Rothschild, and proven by Hindman in 1972, see e.g. [B], [Hi].

Finite Sums Theorem. *If ω is partitioned into finitely many sets $A_0, ..., A_{k-1}$, then for some $i < k$ there exists an infinite $B \subseteq A_i$ so that $\sum F \in A_i$ for any finite $F \subset B$.*

Lemma 3. *Let $\omega = A_0 \cup A_1$, $B_0 = \{4n : n \in A_0\}$ and $B_1 = \{4n : n \in A_1\}$. Then for some $i < 2$, there exists $\{x_n : n \in \omega\} \subseteq B_i$ so that $\sum_{n\in F} x_n \in B_i$ for any finite $F \subset \omega$ and $x_n \geq 4\sum_{j=0}^{n-1} x_j$ for any $n \geq 1$.*

Proof. From the Finite Sums Theorem it follows that for some $i < 2$ there exists $\{y_n : n \in \omega\} \subseteq A_i$ so that $\sum_{n\in F} y_n \in A_i$ for any finite $F \subset \omega$. We can select a subsequence $\{z_n : n \in \omega\} \subseteq \{y_n : n \in \omega\}$ so that $z_n \geq 4\sum_{j=0}^{n-1} z_j$ for any $n \geq 1$.

Define $x_n = 4z_n$, for any $n \in \omega$. Then $\{x_n : n \in \omega\} \subseteq B_i$ as $\{z_n : n \in \omega\} \subseteq A_i$. Since for any finite $F \subset \omega$, $\sum_{n\in F} x_n = \sum_{n\in F} 4z_n = 4\sum_{n\in F} z_n$, and since $\sum_{n\in F} z_n \in A_i$, $\sum_{n\in F} x_n \in B_i$. Also, for any $n \geq 1$, $x_n = 4z_n \geq 16\sum_{j=0}^{n-1} z_j = 4\sum_{j=0}^{n-1} 4z_j = 4\sum_{j=0}^{n-1} x_j$. $\square$

Lemma 4. *Given k so that $2 \leq k < \omega$. Assume that $g(k, r)$ exists for any $k \leq r < \omega$. Then for any r, $k \leq r < \omega$, and any infinite well-ordered set $(X,\preccurlyeq)$, and any $\mathcal{G}$, a spanning graph of $\mathcal{S}(\mathcal{P}_{fin}(X))$, either $\mathcal{G}$ contains a clique of size r or a $\preccurlyeq$-guarded clique of size $k{+}1$.*

Proof. Fix r. Fix $(X,\preccurlyeq)$. Without loss of generality we may assume that $X = \omega$ and that $\preccurlyeq$ is $\leq$. Fix $\mathcal{G}$. Assume that $\mathcal{G}$ does not contain a clique of size r. Our goal is to show that it must contain a guared clique of size $k{+}1$.

By induction construct $X_0 \supseteq X_1 \supseteq ...$ so that

(i) $X_0 = X = \omega$.
(ii) for any $n \in \omega$, $|X_n| = \omega$.
(iii) for any $n \in \omega$, $\mathcal{P}_{fin}(X_n)$ is $4n$-homogenized.

For an $n \in \omega$, consider $a_0 \prec\!\!\prec a_1 \prec\!\!\prec ... \prec\!\!\prec a_{2r+1}$, where each $a_i \in [X_n]^n$. Consider a triple $\{a_0 \cup a_2 \cup a_3 \cup a_4, a_0 \cup a_1 \cup a_4 \cup a_5, a_1 \cup a_2 \cup a_3 \cup a_5\}$, that is a block $\mathcal{S}(\mathcal{P}_{fin}(X_n))$.

If $\{a_0 \cup a_2 \cup a_3 \cup a_4, a_1 \cup a_2 \cup a_3 \cup a_5\}$ is an edge of $\mathcal{G} \restriction \mathcal{S}(\mathcal{P}_{fin}(X_n))$, so is $\{a_l \cup a_r \cup a_{r+1} \cup a_{r+2+l}, a_p \cup a_r \cup a_{r+1} \cup a_{r+2+p}\}$ for any $0 \leq l < p < r$, as $(a_0 \cup a_2 \cup a_3 \cup a_4){:}(a_1 \cup a_2 \cup a_3 \cup a_5) = (a_l \cup a_r \cup a_{r+1} \cup a_{r+2+l}){:}(a_p \cup a_r \cup a_{r+1} \cup a_{r+2+p})$ and $\mathcal{P}_{fin}(X_n)$ is $4n$-homogenized. Thus $\{a_l \cup a_r \cup a_{r+1} \cup a_{r+2+l} : 0 \leq l < r\}$ is a clique of size r, a contradiction.

Thus for any $n \in \omega$ either

(I) for any $a_0, a_1, a_2, a_3, a_4, a_5 \in [X_n]^n$ so that $a_0 \prec\!\!\prec a_1 \prec\!\!\prec a_2 \prec\!\!\prec a_3 \prec\!\!\prec a_4 \prec\!\!\prec a_5$, $\{a_0 \cup a_2 \cup a_3 \cup a_4, a_0 \cup a_1 \cup a_4 \cup a_5\}$ is an edge,

or

(II) for any $a_0, a_1, a_2, a_3, a_4, a_5 \in [X_n]^n$ so that $a_0 \prec\prec a_1 \prec\prec a_2 \prec\prec a_3 \prec\prec a_4 \prec\prec a_5$, $\{a_0 \cup a_1 \cup a_4 \cup a_5, a_1 \cup a_2 \cup a_3 \cup a_5\}$ is an edge.

Define $B_I = \{4n : \text{(I) holds for } n\}$, and $B_{II} = \{4n : \text{(II) holds for } n\}$. By Lemma 3, either

(1) there exists $\{h_n : n \in \omega\} \subseteq B_I$ so that $\sum_{n \in F} h_n \in B_I$ for any finite $F \subset \omega$ and $h_n \geq 4\sum_{i=0}^{n-1} h_i$ for any $n \geq 1$,

or

(2) there exists $\{h_n : n \in \omega\} \subseteq B_{II}$ so that $\sum_{n \in F} h_n \in B_{II}$ for any finite $F \subset \omega$ and $h_n \geq 4\sum_{i=0}^{n-1} h_i$ for any $n \geq 1$.

We have to discuss both cases separately.

Case (1).

Set $t = g(k,r)$. Choose $l \geq g(k,r) \cdot h_{t-1}$. Let $X_l = \{x_n : n \in \omega\}$ be an enumeration of X_l in its natural order (i.e. $x_n \leq x_m$ iff $n \leq m$). Define $u_0 = \{x_n : n < h_0\}$ and $u_{i+1} = \{x_{h_0 + \cdots h_i + n} : n < h_{i+1}\}$ for $i < t-1$. Then $u_0 \prec\prec u_1 \prec\prec \ldots \prec\prec u_{t-1}$ and $|u_n| = h_n$ for any $n < t$. Let $\mathcal{U} = \{u_n : n < t\}$. There is a natural bijection $\phi : \mathcal{P}(\mathcal{U}) \to \{\bigcup F : F \subseteq \mathcal{U}\}$ defined by $\phi(u) = \bigcup u$ for any $u \subseteq \mathcal{U}$. Since $\phi(u \triangle v) = \bigcup(u \triangle v) = (\bigcup u) \triangle (\bigcup v) = \phi(u) \triangle \phi(v)$ for any $u, v \subseteq \mathcal{U}$, we can define $\tilde{\mathcal{G}}$, a spanning graph of $\mathcal{S}(\mathcal{P}(\mathcal{U}))$, by $\{u, v\}$ is an edge of $\tilde{\mathcal{G}}$ iff $\{\bigcup u, \bigcup v\}$ is an edge of $\mathcal{G} \restriction \mathcal{S}(\mathcal{P}_{fin}(X_l))$. Since $|\mathcal{U}| = t = g(k,r)$, either $\tilde{\mathcal{G}}$ contains a clique of size r or a guarded clique of size k. If it is the former and $\{d_0, \ldots, d_{r-1}\}$ is the clique, then $\{\bigcup d_0, \ldots, \bigcup d_{r-1}\}$ is a a clique of $\mathcal{G} \restriction \mathcal{S}(\mathcal{P}_{fin}(X_l))$, a contradiction.

Hence $\tilde{\mathcal{G}}$ contains a guarded clique $\{d_0, \ldots, d_{k-1}\}$. Without loss of generality we may assume that d_0 is its right-guard and d_1 its left-guard. Then $\{\bigcup d_0, \ldots, \bigcup d_{k-1}\}$ is a guarded clique of $\mathcal{G} \restriction \mathcal{S}(\mathcal{P}_{fin}(X_l))$, $\bigcup d_0$ its right-guard, and $\bigcup d_1$ its left-guard. Since $|d_0|$ is a multiple of 4, so is $|\bigcup d_0|$ and thus there are $y_0, y_1, y_2, y_3 \subset X_l$ so that $y_0 \prec\prec y_1 \prec\prec y_2 \prec\prec y_3$ and $|y_0| = |y_1| = |y_2| = |y_3|$ and $d_0 = y_0 \cup y_1 \cup y_2 \cup y_3$. Moreover, $|\bigcup d_0| \in B_I$. Since $h_n \geq 4\sum_{i=0}^{n-1} h_i$ for any $n \geq 1$, $\bigcup d_i \cap \bigcup d_0 \subseteq y_0$ for any $1 \leq i < k$.

Let $n = |y_0|$. Let $a_0, a_1, a_2, a_3, a_4, a_5 \in [X_l]^n$ so that $a_0 \prec\prec a_1 \prec\prec a_2 \prec\prec a_3 \prec\prec a_4 \prec\prec a_5$ and $a_0 = y_0$. Since $|\bigcup d_0| \in B_I$, (I) holds for n. Since $|d_0 \cup \ldots \cup d_{k-1}| \leq g(k,r)$, $|\bigcup d_0 \cup \ldots \cup \bigcup d_{k-1}| \leq g(k,r) \cdot h_{t-1} \leq l$. Thus $n \leq l$ and so $X_n \supseteq X_l$. It follows that $a_0, a_1, a_2, a_3, a_4, a_5 \in [X_n]^n$ and by (I) $\{a_0 \cup a_2 \cup a_3 \cup a_4, a_0 \cup a_1 \cup a_4 \cup a_5\}$ is an edge of $\mathcal{G}$.

For any $1 \leq i < k$, $\{\bigcup d_i, a_0 \cup a_2 \cup a_3 \cup a_4\}$ is an edge of $\mathcal{G}$, as $(\bigcup d_i) {:} (y_0 \cup y_1 \cup y_2 \cup y_3) = (\bigcup d_i) {:} (a_0 \cup a_2 \cup a_3 \cup a_4)$ and $\mathcal{P}_{fin}(X_l)$ is $4l$-homogenized.

For any $1 \leq i < k$, $\{\bigcup d_i, a_0 \cup a_1 \cup a_4 \cup a_5\}$ is an edge of $\mathcal{G}$, as $(\bigcup d_i) {:} (y_0 \cup y_1 \cup y_2 \cup y_3) = (\bigcup d_i) {:} (a_0 \cup a_1 \cup a_4 \cup a_5)$ and $\mathcal{P}_{fin}(X_l)$ is $4l$-homogenized.

Thus $\mathcal{G}$ contains a guarded clique $\{\bigcup d_1, \ldots, \bigcup d_{k-1}, (a_0 \cup a_2 \cup a_3 \cup a_4), (a_0 \cup a_1 \cup a_4 \cup a_5)\}$ of size $k+1$, where $\bigcup d_1$ is its left-guard and $(a_0 \cup a_1 \cup a_4 \cup a_5)$ its right-guard.

Case (2). This case is rather similar to Case (1), nevertheless with some small but necessary changes.

Set $t = g(k,r)$. Choose $l \geq g(k,r) \cdot h_{t-1}$. Let $X_l = \{x_n : n \in \omega\}$. Define $u_0 = \{x_{h_t+n} : n < h_{t-1}\}$ and $u_i = \{x_{h_t+h_{t-1}+\cdots h_{t-i+n}} : n < h_{t-1-i}\}$ for $1 \leq i < t$. Also define $\bar{u} = \{x_n : n < h_t\}$. Then $\bar{u} \prec\prec u_0 \prec\prec ... \prec\prec u_{t-1}$, $\bar{u}, u_0, ..., u_{t-1} \subset X_l$, $|\bar{u}| = h_t$ and $|u_n| = h_{t-1-n}$ for any $n < t$. Let $\mathcal{U} = \{u_n : n < t\}$. As in case (1) define $\tilde{\mathcal{G}}$, a spanning graph of $\mathcal{S}(\mathcal{P}(U))$, by $\{u,v\}$ is an edge of $\tilde{\mathcal{G}}$ iff $\{\bigcup u, \bigcup v\}$ is an edge in $\mathcal{G} \restriction \mathcal{S}(\mathcal{P}_{fin}(X_l))$. $\tilde{\mathcal{G}}$ either contains a clique of size r (which is a contradiction, for in that case $\mathcal{G}$ would contain a clique of size r), or a guarded clique $\{d_0, ..., d_{k-1}\}$. Without loss of generality we may assume that d_0 is its left-guard and d_1 its right-guard. It follows that $\{\bigcup d_0, ..., \bigcup d_{k-1}\}$ is a guarded clique of $\mathcal{G} \restriction \mathcal{S}(\mathcal{P}_{fin}(X_l))$, $\bigcup d_0$ its left-guard and $\bigcup d_1$ its right-guard. Since $|d_0|$ is a multiple of 4, so is $|\bigcup d_0|$. Hence there are $y_0, y_1, y_2, y_3 \subset X_l$, $|y_0| = |y_1| = |y_2| = |y_3|$, $y_0 \prec\prec y_1 \prec\prec y_2 \prec\prec y_3$, so that $\bigcup d_0 = y_0 \cup y_1 \cup y_2 \cup y_3$. Moreover $|\bigcup d_0| \in B_{II}$. Since $h_n \geq 4 \sum_{i=0}^{n-1} h_i$ for any $n \geq 1$, $(\bigcup d_i) \cap (\bigcup d_0) \subseteq y_3$, for any $1 \leq i < k$.
Let $a_0, a_1, a_2, a_3, a_4 \subset \bar{u}$ and $a_5 = y_3$ so that $a_0 \prec\prec a_1 \prec\prec a_2 \prec\prec a_3 \prec\prec a_4 \prec\prec a_5$ and $|a_0| = |a_1| = |a_2| = |a_3| = |a_4| = |a_5|$. Since $|\bigcup d_0| \in B_{II}$, (II) holds for $n = |y_3|$. Since $|d_0 \cup ... \cup d_{k-1}| \leq g(k,r)$, $|\bigcup d_0 \cup ... \cup \bigcup d_{k-1}| \leq g(k,r) \cdot h_{t-1} \leq l$.
Thus $n \leq l$ and so $X_l \subseteq X_n$, $a_0, a_1, a_2, a_3, a_4, a_5 \in [X_n]^n$, and from (II) it follows that $\{a_0 \cup a_1 \cup a_4 \cup a_5, a_1 \cup a_2 \cup a_3 \cup a_5\}$ is an edge of $\mathcal{G} \restriction \mathcal{S}(\mathcal{P}_{fin}(X_n))$, and so of $\mathcal{G} \restriction \mathcal{S}(\mathcal{P}_{fin}(X_l))$.
For $1 \leq i < k$, $\{\bigcup d_i, a_0 \cup a_1 \cup a_4 \cup a_5\}$ is an edge of $\mathcal{G} \restriction \mathcal{S}(\mathcal{P}_{fin}(X_l))$, for $(\bigcup d_i) : (y_0 \cup y_1 \cup y_2 \cup y_3) = (\bigcup d_i) : (a_0 \cup a_1 \cup a_4 \cup a_5)$ and $\mathcal{P}_{fin}(X_l)$ is $4l$-homogenized.
For $1 \leq i < k$, $\{\bigcup d_i, a_1 \cup a_2 \cup a_3 \cup a_5\}$ is an edge of $\mathcal{G} \restriction \mathcal{S}(\mathcal{P}_{fin}(X_l))$, for $(\bigcup d_i) : (y_0 \cup y_1 \cup y_2 \cup y_3) = (\bigcup d_i) : (a_1 \cup a_2 \cup a_3 \cup a_5)$ and $\mathcal{P}_{fin}(X_l)$ is $4l$-homogenized.
Thus, $\{\bigcup d_1, ..., \bigcup d_{k-1}, a_0 \cup a_1 \cup a_4 \cup a_5, a_1 \cup a_2 \cup a_3 \cup a_5\}$ is a guarded clique of $\mathcal{G} \restriction \mathcal{S}(\mathcal{P}_{fin}(X_l))$, where $\bigcup d_1$ is its right-guard and $a_0 \cup a_1 \cup a_4 \cup a_5$ its left-guard. $\square$

Lemma 5. $2 \leq k < \omega$, $k \leq r < \omega$. *Assume that for any infinite well-ordered* $(X, \preccurlyeq)$ *and any* $\mathcal{G}$*, a spanning graph of* $\mathcal{S}(\mathcal{P}_{fin}(X))$*,* $\mathcal{G}$ *contains either a clique of size* r *or a guarded clique of size* k*. Then* $g(k,r)$ *exists.*

Proof. Follows from the Compactness Theorem, see e.g. [J],[KC]. $\square$

Lemma 6. *For any* $2 \leq k < \omega$ *and any* $k \leq r < \omega$*,* $g(k,r)$ *exists.*

Proof. By Lemma 2, $g(2,r)$ exist for every $2 \leq r < \omega$. Assume, by induction on $k \geq 2$, that $g(k,r)$ exist for every $2 \leq r < \omega$. Then by Lemmas 4 and 5, $g(k+1,r)$ exists for every $2 \leq r < \omega$. $\square$

Lemma 7. *For any positive integer* k *there is a positive integer* n *so that for any finite set* X *of size* $\geq n$ *and any* $\mathcal{G}$*, a spanning graph of* $\mathcal{S}(\mathcal{P}(X))$*,* $\mathcal{G}$ *contains a clique of size* k*.*

Proof. Set $n = g(k,k)$. $\Box$

Corollary 8. *For any infinite Boolean algebra $\mathcal{A}$, any infinite $\mathcal{A}_1 \subseteq \mathcal{A}$ closed under $\triangle$, and any $\mathcal{G}$, a spanning graph of $\mathcal{S}(\mathcal{A}_1)$, $\mathcal{G}$ contains cliques of all finite sizes.*

Proof. Follows directly from Lemma 7 as $\mathcal{P}(X)$ for any finite X can be embedded into $\mathcal{A}_1$. $\Box$

Theorem. *For any positive integer k there exists a positive integer $n(k)$ so that for any $n \geq n(k)$, every spanning graph of the projective STS $PG(n,2)$ contains a clique of size k.*

Proof. Just a simple reformulation of Lemma 7. $\Box$

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