COMPLETING THE SPECTRUM OF 2-CHROMATIC S(2,4,v)

F. Franek, *Department of Computing and Software, McMaster University, Hamilton, Ontario, Canada*
T. S. Griggs, *Department of Pure Mathematics, The Open University, Walton Hall, Milton Keynes, United Kingdom*
C. C. Lindner, *Department of Discrete and Statistical Sciences, Auburn University, Auburn, AL, U.S.A.*
A. Rosa, *Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada*

Abstract. We construct 2-chromatic S(2,4,v) for $v = 37, 40$, and 73. This completes the proof of the existence of 2-chromatic Steiner systems S(2,4,v) [equivalently, of Steiner systems S(2,4,v) with a blocking set] for all $v \equiv 1$ or 4 (mod 12).

Introduction

A *Steiner system* S(2,4,v) is a pair $(V, \mathcal{B})$ where V is a v-set, and $\mathcal{B}$ is a collection of 4-element subsets of V called *blocks* such that each 2-subset of V is contained in exactly one block. Thus S(2,4,v) is a BIBD with $k = 4$ and $\lambda = 1$, or a linear space having all lines of size 4. It is well known that a Steiner system S(2,4,v) exists if and only if $v \equiv 1, 4$ (mod 12).

If we replace in the above definition "in exactly one block" with "in exactly λ blocks", we get a definition of the design $S_\lambda(2,4,v)$.

A *colouring* of an S(2,4,v) $(V, \mathcal{B})$ [also called a *weak colouring*] is a mapping $\phi : V \to C$ such that for all $B \in \mathcal{B}$, $|\phi(B)| > 1$ where $\phi(B) = \bigcup_{v \in B} \phi(v)$. The elements of C are called *colours*; if $|C| = k$, we have a k-colouring. For each colour $c \in C$, the set $\phi^{-1}(c) = \{x : \phi(x) = c\}$ is a *colour class*.

In a colouring, each colour class is an independent set (i.e. contains no block), and no block is monochromatic (i.e. no block has all elements the same colour). The chromatic number $\chi(V, \mathcal{B})$ of $(V, \mathcal{B})$ is the smallest integer k for which there exists a k-colouring of $(V, \mathcal{B})$.

Any S(2,4,v) with a 2-colouring is necessarily 2-chromatic. In this case, each of the two colour classes is a *blocking set* (cf. [HLP]), i.e. a set having a nonempty intersection with each block but containing no block.

Hoffman, Lindner, and Phelps [HLP] were apparently the first to investigate the existence of 2-chromatic S(2,4,v). They proved that a 2-chromatic S(2,4,v) exists for all $v \equiv 1, 4$ (mod 12), except possibly when $v \in \{37, 40, 73\}$. In the same paper it is proved that a 2-chromatic $S_2(2, 4, v)$ exists for all $v \equiv 1$ (mod 3) except possibly when $v \in \{19, 34, 37, 46, 58\}$. These exceptions were subsequently removed in [RC]. In a sequel paper [HLPa], the existence of 2-chromatic $S_\lambda(2, 4, v)$ was settled for all $\lambda > 2$.

The purpose of this note is to settle the remaining three exceptional cases when $\lambda = 1$ (cf. Problem 2.4 in [RC]) by providing constructions of 2-chromatic Steiner systems S(2,4,v) for $v \in \{37, 40, 73\}$. Of the three designs, two were constructed by hand but the construction of the 2-chromatic S(2,4,40) required a fairly substantial computer assistance.

$$v = 37$$

Neither of the two cyclic S(2,4,37) [CM] is 2-chromatic. However, assuming an automorphism of order 9 leads to the following 2-chromatic S(2,4,37):
Elements: $V = (Z_9 \times \{1, 2, 3, 4\}) \cup \{\infty\}$.
Blocks: orbits generated by base blocks
$0_1 1_3 7_4 8_4$, $0_1 5_3 7_3 8_3$, $3_1 7_1 2_2 0_3$, $0_1 3_1 6_2 0_4$, $6_1 8_1 4_2 0_4$, $7_1 8_2 0_2 0_4$, $0_1 1_1 4_3 5_4$, $0_2 1_3 5_3 4_4$, $0_2 2_3 2_4 7_4$, $0_2 4_3 6_4 8_4$, $1_2 3_2 6_2 0_3$, $\infty 0_1 0_2 0_3$, $\infty 0_4 3_4 6_4$
(the last orbit is the "short" orbit). The colour classes are $(Z_9 \times \{1, 2\}) \cup \{\infty\}$ and $Z_9 \times \{3, 4\}$.

$$v = 40$$

None of the 10 cyclic S(2,4,40) [CM] is 2-chromatic, nor is any of the 1-rotational S(2,4,40), nor is the S(2,4,40) with a maximal arc given in [MRV]. Constructing a 2-chromatic S(2,4,40) was by far the most difficult of the three cases dealt with in this note.

We first attempted a construction of a 2-chromatic S(2,4,40), say S, with a subsystem S(2,4,13). Since in any 2-colouring of the latter the colour classes must have sizes 7 and 6, respectively, a construction of S would require the existence of a Kirkman triple system (KTS) of order 27 with a certain specified 2-colouring (not proper: some blocks would be monochromatic). Although we cannot completely rule out the existence of such a 2-coloured KTS(27), an extensive computational effort in this direction was unsuccessful, and so we suspect that this object simply may not exist.

Next we assumed an automorphism of order 5, i.e. an automorphism consisting of 8 disjoint 5-cycles $(0_i 1_i 2_i 3_i 4_i), i = 1, 2, \ldots, 8$, and proceeded to "combine" two 4-rotational STS(21) (with the fixed element removed, of course), one of them on the set $X = Z_5 \times \{1, 2, 3, 4\}$, and the other on the set $Y = Z_5 \times \{5, 6, 7, 8\}$. The 4-rotational STS(21) were enumerated in [MR]; there are exactly 1772 nonisomorphic ones. Our 2-chromatic S(2,4,40) to be constructed would consist of 26 block orbits, 12 of which would be of type 3+1 (i.e. each block would have 3 elements of X and 1 element of Y), 12 would have type 1+3, and 2 would have type 2+2.

Our search proceeded as follows: we chose a particular 4-rotational STS(21), deleted its fixed element, thereby obtaining a maximum packing of triples on 20 elements, generated by 12 base blocks of triples under the action of the cyclic group of order 5. These 12 base blocks of triples were combined into 4 sets of three such that within each set the three base blocks were disjoint (shifting a block within an orbit if necessary to attain disjointness). The 3 disjoint base blocks of each of the 4 sets were associated with an element of one of the remaining 4 element orbits $(Z_5 \times \{i\}, i = 5, 6, 7, 8)$. This yielded all 12 block orbits of type 3+1 (which we term here a "starter") of the total of 26 orbits of our S(2,4,40) to be constructed.

For each starter, we next examined all possibilities left for the two orbits of type 2+2. For each of the latter, we attempted to complete the remaining 12 orbits of type 1+3 to an S(2,4,40), by employing a complete backtrack. In this we were, in effect, trying to find another 4-rotational STS(21), to "fit" with the starter.

To initiate a systematic search of possible starters, we first examined the set of 48 nonisomorphic KTS(21), as given in [MR]. For each such KTS, one may consider a subset of all possible starters by dividing the 6 disjoint triples of a base parallel class (the triple contaning the element ∞ is discarded) into two sets of three disjoint triples. Doing this in all possible ways independently for each of the two base parallel classes gives $\binom{6}{3}^2/4 = 100$ starters. The solution (i.e a 2-chromatic S(2,4,40)) given below was obtained for the sixth starter corresponding to the KTS No.F26 (numbering as in [MR]). It may be that other 2-chromatic S(2,4,40) can be found by this method.

Elements: $V = Z_5 \times \{1, 2, \ldots, 8\}$.

Blocks: orbits generated by base blocks

$2_1 3_1 2_3 0_5$, $4_1 0_3 2_4 0_5$, $1_1 0_4 1_4 0_5$, $2_2 1_3 3_3 0_6$, $3_2 4_2 4_4 0_6$, $1_2 4_3 3_4 0_6$, $0_1 3_1 1_2 0_7$, $1_1 3_3 4_3 0_7$, $0_2 2_2 2_3 0_7$, $4_1 3_2 1_4 0_8$, $2_1 4_2 3_4 0_8$, $1_3 2_4 4_4 0_8$, $0_1 1_6 0_8 2_8$, $0_1 2_6 3_6 3_7$, $0_1 4_6 1_7 4_8$, $0_2 1_5 2_7 3_8$, $0_2 2_5 1_7 0_8$, $0_2 3_5 4_5 4_8$, $0_3 2_5 4_7 1_8$, $0_3 1_5 4_5 3_6$, $0_3 0_6 2_8 3_8$, $0_4 2_5 0_6 3_6$, $0_4 1_5 1_7 4_7$, $0_4 4_6 2_7 3_7$, $0_1 0_2 0_5 0_6$, $0_3 0_4 0_7 0_8$. The colour classes are $Z_5 \times \{1, 2, 3, 4\}$ and $Z_5 \times \{5, 6, 7, 8\}$.

$$v = 73$$

Here we utilize a skew Room frame of type 2^6 (for definiton of a skew Room frame, see [S]) which is known to exist by [CZ], and the well known fact that the unique S(2,4,13) has a 2-colouring with colour classes of sizes 7 and 6. Let R be such a skew Room frame on the elements $X = \{1, 2, \ldots, 12\}$ with the holes $\{h_1, h_2, \ldots, h_6\}$, each of size 2. Let $V = (X \times Z_6) \cup \{\infty\}$. Let $\mathcal{B}$ be the following collection of blocks:
(1) for each hole $h_i, i = 1, 2, \ldots, 6$, place the blocks of a 2-coloured S(2,4,13) on the element set $(h_i \times Z_6) \cup \{\infty\}$ in $\mathcal{B}$ where $\{\infty\} \cup (h_i \times \{0, 1, 2\})$ and $(h_i \times \{3, 4, 5\})$ are the two colour classes in a 2-colouring of S(2,4,13).
(2) For any two elements $x, y \in V$ belonging to different holes, place the 6 blocks $\{(x, i), (y, i), (r, 1+i), (c, 4+i)\}$, $i \in Z_6$, (the second coordinates reduced mod 6) in $\mathcal{B}$ whenever $\{x, y\}$ is in the cell (r, c) of R.

[The above construction is essentially *The* $12n + 1$ *Construction* of [LR].]

It is easily seen that $(X \times \{0, 1, 2\}) \cup \{\infty\}$ and $X \times \{3, 4, 5\}$ are the two colour classes of a 2-chromatic S(2,4,73) $(V, \mathcal{B})$.

Main result

Thus we have:

Theorem. *A 2-chromatic Steiner system S(2,4,v) exists if and only if* $v \equiv 1, 4\ (mod\ 12)$.

Equivalently, a Steiner system S(2,4,v) with a blocking set exists if and only if $v \equiv 1, 4\ (mod\ 12)$.

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