TRIANGLES IN 2-FACTORIZATIONS

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1. Introduction

A *2-factor* of a graph G is a factor (i.e. a subgraph containing all vertices of G) which is regular of degree 2. A *2-factorization* of G is a partition (i.e. an edge-disjoint decomposition) of the edge-set of G into 2-factors.

Let v be an odd integer, and let $\mathcal{F} = \{F_1, F_2, \ldots, F_{\frac{(v-1)}{2}}\}$ be a 2-factorization of the complete graph K_v. Two special well solved cases of 2-factorizations of K_v are decompositions of K_v into Hamiltonian cycles, and Kirkman triple systems of order v. While in the former case (when $v > 3$) none of the 2-factors contains a triangle, in the latter case each component in each 2-factor is a triangle. It is the purpose of this article to investigate the intermediate cases between these two extremes.

More precisely, we consider the following problem. Given an arbitrary 2-factorization $\mathcal{F} = \{F_1, \ldots, F_{\frac{(v-1)}{2}}\}$ of K_v, let δ_i be the number of triangles of F_i, and let $\delta = \delta(\mathcal{F}) = \Sigma\delta_i$. Then $\mathcal{F}$ is said to be a 2-factorization with (exactly) δ triangles.

The *triangle-spectrum* for 2-factorizations of K_v is the set $\Delta(v) = \{\delta$: there exists a 2-factorization $\mathcal{F}$ of K_v with $\delta(\mathcal{F}) = \delta\}$.

Since we have obviously $\Delta(3) = \{1\}, \Delta(5) = \{0\}$, we assume from now on $v \geq 7$.

The existence of a Hamiltonian decomposition of K_v shows $min\Delta(v) = 0$, and an easy calculation shows that $M_\Delta(v) = max\Delta(v) \leq M_v$ where $M_v =$

$\frac{(v-1)(v-4)}{6}$ if $v \equiv 1(mod\ 6), = \frac{v(v-1)}{6}$ if $v \equiv 3(mod\ 6)$, and $= \frac{(v-1)(v-5)}{6}$ if $v \equiv 5(mod\ 6)$.

It is an easy observation that $M_\Delta(v) - 1 \notin \Delta(v)$ if $v \equiv 3(mod\ 6)$. Let $P_\Delta(v) = \{0, 1, \ldots, M_\Delta(v)\}$. Then obviously $\Delta(v) \subset P_\Delta(v)$.

We prove in this article that when $v \equiv 1$ *or* $3(mod\ 6)$, apart from some small exceptions, and some additional 11 possible exceptions, actually an equality occurs above, i.e. $\Delta(v) = P_\Delta(v)$.

Even though some of our results pertain also to the case of $v \equiv 5(mod\, 6)$, the problem of determining the set $\Delta(v)$ when $v \equiv 5(mod\ 6)$ is left largely open.

2. Triangle spectra for small v and δ

We start with an easy result.

Lemma 2.1. . $\Delta(7) = \{0, 1, 3\}$.

Proof. The existence of a solution to the Oberwolfach problem OP(7;7) and OP(7;3,4) [A] shows $\{0, 3\} \subset \Delta(7)$. Assume now that in a 2-factorization of K_7, one 2-factor is the 7-cycle (1 2 3 4 5 6 7). Then there is, up to an isomorphism, only one way to choose in our K_7 a triangle edge-disjointly. Let this triangle be, w.l.o.g., (1 3 5); then there is a unique quadrangle (2 6 4 7) which is vertex-disjoint from (1 3 5) and edge-disjoint from (1 2 3 4 5 6 7). The complement of the union of the two 2-factors above is the 7-cycle (1 4 2 5 7 3 6). This proves both $1 \in \Delta(7)$ and $2 \notin \Delta(7)$. $\square$

Let us call a 2-factor whose each component is a triangle a *triangle-factor* or a Δ-*factor*. A 2-factor whose cycles have lengths $c_1, c_2, ..., c_t$ will be said to be of type $c_1 + c_2 + \cdots + c_t$.

Lemma 2.2. . $\Delta(9) = \{0, 1, 2, 3, 4, 5, 6, 8, 12\}$.

Proof. The existence of AG(2,3) (i.e., a Kirkman triple system of order 9) gives $12 \in \Delta(9)$. It is easily seen that the union of (any) two parallel classes of AG(2,3) can be decomposed into two Hamiltonian cycles, or into two 2-factors of type 3+6, respectively. This gives $\{0, 2, 4, 6, 8\} \subset \Delta(9)$. The well-known fact that $K_{3,3,3}$ can be decomposed into Hamiltonian cycles implies $3 \in \Delta(9)$. The two 2-factorizations $\mathcal{F}_1, \mathcal{F}_2$ given below show $1 \in \Delta(9)$, and $5 \in \Delta(9)$, respectively.

$\mathcal{F}_1 : (018)(273546); (021748563); (231405768); (342516078)$

$\mathcal{F}_2 : (012)(345678); (135)(246807); (162375048); (147)(258)(036)$.

On the other hand, there is, up to an isomorphism, a unique set of 3 disjoint Δ-factors; its complement is also a Δ-factor. This shows $i \notin \Delta(9)$ for $i \in \{9, 10, 11\}$. Finally, the complement of the union of two Δ-factors cannot be decomposed into two 2-factors, one of which is a Δ-factor and the other contains just one triangle. This shows $7 \notin \Delta(9)$, which completes the proof. $\square$

Lemma 2.3. $\Delta(11) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Proof. In the Hamiltonian cycle decomposition of K_11 given by $\mathcal{F} = \{F_i : i = 1, 2, 3, 4, 5\}, F_i = \{xy : \delta(xy) = i\}$, replace $F_1 \cup F_2$ with a decomposition into two 2-factors F_1' and F_2' where F_1' has either exactly one, or exactly two triangles, and F_2' is a Hamiltonian cycle (such a decomposition is easily seen to exist). Since $F_3 \cup F_4 \simeq F_1 \cup F_2$, this gives $\{0, 1, 2, 3, 4\} \subset \Delta(11)$. Furthermore (with E representing 11), $\mathcal{G} = \{G_1, G_2, G_3, G_4, G_5\}$ where
$G_1 = (12345)(678)(90E)$, $G_2 = (1697E)(248)(350)$,
$G_3 = (46089)(137)(25E)$, $G_4 = (1470263E859)$,
$G_5 = (18392756E40)$ implies $6 \in \Delta(11)$;
$\{G_1, G_2, G_3, G_4', G_5'\}$ where $G_4' = (140)(2758E639), G_5' = (183E4702659)$ implies $7 \in \Delta(11)$; $\{G_1, G_2, G_3, G_4", G_5"\}$ where $G_4" = (14E63958)(270)$, $G_5" = (19265740)(38E)$ implies $8 \in \Delta(11)$; $\{G_1, G_2^*, G_3^*, G_4^*, G_5^*\}$ where $G_2^* = (13964)(270)(58E)$, $G_3^* = (160)(24E)(38957)$,
$G_4^* = (350)(479)(1826E)$, $G_5^* = (17E36529)(480)$ implies $9 \in \Delta(11)$. The existence of OP(11;3,8) gives $5 \in \Delta(11)$, and, finally, the nonexistence of OP(11;3,3,5) [A] implies $10 \notin \Delta(11)$. $\square$

Lemma 2.4. $\Delta(13) = \{0, 1, \ldots, 18\}$.

Proof. We have $\{0, 1, \ldots, 6\} \subset \Delta(13)$ by Theorem 2.5 below. Any solution to OP(13;3,3,3,4) yields $18 \in \Delta(13)$, and a solution to OP(13;3,3,7) yields $12 \in \Delta(13)$ (cf. [A]). Observing that in this case also $G_{3,6} \simeq G_{1,2}$ (cf. proof of Theorem 2.5 below) gives $\{7, 8, 9\} \subset \Delta(13)$. The decompositions given below complete the proof (here T,E,D represent 10,11,12, respectively). $H_1 = (358)(02T79)(146DE)$, $H_2 = ((15T)(239)(0D7E486)$,
$H_3 = (169)(03E87)(254TD)$, $H_4 = (26E)(137)(0T8D594)$,
$H_5 = (567)(9TE)(01D3428), H_6 = (36T)(05E)(1274D98)$
implies $10 \in \Delta(13)$; $H_1, H_2, H_3' = (459)(8TD)(03E2617)$,
$H_4' = (25D)(04T)93196E87), H_5, H_6$
implies $11 \in \Delta(13)$; $H_1' = (15T)(468)(7ED)(0239)$,
$H_2' = (14E)(358)(06D)(297T), H_3, H_4, H_5, H_6$
implies $13 \in \Delta(13)$;
$H_1', H_2', H_3', H_4', H_5, H_6$ implies $14 \in \Delta(13)$;
$H_1, H_1', H_3" = (26E)(459)(8TD)(0317), H_4" = (25D)(169)(04T)(378E)$,
$H_5' = (34D)(567)(9TE)(0128), H_6' = (36T)(247)(05E)(189D)$
implies $15 \in \Delta(13)$;
$H_1" = (468)(7ED)(02T1539)$, $H_2" = (14E)(06D)(2385T79)$,
$H_3", H_4", H_5', H_6'$ implies $16 \in \Delta(13)$; and
$H_1^* = (17T)(26E)(459)(038D), H_2^* = (14E)(239)(58T)(06D7)$,
$H_3^* = (135)(468)(097EDT2)$, $H_4", H_5', H_6'$
implies $17 \in \Delta(13)$. $\square$

Theorem 2.5. *For each* $v \equiv 1(mod\ 2), v \geq 9, \{0, 1, \ldots, \frac{(v-3)}{2}\} \subset \Delta(v)$.

Proof. For $v = 9$ and 11, see Lemmas 2.2 and 2.3, so we may assume $v \geq 13$. Consider the particular 2-factorization $\mathcal{Q} = \{Q_1, \ldots, Q_{\frac{(v-1)}{2}}\}$ of K_v on $V = Z_v$ given by $Q_i = \{xy : d(xy) = i\}$ where $d(xy) = min(|x - y|), v - |x - y|)$. Let $G_{a,b}$ be the 4-regular subgraph of K_v with $V = Z_v$ and $E = \{xy : d(xy) = aorb\}$. If $v \equiv 1$ or 5 (mod 6), the 4-regular graph $G_{1,2}$ can be decomposed into two 2-factors F_1, F_2 where F_1 contains j triangles $(1, 2, 3), \ldots, (3j-2, 3j-1, 3j)$ and one cycle $(3j+1, 3j+2, 3j+4, 3j+6, \ldots, v-1, v, v-2, v-4, v-6, \ldots, 3j+1)$ of length $v - 3j$, for any

$j \in \{0, 1, \ldots, \frac{(v-\epsilon)}{3}\}$ (where $\epsilon = 4$ or 5 according to whether $v \equiv 1$ or 5 (mod 6)), and F_2 is a Hamiltonian cycle. The graph $G_{4,8}$ if $v \geq 17$, and $G_{4,5}$ if $v = 13$, is isomorphic to $G_{1,2}$ thus $j \in \Delta(v)$ for any $j \in \{0, 1, \ldots, \frac{2}{3}(v-\epsilon)\}$.

If $v \equiv 3 \pmod 6$, the 4-regular graph $G_{1,v/3}$ can be decomposed into two Hamiltonian cycles, one of which is $(1, v/3+1, v/3+2, 2, 3, v/3+3, v/3+4, 4, 5, v/3+5, v/3+6, \ldots, v/3-2, v/3-1, 2v/3, 2v/3+1, 2v/3+2, \ldots, v)$. The graph $G_{2,4}$ is isomorphic to $G_{1,2}$. Moreover, if $v \geq 33$, the graph $G_{8,16}$ ($G_{5,8}$ if $v = 21$, and $G_{8,11}$ if $v = 27$) is isomorphic to $G_{1,2}$. This shows that except when $v = 15$ and $\delta = 4$, $j \in \Delta(v)$ for any $j \in \{0, 1, \ldots, \frac{2}{3}(v-6)\}$. To handle this last remaining case, we observe that when $v = 15$, the graph $G_{1,5}$ can also be decomposed into two 2-factors F_1, F_2 where (say) $F_1 = (1\ 6\ 11)(2\ 7\ 8\ 3\ 4\ 9\ 10\ 5\ 15\ 14\ 13\ 12)$ has exactly one triangle, and F_2 is a Hamiltonian cycle. □

3. Main constructions

An embedding theorem of Rees and Stinson [RS] for Kirkman triple systems (KTS) turns out to be very useful for our purposes.

Theorem 3.1. *Let $v \equiv w \equiv 3 (mod\ 6), w > v$. A KTS(v) can be embedded in a KTS(w) if and only if $w \geq 3v$.*

Theorem 3.2. *Let $v \equiv w \equiv 3 (mod\ 6), w \geq 3v$. Then $\delta \in \Delta(v)$ implies $\delta + \frac{1}{6}[w(w-1) - v(v-1)] \in \Delta(w)$.*

Proof. Consider a KTS(w) with a sub-KTS(v); replace the sub-KTS(v) with a 2-factorization with exactly δ triangles. □

Corollary 3.3. *Let $v \equiv w \equiv 3 (mod\ 6), w \geq 3v$. If $\Delta(v) = P_\Delta(v)$ then $\{\frac{w(w-1)}{6} - \frac{v(v-1)}{6}, \frac{w(w-1)}{6} - \frac{v(v-1)}{6} + 1, \ldots, \frac{w(w-1)}{6} - 2, \frac{w(w-1)}{6}\} \subset \Delta(w)$.*

In other words, if both v and w are congruent to 3 (mod 6), $w \geq 3v$ and the triangle-spectrum $\Delta(v)$ is "complete", then this implies that the "largest" values of $P_\Delta(w)$ do indeed belong to the triangle-spectrum $\Delta(w)$.

Let $v \geq 7$. A 2-factorization $\mathcal{F}$ of K_v is said to be a *2^*-factorization* if there exists a vertex x which in every 2-factor of $\mathcal{F}$ is contained in a triangle.

Let $\Delta^*(v) = \{\delta$: there exists a 2^*-factorization of K_v with exactly δ triangles$\}$.

If $I(v) = \{\frac{(v-1)}{2}, \frac{(v+1)}{2}, \ldots, \frac{v(v-1)}{6}\}$ then clearly $\Delta^*(v) \subset I(v)$.

By comparison with Lemma 2.1 and Lemma 2.2, we see easily that $\Delta^*(v) = \{3\}, \Delta^*(9) = \{4, 6, 8, 12\}$.

Our main construction (the "*PBD-construction*") is a modification of Wilson's construction for resolvable designs [W].

Theorem 3.4. The PBD-construction. *Suppose* $(U,\mathcal{B})$ *is a* $(u,L,1)$*-PBD, and for each* $k\in L$*, there exists a* 2^**-factorization of* K_{2k+1}*. Then there exists a* 2^**-factorization of* K_{2u+1} *on* $U\times\{1,2\}\cup\{\infty\}$*. Furthermore, if* $\delta_B\in\Delta^*(2|B|+1)$ *for a block* $B\in\mathcal{B}$ *then* $\sum_{b\in\mathcal{B}}(\delta_B-|B|)+u\in\Delta^*(2u+1)$.

Proof. Let $V=U\times\{1,2\}\cup\{\infty\}$. We denote (x,i) for brevity by x_i. For a given block $B\in\mathcal{B}$, consider the set $B^*=B\times\{1,2\}\cup\{\infty\}$, and a 2^*-factorization $\mathcal{F}$ of $K_{|B^*|}$ with $\delta_{\mathcal{F}}=\delta_B$ such that ∞ is the element which occurs in every 2-factor of $\mathcal{F}$ in a triangle, and, moreover, that the two other elements of this triangle are x_1,x_2 for some $x\in U$. Let now ${B_1}^x,{B_2}^x,\ldots,B_q^x$ be all blocks of $\mathcal{B}$ that contain x, and let $\mathcal{F}_i^x$ be the corresponding 2^*-factorization on $B_i^{x*}=B_i^x\times\{1,2\}\cup\{\infty\}, i=1,\ldots,q$. Each of these 2^*-factorizations contains a 2-factor with the triangle $\{\infty,x_1,x_2\}$; let this 2-factor of $\mathcal{F}_i^x$ be, say, R_i^x. Then $R_x=\cup_i R_i^x$ is a 2-factor, and $\mathcal{R}=\{R_x : x\in U\}$ is a 2-factorization of K_{2u+1} on V. Clearly, $\delta(\mathcal{R})=\sum_{b\in\mathcal{B}}(\delta_B-|B|)+u$, and the proof is complete. □

4. The sets $\Delta^*(15)$ and $\Delta^*(27)$

In this section we determine (except for two cases that are not needed) the above two sets as these are crucial for the proof of our main result.

Theorem 4.1. $\Delta^*(15)=I(15)=\{7,8,\ldots,35\}$.

Proof. Consider the STS(15) No.61 (cf. [MPR]) which admits a unique Kirkman triple system. This KTS has a cyclic automorphism of order 7 acting on parallel classes. Clearly, any pair of unions of two "consecutive" parallel classes (i.e. δ-factors) is isomorphic, as is any pair of three consecutive parallel classes. The union of two consecutive Δ-factors can be decomposed into two 2-factors of type a) 3+3+3+6, 3+3+3+6, or b) 3+3+9, 3+3+9, or c) 3+12, 3+12, in such a way that one element, say 1, always remains in a triangle. A replacement with such a decomposition reduces the total number of triangles by 4, 6, and 8, respectively. Similarly, the union of 3 consecutive δ-factors can be decomposed into three 2-factors of type a) 3+3+3+6, 3+3+3+6, 3+3+9, or b)3+3+9, 3+3+9, 3+12, or c) 3+12, 3+5+7, 3+3+9, again in such a way that one element always remains in triangle. A replacement with such a decomposition reduces the number of triangles by 7, 10, and 11, respectively. Combining these replacements in all possible ways shows $\{7, 8, \ldots, 26, 27, 29, 31\} \subset \Delta^*(15)$.

Next, the 2-factorization whose 2-factors are
(0 1 2)(3 4 5)(6 7 8)(9 10 11)(12 13 14)
(0 3 6)(1 4 7)(2 9 12)(5 10 13)(8 11 14)
(0 4 9)(1 8 13)(2 3 11)(5 7 14)(6 10 12)
(0 3 6)(1 4 7)(2 9 12)(5 10 13)(8 11 14)
(0 4 9)(1 8 13)(2 3 11)(5 7 14)(6 10 12)
(0 5 12)(1 6 14)(2 7 10)(3 8 9)(4 11 13)
(0 7 11)(1 5 9)(2 6 13)(3 10 14)(4 8 12)
(0 8 10)(2 4 14)(1 11 5 6 9 13 3 7 12)
(2 5 8)(0 13 7 9 14)(1 3 12 11 6 4 10)
shows $28 \in \Delta^*(15)$. Replacing the last two 2-factors with
(1 11 12)(2 5 8)(3 7 13)(0 10 4 6 9 14)
(2 4 14)(5 6 11)(0 8 10 1 3 12 7 9 13)
shows $30 \in \Delta^*(15)$.

Finally, the maximal sets of six Δ-factors No.21 and 28, respectively, of [FMR] yield $32, 33 \in \Delta^*(15)$. The existence of a KTS(15) implies $35 \in \Delta^*(15)$. This completes the proof. □

Theorem 4.2. $I(27) \setminus \{14, 16\} \subset \Delta^*(27)$.

Proof. Let us start with applying Theorem 3.4 to S(2,4,13). Since the latter has 13 blocks, and $\Delta^*(9) = \{4, 6, 8, 12\}$ (cf. Section 3 above), this implies $\{13, 15, 17, \ldots, 111, 113, 117\} \subset \Delta^*(27)$. Next consider a resolvable transversal design TD(3,9) on the set $V \times \{1, 2, 3\}$, where V is any 9-set, with $V \times \{i\}, i = 1, 2, 3$ as groups. Construct a 2^*-factorization of K_{27} on $V \times \{1, 2, 3\}$ by taking the 9 parallel classes of our TD(3,9) as triangle-factors, together with three 2-factors $F_i \cup G_i \cup H_i, i = 1, 2, 3$ where $\mathcal{F} = \{F_1, F_2, F_3\}$ is a 2^*-factorization of K_9 on $V \times \{1\}$, and $\mathcal{G} = \{G_1, G_2, G_3\}, \mathcal{H} = \{H_1, H_2, H_3\}$ are 2-factorizations of K_9 on $V \times \{2\}$, and on $V \times \{3\}$, respectively. Since $\Delta(9) = \{0, 1, 2, 3, 4, 5, 6, 8, 12\}$ (cf. Lemma 2.2), this yields $\{85, 86, \ldots, 110, 11, 113, 117\} \subset \Delta^*(27)$.

Consider now the solution to the Oberwolfach problem OP(27;3,3,3,3,3,4,8) on the set $\{0,1,\ldots,26\}\cup\{\infty\}$ obtained by developing the base 2-factor $(\infty\ 2\ 24)(1\ 3\ 9)(6\ 7\ 10)4\ 14\ 16)(13\ 17\ 25)$ $(0\ 18\ 11\ 19)\ (5\ 20\ 23\ 12\ 15\ 21\ 8\ 22)$ modulo 13 under the automorphism $(\infty)(01\ldots12)(1314\ldots25)$; clearly, this is a 2^*-factorization. Union of this base 2-factor with a 2-factor obtained by adding to it 1 modulo 13 can be decomposed into two 2-factors F_1, F_2 as follows:

(i) $F_1 = (\infty\ 2\ 24)(1\ 3\ 9)(6\ 7\ 10)(0\ 16\ 4\ 14\ 13\ 25\ 17\ 15\ 21\ 8\ 22\ 5\ 20\ 23\ 12\ 19\ 11\ 13)$,
$F_2 = (\infty\ 3\ 25)(2\ 4\ 10)(7\ 8\ 11)\ (0\ 19\ 1\ 20\ 12\ 15\ 5\ 17\ 13\ 18\ 14\ 16\ 22\ 9\ 23\ 6\ 21\ 24)$;

(ii) $F_1 = (\infty\ 2\ 24)(1\ 3\ 9)(4\ 14\ 16)(0\ 18\ 13\ 25\ 17\ 15\ 12\ 23\ 20\ 5\ 22\ 8\ 21\ 6\ 10\ 7\ 11\ 19)$,
$F_2 = (\infty\ 3\ 25)(2\ 4\ 10)(1\ 19\ 12\ 20)\ (0\ 16\ 22\ 9\ 23\ 6\ 7\ 8\ 11\ 18\ 14\ 13\ 17\ 5\ 15\ 21\ 24)$;

(iii) $F_1 = (\infty\ 2\ 24)(1\ 3\ 9)(4\ 14\ 16\ 22\ 8\ 21\ 15\ 12\ 23\ 20\ 5\ 17\ 25\ 13\ 18\ 0\ 19\ 11\ 7\ 6\ 10)$,
$F_2 = (\infty\ 3\ 25)(1\ 19\ 12\ 20)\ (0\ 16\ 4\ 2\ 10\ 7\ 8\ 11\ 18\ 14\ 13\ 17\ 15\ 5\ 22\ 9\ 23\ 6\ 21\ 24)$;

(iv) $F_1 = (\infty\ 2\ 24)(0\ 18\ 11\ 7\ 6\ 10\ 4\ 16\ 14\ 13\ 25\ 17\ 5\ 20\ 23\ 12\ 15\ 21\ 8\ 22\ 9\ 3\ 1\ 19)$,
$F_2 = (\infty\ 3\ 25)(0\ 16\ 22\ 5\ 15\ 17\ 13\ 18\ 14\ 4\ 2\ 10\ 7\ 8\ 11\ 19\ 12\ 20\ 1\ 9\ 23\ 6\ 21\ 24)$.

Replacing two "consecutive" 2-factors in the way described above reduces the number of triangles by 4, 5, 7, and 8, respectively, while clearly preserving the property of being a 2^*-factorization. Combining these replacements gives $\{17,18,\ldots,58,60,61,65\}\subset\Delta^*(27)$.

Next consider the 2^*-factorization of K_{27} on the set $Z_{10}\times\{1,2\}\cup\{\infty_i : i=1,2,\ldots,7\}$ whose first ten 2-factors are obtained by applying repeatedly the mapping $x_j \to (x+1)_j = 1,2$ (each ∞_i is a fixed point) to the 2-factor $(\infty_1\ 2_1\ 6_2)(\infty_2\ 3_1\ 2_2)(\infty_3\ 5_1\ 8_2)(\infty_4\ 6_1\ 7_2)(\infty_5\ 7_1\ 9_2)(\infty_6\ 8_1\ 3_2)$ $(\infty_7\ 9_1\ 5_2)(0_1\ 1_1\ 4_1)(0_2\ 1_2\ 4_2)$,
and whose the remaining three 2-factors are obtained by taking a disjoint

union of the 2-factors of the (unique) 2^*-factorization of K_7 on the set $\{\infty_i : i = 1, 2, \dots, 7\}$, and of the following three disjoint 2-factors on the set $Z_{10} \times \{1, 2\}$:
$(0_1\ 2_1\ 0_2)((1_1\ 6_1\ 4_2\ 6_2\ 1_2)(3_1\ 5_1\ 7_1\ 5_2\ 8_1\ 8_2\ 3_2)(4_1\ 9_9\ 9_2\ 7_2\ 2_2)$;
$(0_1\ 5_1\ 5_2\ 3_2\ 6_1\ 8_1\ 0_2\ 3_1\ 1_1\ 9_1\ 7_1\ 7_2)(2_1\ 4_1\ 1_2\ 9_2 4_2\ 2_2)$;
$(0_1\ 8_1\ 3_1\ 1_2\ 3_2\ 5_1\ 0_2\ 5_2\ 7_2\ 9_1\ 6_2\ 6_1\ 4_1\ 4_2\ 7_1\ 2_1\ 9_2\ 1_1\ 8_2)$.

It is easily verified that this is indeed a 2^*-factorization, and that it has a total of 94 triangles.

The first two (or any two of the first ten) "consecutive" 2-factors can be decomposed as follows:
(i) $(0_1\ 1_1\ 4_1\ \infty_2\ 3_1\ 2_2\ 5_2\ 9_1\ \infty_7)(\infty_1\ 2_1\ 6_2)(\infty_3\ 5_1\ 8_2)(\infty_4\ 6_1\ 7_2)(\infty_5\ 7_1\ 9_2)$ $(\infty_6\ 8_1\ 3_2)(0_2\ 1_2\ 4_2)$,
$(0_1\ 4_1\ 3_2\ \infty_2\ 2_2\ 1_2\ 5_2\ \infty_7\ 6_2)(\infty_1\ 3_1\ 7_2)(\infty_3\ 6_1\ 9_2)(\infty_4\ 7_1\ 8_2)(\infty_5\ 8_1\ 0_2)$ $(\infty_6\ 9_1\ 4_2)(1_1\ 2_1\ 5_1)$;
(ii) $(0_1\ 1_1\ 5_1\ \infty_3\ 8_2\ 7_1\ 9_2\ \infty_5\ 8_1\ \infty_6\ 3_2\ 4_1)(\infty_1\ 2_1\ 6_2)(\infty_2\ 3_1\ 2_2)(\infty_4\ 6_1\ 7_2)$ $(\infty_7\ 9_1\ 5_2)(0_2\ 1_2\ 4_2)$,
$(1_1\ 2_1\ 5_1\ 8_2\ \infty_4\ 7_1\ \infty_5\ 0_2\ 8_1\ 3_2\ \infty_2\ 4_1)(\infty_1\ 3_1\ 7_2)(\infty_3\ 6_1\ 9_2)(\infty_6\ 9_1\ 4_2)$ $(\infty_7\ 0_1\ 6_2)(1_2\ 2_2\ 5_2)$;
(iii) $(0_1\ 4_1\ 1_1\ 5_1\ \infty_3\ 8_2\ 7_1\ 9_2\ \infty_5\ 8_1\ 3_2\ \infty_6\ 9_1\ 5_2\ \infty_7)(\infty_1\ 2_1\ 6_2)(\infty_2\ 3_1\ 2_2)$ $(\infty_4\ 6_1\ 7_2)(0_2\ 1_2\ 4_2)$,
$(0_1\ 1_1\ 2_1\ 5_1\ 8_2\ \infty_4\ 7_1\ \infty_5\ 0_2\ 8_1\ \infty_6\ 4_2\ 9_1\ \infty_7\ 6_2)(\infty_1\ 3_1\ 7_2)(\infty_2\ 4_1\ 3_2)$ $(\infty_3\ 6_1\ 9_2)(1_2\ 2_2\ 5_2)$.

Replacing two consecutive 2-factors by the decomposition (i), (ii), or (iii) reduces the number of triangles by 6, 8, or 10, respectively. Thus $\{44, 46, 48, \dots, 88\} \subset \Delta^*(27)$.

The following 2^*-factorization of K_{27} shows that $112 \in \Delta^*(27)$:
(1 2 3)(4 5 6)(7 8 9)(10 11 12)(13 14 15)(16 17 18)(19 20 21) (22 23 24)(25 26 27),
(1 4 7)(2 6 8)(3 5 9)(10 13 16)(11 15 17)(12 14 18)(19 22 25) (20 24 26)(21 23 27),
(1 5 8)(2 4 9)(3 6 7)(10 14 17)(11 13 18)(12 15 16)(19 23 26) (20 22 27)(21 24 25),
(1 6 9)(2 5 7)(3 4 8)(10 15 18)(11 14 16)(12 13 17)(19 24 27) 20 23 25)(21

22 26),
(1 15 24)(2 16 21)(3 17 22)(4 14 19)(5 12 25)(6 11 27)(7 18 23) 8 10 26)(9 13 20),
(1 16 25)(2 13 27)(3 14 24)(4 17 21)(5 10 23)(6 18 20)(7 12 26) (8 11 19)(9 15 22),
(1 17 26)(2 18 22)(3 15 23)(4 12 20)(5 13 19)(6 10 24)(7 11 21) (8 16 27)(9 14 25),
(1 18 27)(2 14 26)(3 10 21)(4 16 23)(5 11 22)(6 17 25)(7 13 24) (8 15 20)(9 12 19),
(1 10 19)(2 11 20)(3 12 27)(4 13 25)(5 15 21)(6 14 23)(7 16 22) (8 17 24)(9 18 26),
(1 12 23)(2 10 25)(3 18 19)(4 11 26)(5 17 20)(6 13 22)(7 15 27) (8 14 21)(9 16 24),
(1 13 21)(2 12 24)(3 16 20)(4 10 22)(5 14 27)(6 15 26)(7 17 19) (8 18 25)(9 11 23),
(1 14 22)(2 15 19)(3 11 25)(4 18 24)(5 16 26)(6 12 21)(7 10 20) (8 13 23)(9 17 27),
(1 11 24 5 18 21 9 10 27 4 15 25 7 14 20)(2 17 23)(3 13 16) (6 16 19)(8 12 22).

To show $114 \in \Delta^*(27)$, consider the following 2^*-factorization of K_{21}: take the first six 2-factors and the eighth 2-factor as above, and also the 2-factors (1 13 22)(2 17 19)(3 15 23)(4 12 20)(5 18 26)(6 10 24)(7 11 21) (8 16 27)(9 14 25),
(1 10 19)(2 18 22)(3 12 27)(4 13 25)(5 16 24)(6 15 26)(7 17 20) 8 14 21)(9 11 23),
(1 11 20)(2 12 23)(3 13 26)(4 10 22)(5 15 21)(6 16 19)(7 14 27) (8 18 25)(9 17 24),
(1 12 21)(2 10 20)(3 11 25)(4 18 24)(5 17 27)(6 14 22)(7 15 19) (8 13 23)(9 16 26),
(1 17 23)(2 15 25)(3 18 19)(4 11 26)(5 14 20)(6 13 21)(7 16 22) (8 12 24)(9 10 27),
(1 14 23 6 12 22 8 17 26)(2 11 24)(3 16 20)(4 15 27)(5 13 19) (7 10 25)(9 18 21).

Finally, to show $115 \in \Delta^*(27)$, consider the following 2^*-factorization of K_{21}: take the first five 2-factors and the eighth 2-factor as above, and also the 2-factors (1 14 23)(2 15 25)(3 18 19)(4 11 26)(5 16 24)(6 12 22)(7 17 20) (8 13 21)(9 10 27),
(1 16 25)(2 10 20)(3 14 24)(4 18 21)(5 17 27)(6 13 23)(7 12 26) (8 11 19)(9 15 22),
(1 17 26)(2 18 22)(3 16 20)(4 15 27)(5 13 19)(6 14 21)(7 10 25) (8 12 23)(9 11 24),
(1 10 19)(2 11 23)(3 12 27)(4 13 25)(5 14 20)(6 15 26)(7 16 22) (8 18 24)(9 17 21),
(1 12 21)(2 13 27)(3 11 25)(4 17 24)(5 10 23)(6 18 20)(7 15 19) (8 14 22)(9 16 26),
(1 13 22)(2 17 19)(3 15 23)(4 12 20)(5 18 26)(6 10 24)(7 11 21) (8 16 27)(9 14 25),
(1 11 20)(2 12 24)(3 13 26)(4 10 22)(5 15 21)(6 16 19)(7 14 27) (8 17 23 9 18 25).

This completes the proof. $\square$

Corollary 4.3. $\Delta(27) = P_\Delta(27)$.

Proof. Combine Theorem 4.2 with Theorem 2.5. $\square$

5. More triangle spectra for small v

Our first lemma in this section is auxiliary as it is needed in the proof of Lemma 5.3.

Lemma 5.1. $\{6, 11, 14, 15, 16, 17, 18\} \subset \Delta^*(13)$.

Proof. An inspection of the proof of Lemma 2.4 shows $\{11, 14, 15, 16, 17\} \subset \Delta^*(13)$. For $6 \in \Delta^*(13)$, consider the following solution to OP(13;3,10). The vertex-set of K_{13} is $Z_3 \times \{1, 2, 3, 4\} \cup \{\infty\}$, and the two base 2-factors are $F_1 = (\infty 0_3 2_1)(0_1 0_2 1_4 2_2 1_2 2_3 1_3 2_4 1_1 0_4) F_2 = (\infty 0_2 0_4)(0_1 2_1 1_2 0_3 2_4 1_4 1_3 1_1 2_2 2_3)$ (the remaining 2-factors are obtained by developing F_1, F_2 modulo 3). Finally, for $18 \in \Delta^*(13)$, consider the following solution to OP(13;3,3,3,4):
$(168)(45T)(7ED)(0239)$
$(14E)(358)(06D)(297T)$
$(159)(26E)(8TD)(0347)$
$(01T)(25D)(469)(378E)$
$(13D)(567)(9TE)(0428)$
$(127)(36T)(05E)(489D)$ $\square$

Lemma 5.2. $\Delta(21) = P_\Delta(21)$.

Proof. When $v = 21$, the graph $G_{1,2}$ (cf. Theorem 2.5) can be decomposed into a Hamiltonian cycle and a 2-factor F having i triangles where $i \in \{0, 1, 2, 3, 4, 5, 7\}$. Further, the graphs $G_{4,8}$ and $G_{5,10}$ are isomorphic to $G_{1,2}$. Moreover, the graph $G_{3,6,9}$ (defined in analogy with $G_{a,b}$ in an obvious way) can be decomposed into three 2-factors in such a way that the total number of triangles in the three 2-factors is j where $j \in \{0, 1, \ldots, 7, 9\}$. Clearly, the 2-factor consisting of edges of length 7 contains exactly 5 triangles. This shows $\{0, 1, \ldots, 33, 35\} \subset \Delta(21)$.

Consider now a resolvable transversal design equivalent to the pair of orthogonal cyclic latin squares of order 7. The union of two of its parallel classes is easily seen to be decomposable into two Hamiltonian cycles, or into two 2-factors of type 3+3+3+12, respectively. The corresponding replacement decreases the number of triples by 14, and by 8, respectively. We can complete to a 2-factorization by taking a 2-factorization of K_7 on each of the three groups, taking into account Lemma 2.1. This gives $\{33, 34, \ldots, 48, 50\} \in \Delta(21)$.

Next consider the set of 7 disjoint 2-factors obtained by developing modulo 20 the 2-factor
(0 5 11)(1 9 13)(2 6 15)(3 10 17)(4 14 19)(7 12 18)(8 16 20). The complement of this set in K_{21} on Z_{21} is the graph $G_{1,2,3}$. Since $G_{1,2}$ can be decomposed into a Hamiltonian cycle and a 2-factor containing i triangles, $i \in \{0, 1, 2, 3, 4, 5, 7\}$ (cf. Theorem 2.5), we get right away
$\{49, 50, 51, 52, 53, 54, 56\} \in \Delta(21)$. The graph $G_{1,2,3}$ can be decomposed into three 2-factors
(0 1 2)(3 4 5)(6 7 8)(9 10 11)(12 13 14)(15 16 18 20 19 17),
(0 18 19)(1 3 2 4 6 5 7 9 8 10 12 11 13 15 14 16 17 20),
(0 3 6 9 12 15 18 17 14 11 8 5 2 20)(1 4 7 10 13 16 19)
which shows $55 \in \Delta(21)$. Another decomposition of $G_{1,2,3}$ into three 2-factors
$F =$ (0 1 3)(2 4 5)(6 7 9)(8 10 11)(12 13 15)(14 16 17)(18 19 20), (0 20 2 1 19 17 18)(3 4 6)(5 7 10 12 9 8)(11 13 16 15 14),
(0 2 3 5 6 8 7 4 1 20 17 15 18 18 19)(9 10 13 14 12 11) shows $57 \in \Delta(21)$,

and the decomposition
(0 2 1 19 16 18 15 17 20)(3 4 7 8 6 5)(9 10 12)(11 13 14),
(0 18 17 19)(1 4 6 3 2 20)(5 7 10 13 16 15 14 12 11 9 8),
together with F as above, shows that $58 \in \Delta(21)$. Yet another decomposition of $G_{1,2,3}$ into three 2-factors (0 2 20)((3 5 6)(1 4 7 8 9 10 12 14 11 13 16 15 18 17 19),
(0 18 16 19)(1 2 3 4 6 8 5 7 10 13 14 15 17 20)(9 11 12),
and F as above, shows $59 \in \Delta(21)$. Similarly, the decomposition
(0 2 20)((1 4 6 3 5 7 8 9 10 13 16 19)(11 12 14)(15 17 18)
(0 18 16 15 14 13 11 9 12 10 7 4 3 2 1 20 17 19)(5 6 8),
together with F as above, shows $60 \in \Delta(21)$.

Consider now the set of 7 disjoint 2-factors obtained by developing modulo 21 the 2-factor
(0 2 10)(1 13 19)(3 14 16)(4 11 18)(5 8 20)(6 12 15)(7 9 17). The complement of this set in K_{21} on Z_{21} is the graph $G_{1,4,5}$ which can be decomposed into three 2-factors
(0 1 5)(2 3 7)(4 8 9)(6 10 11)(12 13 17)(14 18 19)(15 16 20)
(0 16 17)(1 2 18 13 8 12 7 11 15 19 20 3 4 5 6)(9 10 14),
(0 4 20)(1 17 18)(2 6 7 8 3 19)(5 9 13 14 15 10)(11 12 16); this shows $61 \in \Delta(21)$.

The existence of a Kirkman triple system of order 21 implies $70 \in \Delta(21)$. Consider now the KTS(21) on the set $Z_7 \times \{1, 2, 3\}$ with base parallel classes
$R = (0_1 1_2 2_3)(1_1 2_1 6_1)(3_1 0_2 4_3)(4_1 2_2 0_3)(5_1 4_2 3_3)(3_2 5_2 6_2)(1_3 5_3 6_3)$,
$S_1 = \{(i_1 (i+2)_2 (i+4)_3) : i \in Z_7\}$
$S_2 = \{(i_1 (i+3)_2 (i+6)_3) : i \in Z_7\}$
$S_3 = \{(i_1 i_2 i_3) : i \in Z_7\}$.
(Developing R yields 7 parallel classes while each of S_i is a parallel class on its own). The union of S_1 and S_3 can be decomposed into two 2-factors each of which is of type (i) 3+3+3+3+3+6, or (ii) 3+3+3+3+9, or (iii) 3+3+3+12:
(i) $(0_1 0_2 5_1 5_3 5_2 0_3)(1_1 1_2 1_3)(2_1 2_2 2_3)(3_1 3_2 3_3)(4_1 4_2 4_3)(6_1 6_2 6_3)$,
$(0_1 2_2 4_3)(1_1 3_2 5_3)(2_1 4_2 6_3)(3_1 5_2 5_1 2_3 0_2 0_3)(4_1 6_2 1_3)(6_1 1_2 3_3)$;

(ii) $(0_10_20_3)(1_11_26_16_34_24_34_16_21_3)(2_12_22_3)(3_13_23_3)(5_15_25_3)$,
$(0_12_24_3)(1_13_25_3)(2_14_24_11_31_23_36_16_26_3)(3_15_20_3)(5_10_22_3)$;
(iii) $(1_11_21_36_24_14_34_26_36_13_33_13_2)(0_10_20_3)(2_12_22_3)(5_15_25_3)$,
$(1_11_34_14_22_16_36_26_11_23_33_25_3)(0_12_24_3)(3_15_20_3)(5_10_22_3)$.

Replacing S_1 and S_3 with two 2-factors of type (i), (ii), or (iii) decreases the number of triangles by 4,6, or 8, respectively. Thus $62, 64, 66 \in \Delta(21)$.

The following 2-factorization of K_{21} shows $63 \in \Delta(21)$.
(1 2 3)(4 5 6)(7 8 9)(10 11 12)(13 14 15)(16 17 18)(19 20 21),
(1 4 7)(2 18 20)(3 6 21)(5 8 10)(9 11 13)(12 14 16)(15 17 19),
(1 9 15)(2 6 12)(3 7 18)(4 16 21)(5 11 14)(8 13 19)(10 17 20),
(1 11 21)(2 9 17)(3 8 15)(4 10 19)(5 12 18)(6 7 14)(13 16 20),
(1 12 13)(2 8 16)(3 5 20)(4 9 18)(6 11 19)(7 10 15)(14 17 21),
(1 18 19)(2 10 21)(3 9 14)(4 8 12)(5 15 16)(6 13 17)(7 11 20),
(1 5 17)(2 4 11)(3 12 19)(6 15 18)(7 13 21)(8 14 20)(9 10 16),
(1 6 8)(2 14 19)(3 11 16)(4 15 20)(5 9 21)(7 12 17)(10 13 18),
(1 10 14)(2 5 13)(3 4 17)(6 9 20)(7 16 19)(8 11 18)(12 15 21),
(1 20 12 9 19 5 7 2 15 11 17 8 21 18 14 4 13 3 10 6 16).

The following 2-factorization of K_{21} shows $65 \in \Delta(21)$: the first two 2-factors are as in the previous case, and the remaining eight 2-factors are
(1 16 21)(2 7 13)(3 4 19)(5 9 18)(6 10 20)(8 12 15)(11 14 17),
(1 10 14)(2 8 21)(3 12 17)(4 13 18)(5 11 19)(6 7 15)(9 16 20),
(1 9 15)(2 4 11)(3 5 16)(6 12 18)(7 14 20)(8 13 19)(10 17 21),
(1 11 20)(2 5 14)(3 15 18)(4 8 16)(6 13 17)(7 12 21)(9 10 19),
(1 5 17)(2 10 15)(3 13 20)(4 9 12)(6 8 11)(7 16 19)(14 18 21),
(1 6 19)(2 9 17)(3 8 14)(4 15 21)(5 12 20)(7 11 18)(10 13 16),
(1 8 18)(2 12 19)(3 7 10)(4 17 20)(5 13 21)(6 9 14)(11 15 16),
(1 12 13)(2 6 16)(3 9 21 11)(4 10 18 19 14)(5 7 17 8 20 15).

The following 2-factorization of K_{21} shows $67 \in \Delta(21)$:
(1 2 4)(3 8 21)(5 10 16)(6 11 17)(7 12 18)(9 13 14)(15 19 20),
(1 3 7)(2 8 19)(4 12 20)(5 14 17)(6 9 21)(10 11 13)(15 16 18),
(1 5 6)(2 12 17)(3 13 18)(4 14 19)(7 10 15)(8 9 11)(16 20 21),
(1 8 15)(2 9 16)(3 10 17)(4 5 7)(6 13 20)(11 12 14)(18 19 21),
(1 9 20)(2 14 21)(3 6 12)(4 11 15)(5 13 19)(7 16 17)(8 10 18),

(1 11 16)(2 3 5)(4 10 21)(6 14 15)(7 9 19)(8 12 13)(17 18 20),
(1 12 21)(2 13 15)(3 14 16)(4 8 17)(5 9 18)(6 10 19)(7 11 20),
(1 13 17)(2 10 20)(3 9 15)(4 6 18)(5 11 21)(7 8 14)(12 16 19),
(1 14 18)(2 6 7)(3 11 19)(4 13 16)(5 8 20)(9 10 12)(15 17 21),
(1 10 14 20 3 4 9 17 19)(2 11 18)(5 12 15)(6 8 16)(7 13 21).

Finally, the following 2-factorization of K_{21} shows that $68 \in \Delta(21)$:
(1 2 4)(3 10 17)(5 11 16)(6 9 18)(7 14 21)(8 12 13)(15 19 20),
(1 5 6)(2 8 20)(3 12 16)(4 11 18)(7 10 19)(9 13 14)(15 17 21),
(1 7 13)(2 9 16)(3 11 19)(4 10 15)(5 20 21)(6 12 17)(8 14 18),
(1 8 15)(2 10 18)(3 9 21)(4 14 16)(5 12 19)(6 13 20)(7 11 17),
(1 9 17)(2 11 15)(3 7 18)(4 12 20)(5 10 14)(6 8 19)(13 16 21),
(1 10 21)(2 13 19)(3 4 6)(5 8 17)(7 9 20)(11 12 14)(15 16 18),
(1 11 20)(2 3 5)(4 13 17)(6 14 15)(7 8 16)(9 10 12)(18 19 21),
(1 12 18)(2 6 7)(3 14 20)(4 8 21)(5 9 15)(10 11 13)(16 17 19),
(1 14 19)(2 12 21)(3 13 15)(4 5 7)(6 10 16)(8 9 11)(17 18 20),
(1 3 8)(10 20 16)(2 14 17)(4 9 19)(5 13 18)(6 11 21)(7 12 15). □

Lemma 5.3. *If $v \in \{49, 55, 73\}$ then $\Delta(v) = P_\Delta(v)$.*

Proof. By [RS] (cf.also [KS], [MG]), there exists a 4-GDD of type $3^4 6^2$. Taking the groups of this GDD as blocks results in a PBD(24,{3,4,6},1) where each element is in exactly one block of size 3 or 6. Apply now Theorem 3.4 while taking into account Lemma 5.1. This proves the statement for $v = 49$. Taking instead a 4-GDD of type $3^1 6^4$ which also exists by [RS] (cf. [KS], [MG]) and proceeding as above proves the statement for $v = 55$. For $v = 73$, consider a 4-GDD of type 6^6 which exists by [BSH] (cf. also [MG]). Taking the groups of this GDD as blocks results in a PBD(36,{4,6},1) with a parallel class of blocks of size 6. Apply again Theorem 3.4 taking into account Lemma 5.1. □

Lemma 5.4. *If $v \in \{51, 75\}$ then $\Delta(v) = P_\Delta(v)$.*

Proof. Extending the groups of the 4-GDD of type $3^4 6^2$ from the proof of the previous lemma by a common new element ∞ yields a PBD(25,{4,7},1). Applying now Theorem 3.4 and taking into account Theorem 4.1 proves the statement for $v = 51$. Proceeding in the same fashion but starting instead with the 4-GDD of type 6^6 proves the statement for $v = 75$. □

Lemma 5.5. $\Delta(57) = P_\Delta(57)$.

Proof. Apply Theorem 3.4 to a PBD$(28, \{4, 7\}, 1)$ (obtained from a transversal design TD(4,7) by simply taking the groups of size 7 as blocks), employing also Theorem 4.1 (giving $\Delta^*(15)$). □

6. Main results

Theorem 6.1. *For all* $v \equiv 3(mod\ 6), v \geq 81$*, or* $v \in \{45, 63, 69\}$*,* $\Delta(v) = P_\Delta(v)$*.*

Proof. Steiner systems S(2,4,u) with a subsystem S(2,4,13) are known to exist for all $u \equiv 1, 4(mod\ 12), u \geq 40$ [RS]. Taking now as our PBD in Theorem 3.4 any PBD$(u, \{4, 13^*\}, 1)$, and using it together with Theorems 2.5 and 4.2 shows that the statement holds for all $v \equiv 3, 9(mod\ 24), v \geq 81$. When u ≡7, 10 (mod 12), taking instead any PBD$(u, \{4, 7^*\}, 1)$ known to exist for all such $u \geq 22$ (cf. [RS]), and using Theorem 3.4, together with Theorems 2.5 and 4.1 shows that the statement holds for all $v \equiv 15, 21(mod 24), v \geq 45$. □

Theorem 6.2. *For all* $v \equiv 1(mod\ 6), v \geq 79$*, or* $v \in \{43, 61, 67\}, \Delta(v) = P_\Delta(v)$*.*

Proof. Consider an S(2,4,w) with a sub-S(2,4,13) from the previous theorem, and delete an element *not* in the subsystem. This results in a $\{4, 13\}$-GDD of type $3^{(w-1)/3}$ with a unique block of size 13, or, equivalently, in a PBD$(u = w - 1, \{3, 4, 13^*\}, 1)$ with a parallel class of blocks of size 3. Such a PBD exists for all $v \equiv 0, 3(mod\ 12), v \geq 39$. It is essential to note that every element of this PBD occurs in a unique block of size 3. Applying now Theorem 3.4 to this PBD, together with Theorems 2.5 and 4.2, shows that the statement holds for all $v \equiv 1, 7(mod\ 24), v \geq 79$. Similarly, deleting an element of the PBD$(w, \{4, 7^*\}, 1)$ not in the unique block of size 7 (cf. Theorem 6.1) results in a PBD$(u, \{3, 4, 7^*\}, 1)$ with a parallel class of blocks of size 3; such a PBD exists for all $u \equiv 6, 9(mod\ 12), v \geq 21$. Applying Theorem 3.4, together with Theorems 2.5 and 4.1, shows that the statement holds for all $v \equiv 13, 19(mod\ 24), v \geq 43$. □

Combining now Theorems 6.1 and 6.2 with the lemmas of Section 5 gives our main result.

Theorem 6.3. *Let* $v \equiv 1, 3(mod\ 6), v \geq 43$ *or* $v \in \{13, 15, 21, 27\}$*. Then* $\Delta(v) = P_\Delta(v)$*.*

For the "remaining" orders $v = 19, 25, 31, 33, 37, 39$, we were so far unable to determine the set $\Delta(v)$ completely. We were able to show, however, the following.

(i) $P_\Delta(19) \setminus \{43, 44\} \subset \Delta(19)$.
(ii) $P_\Delta(25) \setminus \{83\} \subset \Delta(25)$.
(iii) $P_\Delta(31) \setminus \{134\} \subset \Delta(31)$.
(iv) $P_\Delta(33) \setminus \{171, 173, 174\} \subset \Delta(33)$.
(v) $P_\Delta(37) \setminus \{197\} \subset \Delta(37)$.
(vi) $P_\Delta(39) \setminus \{242, 244, 245\} \subset \Delta(39)$.

More precisely, there are 11 pairs (v, δ) for which we could not decide whether $v \in \Delta(v)$. These are the pairs $(v, \delta) = (19, 43), (19, 44), (25, 83)$, (31,134),(33,171),(33,173),(33,174),(37,197),(39,242), (39,244),(39,245).

The proof of (i)-(vi) above is fairly complicated and would necessitate introducing tools, such as frames, not needed in the proof of the main results; it is therefore omitted at present.

As mentioned in the introduction, determining the sets $\Delta(v)$ for $v \equiv 5(mod\ 6)$ remains an open problem.

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