How many double squares can a string contain?

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Motivation and background

We are dealing with finite strings over finite alphabets. There is no particular requirement about the order of the alphabet.

What is the *maximum number of distinct squares problem*?

We are counting types of squares rather than their occurrences.

```
 a a b a a b a a
```

has 6 occurrences of squares, but only 4 distinct squares, *aa*, *aabaab*, *abaaba*, and *baabaa*.
A trivial bound: the number of all occurrences of primitively rooted squares in a string of length $n$ is bounded by $O(n \log n)$ (Crochemore 1978) and the number of distinct non-primitively rooted squares is $O(n)$ (Kubica et al. 2013)

Could it be $O(n)$? And if so, what would be the constant?

Why this is not simple? In a string of length $n$, $O(\log n)$ squares can start at the same position!

How many double squares can a string contain?
It is easy to compute it for short strings, so why induction cannot be used?

\[ \text{a a b a a b} + \text{a a b a a b} = \text{a a b a a b a a b a a b} \]

Concatenation does both “destroys” existing types through multiple-occurrences and “creates” new types. Of course, same holds true for the reverse process - partitioning of strings.
Theorem (Fraenkel-Simpson, 1998)

There are at most \(2n\) distinct squares in a string of length \(n\).

Count only the rightmost occurrences. Fraenkel-Simpson showed that if there are three rightmost squares \(uu, vv,\) and \(ww\) starting at the same position so that \(|u| < |v| < |w|\), then \(ww\) contains a farther copy of \(uu\), based on Crochemore-Rytter (1995) Lemma showing that in such a case, \(|w| \geq |u| + |v|\).
Fraenkel-Simpson hypothesized that the number of distinct squares should be bounded by $n$, i.e.

$$\sigma(n) \leq n$$

where $\sigma(n) = \max \{ s(x) : x \text{ is a string of length } n \}$.

Fraenkel-Simpson gave an infinite sequence of strings $\{x_n\}_{n=1}^{\infty}$ so that $|x_n| \uparrow \infty$ and

$$\frac{s(x_n)}{|x_n|} \uparrow 1$$

where $s(x) =$ number of distinct squares in $x$. 
In 2005 Ilie provided a simpler proof of Fraenkel-Simpson’s Theorem and in 2007 presented an asymptotic upper bound of $2n - \theta(\log n)$.

In 2011 Deza-F. proposed a $d$-step approach to the problem and conjectured that $\sigma_d(n) \leq n - d$, where $\sigma_d(n) = \max \{ s(x) : x \text{ is a string of length } n \text{ with } d \text{ distinct symbols} \}$. 
Basic notions and tools

Definition

non-trivial power of a string $x$ is a concatenation of $m$ copies of $x$ denotes as $x^m$; $x^2$ is a square, $x^3$ a cube.

A string $x$ is primitive if $x \neq y^n$ for any $y$ and any $n \geq 2$.

primitive root of $x$ is the smallest primitive $y$ so that $x = y^n$.

$x$ and $y$ are conjugates if $x = uv$ and $y = vu$ for some $u, v$. 
Lemma (Synchronization principle)

Given a primitive string $x$, a proper suffix $y$ of $x$, a proper prefix $z$ of $x$, and $m \geq 0$, there are exactly $m$ occurrences of $x$ in $yx^mz$.

Lemma (Common factor lemma)

For any strings $x$ and $y$, if a non-trivial power of $x$ and a non-trivial power of $y$ have a common factor of length $|x| + |y|$, then the primitive roots of $x$ and $y$ are conjugates. In particular, if $x$ and $y$ are primitive, then $x$ and $y$ are conjugates.
**Double squares**

- **Fraenkel-Simpson**: only two rightmost squares can start at the same position. Thus, only one rightmost square or two rightmost squares may start at any position.

- **Lam (2009 – unpublished)** tried bounding the number of **double squares** and hence bound the number of distinct squares. His approach is based on a taxonomy of all possible configurations of two double squares yielding a bound of $\frac{94}{48} n \approx 1.98n$. 

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A configuration of two squares

\[
\begin{array}{|c|c|}
\hline
U & U \\
\hline
u & u \\
\hline
\end{array}
\]

has been investigated in many different contexts:

- **Smyth et. al.**: with intention to find a position for amortization argument for runs conjecture.
- in computational framework by **Deza-F.-Jiang**: such configurations are used in **Liu**’s Ph.D. thesis to speed up computation of \( \sigma_d(n) \).
- **Lam**: two rightmost squares have a unique structure.
Lemma

Let $uu$ and $UU$ be two squares in a string $x$ starting at the same position with $|u| < |U|$ such that either

(b) both $uu$ and $UU$ are rightmost occurrences, or

(a) $|U| < |uu|$ and either $uu$ or $UU$ is primitively rooted.

Then $|u| < |U| < |uu| < |UU|$ and there is a unique primitive string $u_1$, a unique proper prefix $u_2$ of $u_1$, and unique integers $e_1$ and $e_2$ satisfying $1 \leq e_2 \leq e_1$ such that $u = u_1^{e_1} u_2$ and $U = u_1^{e_1} u_2 u_1^{e_2}$; i.e. $uu$ and $UU$ form a double square.
Thus, only strings of length at least 10 may contain a double square: $|UU| = 2((u(1) + u(2))|u_1| + |u_2|) \geq 2((1 + 1)2 + 1) = 10$. 

$u_1 \ u^{(1)} \ u_2 \ u_1 \ u^{(2)} \ u_1 \ u^{(1)} \ u_2 \ u_1 \ u^{(2)}$
Cyclic shift (rotation) to the right is controlled by

\[ lcp(u_1, \overline{u}_1) \]

while cyclic shift to the left is controlled by

\[ lcs(u_1, \overline{u}_1) \]

\( lcp \) = largest common prefix
\( lcs \) = largest common suffix
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How many double squares can a string contain?

\[ u_1 = aaabaa, \ u_2 = aaab, \ \overline{u}_2 = aa, \ u(1) = u(2) = 2 \]
How many double squares can a string contain?

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\[ u_1 = aaabaa, \quad u_2 = aaab, \quad \overline{u}_2 = aa, \quad u(1) = 2, \text{ and } u(2) = 1. \]
Definition

For a double square $U, \bar{v}v\bar{v}$ where $|v| = |\bar{u}_2|$ and $|v| = |u_2|$ is an inversion factor

$$U = u_1 u^{(1)} u_2 u_1 u^{(2)} + u^{(1)} u_2 u_1 u^{(2)} =$$

$$u_1 (u^{(1)} - 1) u_2 \bar{u}_2 u_2 \bar{u}_2 u_1 u^{(2)} + u^{(1)} - 2 u_2 \bar{u}_2 u_2 u_2 \bar{u}_2 u_1 (u^{(2)} - 1)$$

$N_1$  $N_2$

natural inversion factors

How many double squares can a string contain?
A cyclic shift of an inversion factor is an inversion factor, also controlled by $lcp(u_1, \overline{u_1})$ and $lcs(u_1, \overline{u_1})$. 
All inversion factors are cyclic shifts of the natural ones:

**Lemma (Inversion factor lemma)**

*Given a double square $\mathcal{U}$, there is an inversion factor of $\mathcal{U}$ within the string $UU$ starting at position $i$ $\iff i \in [L_1, R_1] \cup [L_2, R_2]$.***
Inversion factor lemma for distinct squares

Theorem (*Fraenkel-Simpson, Ilie*)

*At most two rightmost squares can start at the same position.*

Let us assume that 3 rightmost squares $uu$, $UU$, $vv$ start at the same position.

By item (c) of Inversion factor lemma, $uu$ and $UU$ form a double square $U: u = u_1^U u_2$ and $U = u_1^U u_2 u_1^U u(2)$.

Since the first $v$ contains an inversion factor, the second $v$ must also contain an inversion factor.

*Cont. on the next slide*
If the inversion factor in the second $v$ were from $[L_2, R_2]$, then $|v| = |U|$, a contradiction. Hence $v$ must not contain an inversion factor from $[L_2, R_2]$ and so $u_1^{U(1)} u_2 u_1^{U(1)+U(2)-1} u_2$ must be a prefix of $v$. Therefore $vv$ contains another copy of $u_1^{U(1)} u_2 u_1^{U(1)} u_2 = uu$, a contradiction.
**Fundamental Lemma:**

**Lemma**

Let $x$ be a string starting with a double square $U$. Let $V$ be a double square with $s(U) < s(V)$, then either

(a) $s(V) < R_1(U)$, in which case either
   (a₁) $V$ is an $\alpha$-mate of $U$ (cyclic shift), or
   (a₂) $V$ is a $\beta$-mate of $U$ (cyclic shift of $U$ to $V$), or
   (a₃) $V$ is a $\gamma$-mate of $U$ (cyclic shift of $U$ to $V$), or
   (a₄) $V$ is a $\delta$-mate of $U$ (big tail),

or

(b) $R_1(U) \leq s(V)$, then
   (b₁) $V$ is an $\varepsilon$-mate of $U$ (big gap).
$\alpha$-mate (cyclic shift):

$$R_1$$

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\( \beta \)-mate (cyclic shift of \( U \) to \( V \))

\[
\begin{align*}
\text{How many double squares can a string contain? LIGM, Paris-Est, February 2014}
\end{align*}
\]
γ-mate (cyclic shift of $U$ to $v$)

```
[ ] [ ] ( [ ] )
aabaabaabaabaabaabaabaabaabaabaabaabaabaabab
[ ] [ ] ( [ ] )
aabaabaabaabaabaabaabaabaabaabaabaabaabaabab
```
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\[ \delta \text{-mate (big tail)} \]

sufficiently big tail

\[
\begin{array}{c}
R_1 \\
\end{array}
\]

[ ] [ ] ( ) ( ]

\[
\begin{array}{c}
aabaabaabaabaabaabaabaabaaabaabaaabaabaabaaabaab \\
\end{array}
\]

[ ] [ ] ( ) ( ]

\[
\begin{array}{c}
aabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaaabaabaaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaabaab \\
\end{array}
\]

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An upper bound for the number of double squares

We show by induction a bound $\delta(x) \leq \frac{5}{6}|x| - \frac{1}{3}|u|$, where $uu$ is the shorter square of the leftmost double square of $x$.

<table>
<thead>
<tr>
<th>$u$</th>
<th>$u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$</td>
<td>$v$</td>
</tr>
</tbody>
</table>

The fundamental lemma basically says that either the gap $G(U, V)$ is "big" or the tail $T(U, V)$ is "big" (for $\delta$-mate and $\varepsilon$-mate), or it is case of $\alpha$-mate, $\beta$-mate, or $\gamma$-mate.
Lemma (Gap-Tail lemma)

\[ \delta(x') \leq \frac{5}{6} |x'| - \frac{1}{3} |v| \text{ implies } \]

\[ \delta(x) \leq \frac{5}{6} |x| - \frac{1}{3} |u| + d - \frac{1}{2} |G(U, V)| - \frac{1}{3} |T(U, V)| \]
We deal with $\alpha$-mates, $\beta$-mates, and $\gamma$-mates separately.

It is possible as they form families, either a pure $\alpha$-family, or $\alpha+\beta$-family, or $\alpha+\beta+\gamma$-family.
**\( \mathcal{U} \)-family consists only of \( \alpha \)-mates**

Illustration of \( \alpha \)-family with \( \mathcal{U}(1) = \mathcal{U}(2) \)

```
aaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaaaabaaa
```
Illustration of $\alpha$-family with $U(1) > U(2)$

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How many double squares can a string contain?  LIGM, Paris-Est, February 2014
It is easy to estimate the size of $\alpha$-family, as it is controlled by $lcp(u_1, \overline{u}_1)$ and $lcp(y, u_2)$ where $y$ is $x$ without $UU$: the size $\leq |u_1|$.

- Either there are no other double squares, and then it can be shown directly that the bound holds, or

- There is a $\gamma$ underneath, and we can use induction using the Gap-Tail lemma. $\gamma$ must be either $\gamma$-mate, or $\delta$-mate, or $\varepsilon$-mate, and the Gap-Tail lemma can be applied to propagate the bound.
**Illustration of** $\alpha+\beta$-**family**

\[
\begin{align*}
\text{U-family consists of } \alpha\text{-mates and } \beta\text{-mates.} \\
\text{How many double squares can a string contain? LIGM, Paris-Est, February 2014}
\end{align*}
\]
It is more complicated to estimate the size of a $\alpha+\beta$-family:

\[
\sum_{\mathcal{U}} |u_1| \begin{cases} 
\left\lceil \frac{\mathcal{U}(1) - \mathcal{U}(2)}{2} \right\rceil |u_1| & \text{if } \mathcal{U}(2) = 1 \\
\frac{\mathcal{U}(1) - \mathcal{U}(2)}{2} |u_1| & \text{if } \mathcal{U}(2) > 1
\end{cases}
\]

- Either there are no other double squares, and then it can be shown directly that the bound holds, or
- There is a $\mathcal{V}$ underneath, and we can use induction using the Gap-Tail lemma. $\mathcal{V}$ must be either $\delta$-mate, or $\varepsilon$-mate, and the Gap-Tail lemma can be applied to propagate the bound. (Special care needed for $\varepsilon$-mate case and super-$\varepsilon$-mate must be put in play!)

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$\mathcal{U}$-family consists of $\alpha$-mates, $\beta$-mates, and $\gamma$-mates

**Illustration of $\alpha+\beta+\gamma$-family**

How many double squares can a string contain? LIGM, Paris-Est, February 2014
It is quite complex to estimate the size of a $\alpha+\beta+\gamma$-family:

$$\leq \frac{2}{3} (u(1) + 1)|u_1|$$

- Either there are no other double squares, and then it can be shown directly that the bound holds, or
- There is a $\mathcal{V}$ underneath, and we can use induction using the Gap-Tail lemma. $\mathcal{V}$ must be either $\delta$-mate, or $\varepsilon$-mate, and the Gap-Tail lemma can be applied to propagate the bound.
Main theorems

**Theorem**

The number of double squares in a string of length $n$ is bounded by $\left\lfloor \frac{5n}{6} \right\rfloor$.

**Corollary**

The number of distinct squares in a string of length $n$ is bounded by $\left\lfloor \frac{11n}{6} \right\rfloor$. 
We presented a universal upper bound of $\frac{11n}{6}$ for the maximum number of distinct squares in a string of length $n$.

A bound of $\frac{5n}{6}$ for the maximum number of double squares.

It improves the universal bound of $2n$ by Fraenkel-Simpson.

It improves the asymptotic bound of $2n - \Theta(n)$ by Ilie.

The combinatorics of double squares is interesting on its own and possibly can be used for some other problems.
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