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Semantics of Inhibitor Nets

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We discuss an abstract semantics of concurrent systems generalising causal partial orders. The new semantics employs relational structures —called stratified order structures—which comprise causal partial orders and weak causal partial orders. Stratified order structures can be represented by certain equivalence classes of step sequences—comtraces—directly generalising Mazurkiewicz traces. We use Elementary Net Systems with inhibitor arcs as a system model and show that stratified order structures can provide an abstract semantics which is consistent with their operational semantics expressed in terms of step sequences. Two different types of operational rules are considered. We also construct occurrence nets to enable the generation of stratified order structures for a given run of the net. © 1995 Academic Press, Inc.

1. INTRODUCTION

In the development of mathematical models for concurrent systems, the concepts of partial and total order undoubtedly occupy a central position. Interleaving models use total orders of event occurrences, while so-called "true concurrency" models use step sequences or causal partial orders (compare [3, 11, 28]). Even more complex structures, such as failures [11] or event structures [32], are in principle based on the concept of total or partial orders. While interleavings and step sequences usually represent executions or observations and can be regarded as directly representing operational behaviour of a concurrent system, a causal partial order represents a set of executions or observations. The lack of ordering between two event occurrences in the case of a step sequence is interpreted as simultaneity, while in the case of a causal partial order it is interpreted as concurrency. The latter means that the event occurrences can be executed (observed) in either order or simultaneously. In other words, a causal partial order is an invariant describing an abstract history of a concurrent system. Both interleaving and true concurrency models have been developed to a high degree of sophistication providing a successful specification and verification framework. However, there are some problems; for instance, the specification of priorities using partial orders alone is often problematic (see [12, 18, 20]). Another example are inhibitor nets (see [27]), which are greatly admired by practitioners, and almost completely rejected by theoreticians even in the bounded case, in our opinion mainly because their full concurrent behaviour cannot be adequately specified in terms of causality based structures [16]. We think that these kind of problems follow from the general assumption that all behavioural properties of a concurrent system can be modelled in terms of causal partial orders or causality-based relations. We claim [14, 16] that the structure of concurrency phenomena is richer and there are other invariants which can be used to represent abstract histories of a concurrent system.

In this paper we consider one of such invariants, called *weak causality*. More precisely, we employ *stratified order structures* [17] to provide one such invariant semantics of concurrent systems modelled by Elementary Net Systems with inhibitor arcs. Each stratified order structure is a relational structure comprising causality and weak causality invariant. We introduce a representation of stratified order structures using a novel concept of *comtrace*—a direct extension of Mazurkiewicz trace [1, 22, 23]. Stratified order structures correspond to posets, comtraces correspond to traces, while step sequences play the role of interleaving sequences. The independence relation on events used to define Mazurkiewicz traces in [22, 23] is replaced by two new relations on events, simultaneity and serialisability. The former specifies which events may be executed in one step; the latter comprises pairs of events (e, f) such that if $\{e, f\}$ is a possible step then e and f can also be executed in the order e followed by f (but not necessarily in the order f followed by e).

The paper is organised as follows. In the next section two different operational rules involving inhibitor arcs are informally discussed. In Section 3 we introduce stratified order

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structures. Section 4 contains the development of the comtrace model. Section 5 shows how comtraces can provide an invariant semantics for the Elementary Net Systems with Inhibitor Arcs (ENI-systems). In Section 6 a different, more restrictive, but easier to handle semantics is analysed. Section 7 contains final comments.

This paper is a revised and full version of [15].

2. SIMULTANEITY AND INHIBITOR ARCS

The standard execution rule for inhibitor arcs says that an inhibitor arc between a condition (place) s and an event (transition) e means that e can only be fired if s is unmarked [2, 25, 27]. Such a rule is sufficient if one is to define purely sequential (interleaving) semantics, since a non-interleaving semantics of any kind of nets requires (explicitly or implicitly) the definition of a *simultaneous step* of events (transitions). There seems to be no general agreement on an exact definition of a simultaneous step for nets with inhibitor arcs. To explain this let us consider the net from Fig. 1. There is no problem with its interleavings, the net can only produce three non-empty sequences: e, f and fe. Moreover, one can observe that while the firing of e is completely independent of the firing of f, the firing of f depends on the behaviour of e since e may disable f by firing first. Hence the intuitive independence relation is not symmetric. A fundamental question which one must now answer is: *Can this net fire simultaneously the step* $\{e, f\}$?

Let us call the step sequence semantics which allows the step $\{e, f\}$ *type*-1, and the step sequence semantics which disallows the step $\{e, f\}$ *type*-2. What is interesting is that both can be found in the literature. Type-1 semantics is assumed in [4, 15, 16, 18], while type-2 is assumed in [7, 24] and (implicitly) [27]. Both types are considered in [6]: the first was called the "*a-priori* concurrent semantics," and the second the "*a-posteriori* concurrent semantics." In this paper we will adopt this terminology.

We think that, in principle, both types of semantics are admissible, e.g., depending on whether the events (transitions) involved are viewed as taking time or not:[1]

(1) *Events (transitions) are interpreted as representations of activities whose completion takes some time.* This interpretation seems to be frequently present in both software and hardware applications [2, 19, 25]. If the event

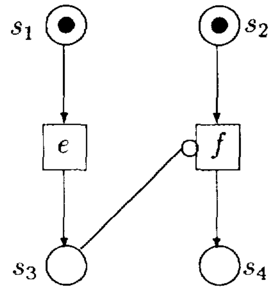

FIG. 1. Net with inhibitor arcs.

e takes a non-zero time to complete, then there should be some moment during the occurrence of e in which the event f is still enabled (so it can be fired) because the occurrence of e has not yet finished and s_3 is still empty. Hence, in this case, there is intuitively nothing that would seem to disallow the simultaneous occurrence of e and f.

(2) *Events (transitions) are instantaneous, their occurrence takes zero time.* In such a case the simultaneous execution of e and f ought to be excluded. One may argue that firing an instantaneous event e places a token in s_3 in the very moment it removes a token from s_1.

The problem is that under the firing rules of the a-priori semantics, the process level cannot be defined in terms of partial orders ($\{e, f\}$ and $\{f\}\{e\}$ are allowed, but $\{e\}\{f\}$ is not, see [16] for a detailed discussion). We need a more powerful apparatus, and we shall use stratified order structures instead of partial orders. For the a-posteriori semantics the standard posets are sufficient.[2] In this paper we assume the a-priori semantics everywhere except Section 6. In Section 6 we show that on the process level, in our approach, the a-posteriori semantics can be treated as though it was a special (and much simpler) case of the a-priori semantics. In particular, no new definitions or concepts are needed. Thus the theory developed in the sequel covers both the a-priori and a-posteriori semantics.

3. BASIC CONCEPTS AND NOTATION

We use the standard mathematical notation. In particular, if P and Q are binary relations over a set X then their composition $P \circ Q$ is defined thus:

$$P \circ Q = \{(x, y) \in X \times X \mid (\exists z \in X)(x, z) \in P \wedge (z, y) \in Q\}.$$

[1] As pointed out by one of the referees, an alternative (and more general) interpretation of the a-posteriori semantics is that a step in this semantics is only allowed if no problem may ever arise *during* its execution (hence a duration is allowed) until the very end, of the kind "a token occurs in a place controlling the inhibition of a fired transition."

[2] For this reason the a-posteriori semantics was chosen in [7]; in their terminology, the a-posteriori semantics satisfies the "diagonal rule," while the a-priori semantics does not.

We also define

$$P^0 = id_X$$
$$P^i = P^{i-1} \circ P \qquad (i \geqslant 1)$$
$$P^+ = \bigcup_{i \geqslant 1} P^i$$
$$P^* = \bigcup_{i \geqslant 0} P^i.$$

3.1. *Partially Ordered Sets*

A *partially ordered* set (poset) is a pair $po = (X, \prec)$ such that X is a set and $\prec$ is an irreflexive transitive relation on X. In this paper we will always assume that X is *finite*.[3] We will use $a \frown b$ to denote that a and b are distinct incomparable elements of X. If necessary, we will write $\prec_{po}$ and $\frown_{po}$ to denote $\prec$ and $\frown$. The poset is *total* if $\frown$ is empty, and *stratified* [9] if $\frown \cup id_X$ is an equivalence relation. In the latter case X can be partitioned onto a unique sequence of non-empty sets of elements $\mho_{po} = B_1 \cdots B_k$ $(k \geqslant 0)$ such that

$$\prec = \bigcup_{i<j} B_i \times B_j \qquad \text{and} \qquad \frown = \bigcup_i B_i \times B_i - id_X.$$

Conversely, if X can be partitioned onto non-empty sets $B_1, \ldots, B_k$ satisfying the above conditions then po is stratified. We will sometimes identify po with $\mho_{po}$.[4] A poset $(X, \prec')$ is an *extension* of $po = (X, \prec)$ if $\prec \subseteq \prec'$.

In Fig. 2, po' and po'' are stratified extensions of po, $\mho_{po'} = \{a\}\{b, c\}$, and $\mho_{po''} = \{a, c\}\{b\}$.

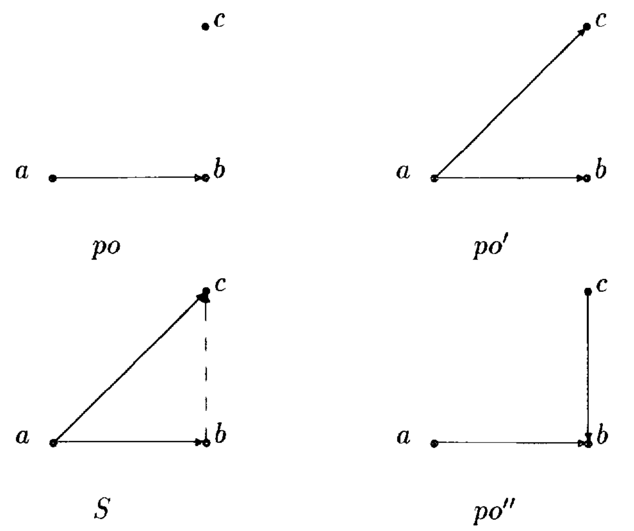

FIG. 2. Partial orders and a stratified order structure.

PROPOSITION 3.1. *For every poset $po = (X, \prec_{po})$ there is exactly one stratified poset $q = B_1 \cdots B_k$ which is an extension of po such that*

(∗) *If $i \geqslant 2$ and $b \in B_i$ then there is $a \in B_{i-1}$ satisfying $a \prec_{po} b$.*

Proof. Although the result follows from [5], we include a simple proof to make the paper self-contained.

For every po, let Λ_{po} be the set of all stratified extensions $q = B_1 \cdots B_k$ of po for which (∗) holds. We want to show that $|\Lambda_{po}| = 1$ by applying induction on the size of X.

If $X = \varnothing$ then $\Lambda_{po} = \{\varepsilon\}$ and the thesis holds. The induction step for $X \neq \varnothing$ follows from the following observation: Let B be the set of all minimal elements of po (i.e., $a \in B$ iff there is no b such that $b \prec a$), and let po' be po restricted to $X - B$. Then

(1) If $B_1 \cdots B_k \in \Lambda_{po}$ then $B_1 = B$ and $B_2 \cdots B_k \in \Lambda_{po'}$.

(2) If $B_1 \cdots B_k \in \Lambda_{po'}$ then $BB_1 \cdots B_k \in \Lambda_{po}$.

Both (1) and (2) follow directly from (∗) and the definition of a minimal element of a poset. Then (1) can be used to show that $|\Lambda_{po}| \leqslant 1$, and (2) to show that $|\Lambda_{po}| > 0$. The induction hypothesis can be applied for po' since $B \neq \varnothing$ as X is finite. ∎

The stratified poset q from Proposition 3.1 will be called *canonical* for po and denoted by can_{po}. In Fig. 2, $can_{po} = po''$.

3.2. *Stratified Order Structures*

By a *relational structure* we will mean a triple $S = (X, \prec, \sqsubset)$, where $\prec$ and $\sqsubset$ are binary relations on X. We will sometimes use $\prec_S$ and $\sqsubset_S$ to respectively denote $\prec$ and $\sqsubset$. A relational structure $S' = (X, \prec', \sqsubset')$ is an *extension* of S, $S \subseteq S'$, if $\prec \subseteq \prec'$ and $\sqsubset \subseteq \sqsubset'$. S is a *stratified order structure* if for all $a, b, c \in X$ the following hold:

C1 $\quad a \not\sqsubset a$

C2 $\quad a \prec b \Rightarrow a \sqsubset b$

C3 $\quad a \sqsubset b \sqsubset c \wedge a \neq c \Rightarrow a \sqsubset c$

C4 $\quad a \sqsubset b \prec c \vee a \prec b \sqsubset c \Rightarrow a \prec c.$

C1–C4 imply[5] that $(X, \prec)$ is a poset and $a \prec b \Rightarrow b \not\sqsubset a$. Figure 2 shows a stratified order structure S such that

[3] This is mainly due to the fact that we only define finite comtraces (see Section 4). A smooth extension to the infinite ones is a non-trivial problem, even for the restricted case of Mazurkiewicz traces [8], and is left as a topic for further research.

[4] If X is empty, $\mho_{po}$ is the empty sequence, ε.

[5] $a \not\prec a$ follows from C1 and C2; $a \prec b \prec c \Rightarrow a \prec c$ from C2 and C4; and $a \prec b \Rightarrow b \not\sqsubset a$ from C4 and $a \not\prec a$.

$a \prec b \sqsubset c$ and $a \prec c$. In the diagrams, solid arrows represent $\prec$ (causality) and dashed arrows represent $\sqsubset$ (weak causality). We do not draw a dashed arrow between a and b if both $a \prec b$ and $a \sqsubset b$ hold; arrows which can be deduced from those which are drawn using C3 and C4 can be omitted.

The relationship between stratified order structures and stratified posets is very much like that between partial orders and their total extensions (cf. Section 3.4). In particular, a poset $po = (X, \prec)$ is stratified if and only if $(X, \prec, \prec \cup \frown)$ is a stratified order structure [17].

Stratified order structures were independently introduced in [10] (as "prossets"—preorder specification sets) and in [14] (as "composets"—combined posets). In both [10] and [14] the defining axioms are slightly different from C1–C4, although equivalent.

Stratified order structures are a special case of (general) order structures, which were introduced and analysed in [17]. Besides stratified order structures and (general) order structures, Janicki and Koutny consider interval order structures and total order structures in [17]. Interval order structures are a refined version of the relational structures defined in [21]. For more details the reader is referred to [17]. In this paper we shall use only stratified order structures.

3.3. *Closure Properties of Relational Structures*

In this section we develop the notion of $\diamond$-closure of a relational structure. It roughly corresponds to the notion of transitive closure of a binary relation which is often used in the construction of partial orders. For a relational structure $S = (X, \prec, \sqsubset)$, its $\diamond$-closure will be defined as $(X, \prec', (\prec \cup \sqsubset)^* - id_X)$, where $\prec'$ is the composition of three relations: $(\prec \cup \sqsubset)^*$ and $\prec$ and (again) $(\prec \cup \sqsubset)^*$. In other words, $a \prec' b$ if $aR_1a_1R_2 \cdots R_{k-1}a_{k-1}R_kb$, where each R_i is $\prec$ or $\sqsubset$, and at least one R_i is $\prec$. The $\diamond$-closure will be used to construct stratified order structures.

Formally, the *$\diamond$-closure* of a relational structure $S = (X, \prec, \sqsubset)$ is defined as

$$S^{\diamond} = (X, \varrho \circ \prec \circ \varrho, \varrho - id_X)$$

where $\varrho = (\prec \cup \sqsubset)^*$. We will also use ϱ_S to denote ϱ. The terminology is justified by part (2) and (3) of the following result. (Below $S = (X, \prec, \sqsubset)$ is a fixed relational structure.)

PROPOSITION 3.2.

(1) $((\varrho \circ \prec \circ \varrho) \cup (\varrho - id_X))^* = \varrho^*$.

(2) *If* $\sqsubset$ *is irreflexive then* $S \subseteq S^{\diamond}$.

(3) $(S^{\diamond})^{\diamond} = S^{\diamond}$.

Proof. (1) The $\subseteq$ inclusion follows from $\varrho \circ \prec \circ \varrho \subseteq \varrho$. To show the $\supseteq$ inclusion we first observe that

$$\begin{aligned}\varrho - id_X &= (\prec \cup \sqsubset)^+ - id_X \\ &= (\prec \cup \sqsubset)^* \circ (\prec \cup \sqsubset) \circ (\prec \cup \sqsubset)^* - id_X \\ &= (\varrho \circ \prec \circ \varrho) \cup (\varrho \circ \sqsubset \circ \varrho) - id_X \\ &\subseteq (\varrho \circ \prec \circ \varrho) \cup \varrho - id_X \\ &\subseteq (\varrho \circ \prec \circ \varrho) \cup (\varrho - id_X).\end{aligned}$$

Hence $\varrho^* = (\varrho - id_X)^* \subseteq ((\varrho \circ \prec \circ \varrho) \cup (\varrho - id_X))^*$.

(2) We have $\prec = id_X \circ \prec \circ id_X \subseteq \varrho \circ \prec \circ \varrho$. Moreover, since $\sqsubset$ is irreflexive, $\sqsubset \subseteq \sqsubset - id_X \subseteq \varrho - id_X$.

(3) Follows from

$$\begin{aligned}\varrho_{S^\diamond} \circ \prec_{S^\diamond} \circ \varrho_{S^\diamond} = &\ ((\varrho \circ \prec \circ \varrho) \cup (\varrho - id_X))^* \\ &\circ \varrho \circ \prec \circ \varrho \circ ((\varrho \circ \prec \circ \varrho) \cup (\varrho - id_X))^* \\ =_{(1)} &\ \varrho^* \circ \varrho \circ \prec \circ \varrho \circ \varrho^* = \varrho \circ \prec \circ \varrho. \\ \varrho_{S^\diamond} - id_X = &\ ((\varrho \circ \prec \circ \varrho) \cup (\varrho - id_X))^* - id_X \\ =_{(1)} &\ \varrho^* - id_X = \varrho - id_X. \quad \blacksquare\end{aligned}$$

A necessary and sufficient condition for $S^{\diamond}$ to be a stratified order structure is given next.

PROPOSITION 3.3. *$S^{\diamond}$ is a stratified order structure if and only if $\varrho \circ \prec \circ \varrho$ is irreflexive.*

Proof. ($\Rightarrow$) Follows from C1 and C2.

($\Leftarrow$) C1 clearly holds, and C2 follows from $\varrho \circ \prec \circ \varrho$ being irreflexive. Moreover, C3 and C4 follow from

$$\begin{aligned}((\varrho - id_X) \circ (\varrho - id_X)) - id_X &\subseteq \varrho - id_X \\ (\varrho - id_X) \circ (\varrho \circ \prec \circ \varrho) &\subseteq \varrho \circ \prec \circ \varrho \\ (\varrho \circ \prec \circ \varrho) \circ (\varrho - id_X) &\subseteq \varrho \circ \prec \circ \varrho. \quad \blacksquare\end{aligned}$$

In the rest of this section we prove some useful properties of the $\diamond$-closure.

PROPOSITION 3.4. *If S is a stratified order structure then* $S = S^{\diamond}$.

Proof. Follows from $(\prec \cup \sqsubset)^* \circ \prec \circ (\prec \cup \sqsubset)^* =_{C2} \sqsubset^* \circ \prec \circ \sqsubset^* =_{C4} \prec$ and $(\prec \cup \sqsubset)^* - id_X =_{C2} \sqsubset^* - id_X =_{C3} \sqsubset - id_X =_{C1} \sqsubset$. ∎

PROPOSITION 3.5. *If $S = (X, \prec, \sqsubset)$ is a relational structure such that $S^{\diamond}$ is a stratified order structure, and $S_0 = (X, \prec_{S_0}, \sqsubset_{S_0})$ is a relational structure such that*

$$\prec_{S_0} \subseteq \varrho \circ \prec \circ \varrho \quad \text{and} \quad \sqsubset_{S_0} \subseteq \varrho - id_X,$$

then $S_0^{\diamond}$ is a stratified order structure satisfying $S_0^{\diamond} \subseteq S^{\diamond}$.

Proof. We have

$$\varrho_{S_0} \circ \prec_{S_0} \circ \varrho_{S_0} \subseteq \quad ((\varrho \circ \prec \circ \varrho) \cup (\varrho - id_X))^* \circ \varrho \circ \prec \circ \varrho \circ ((\varrho \circ \prec \circ \varrho) \cup (\varrho - id_X))^*$$
$$=_{3.2(1)} \varrho^* \circ \varrho \circ \prec \circ \varrho \circ \varrho^* \;=\; \varrho \circ \prec \circ \varrho.$$
$$\varrho_{S_0} - id_X \subseteq \quad ((\varrho \circ \prec \circ \varrho) \cup (\varrho - id_X))^* - id_X$$
$$=_{3.2(1)} \varrho^* - id_X = \varrho - id_X.$$

Using (twice) Proposition 3.3, the first inclusion and the assumption made about $S^\diamond$, we obtain that $S_0^\diamond$ is a stratified order structure. $S_0^\diamond \subseteq S^\diamond$ follows from both inclusions. ▮

The next result shows two different ways of augmenting a stratified order structure. Note that the S in Fig. 2 could be augmented by adding either $b \prec c$ or $c \sqsubset b$.

Proposition 3.6. *Let S be a stratified order structure and $a, b \in X$, $a \neq b$.*

(1) *If $a \not\sqsubset b$ then $T^\diamond$ is a stratified order structure, where*

$$T = (X, \prec \cup \{(b, a)\}, \sqsubset).$$

(2) *If $a \not\prec b$ then $T^\diamond$ is a stratified order structure, where*

$$T = (X, \prec, \sqsubset \cup \{(b, a)\}).$$

Moreover, in both cases $S \subseteq T^\diamond$.

Proof. (1) By Proposition 3.3, it suffices to show that $\varrho_T \circ \prec_T \circ \varrho_T$ is irreflexive. From C2 for S it follows that

(a) $\varrho_T \circ \prec_T \circ \varrho_T = (\sqsubset \cup \{(b, a)\})^* \circ (\prec \cup \{(b, a)\}) \circ (\sqsubset \cup \{(b, a)\})^*$.

(b) $\varrho \circ \prec \circ \varrho = \sqsubset^* \circ \prec \circ \sqsubset^*$.

Suppose $(c, c) \in \varrho_T \circ \prec_T \circ \varrho_T$. Since $(c, c) \notin \sqsubset^* \circ \prec \circ \sqsubset^*$ (follows from (b) and Propositions 3.3 and 3.4), we must have, by (a) and C2 for S, $(c, b) \in \sqsubset^*$ and $(a, c) \in \sqsubset^*$. Hence, by $a \neq b$, $(a, b) \in \sqsubset^* - id_X$. Since, by Proposition 3.4, $\sqsubset = (\prec \cup \sqsubset)^* - id_X = \sqsubset^* - id_X$, this yields $a \sqsubset b$, a contradiction.

(2) We proceed as before and obtain

(c) $\varrho_T \circ \prec_T \circ \varrho_T = (\sqsubset \cup \{(b,a)\})^* \circ \prec \circ (\sqsubset \cup \{(b,a)\})^*$.

Suppose $(c, c) \in \varrho_T \circ \prec_T \circ \varrho_T$. Since $(c, c) \notin \sqsubset^* \circ \prec \circ \sqsubset^*$ at least one of the following three cases holds:

$(c, b) \in \sqsubset^*$ and
$(a, b) \in \sqsubset^* \circ \prec \circ \sqsubset^*$ and
$(a, c) \in \sqsubset^*$,

or $(c, b) \in \sqsubset^*$ and
$(a, c) \in \sqsubset^* \circ \prec \circ \sqsubset^*$,

or $(c, b) \in \sqsubset^* \circ \prec \circ \sqsubset^*$ and
$(a, c) \in \sqsubset^*$.

In either case, $(a, b) \in \sqsubset^* \circ \prec \circ \sqsubset^*$.

Since, by Proposition 3.4, $\prec = (\prec \cup \sqsubset)^* \circ \prec \circ (\prec \cup \sqsubset)^* = \sqsubset^* \circ \prec \circ \sqsubset^*$, we obtain $a \prec b$, a contradiction.

Finally, by Proposition 3.4, $S = S^\diamond$. Moreover, by the definition of the $\diamond$-closure, $S^\diamond \subseteq T^\diamond$. Hence $S \subseteq T^\diamond$. ▮

The last result provides a means of proving equality of stratified order structures obtained through simple composition.

Proposition 3.7. *Let S_1 and S_2 (resp. T_1 and T_2) be relational structures with the same domain X (resp. Y). Moreover, let $X \cap Y = \varnothing$, $\prec \subseteq X \times Y$ and $\sqsubset \subseteq X \times Y$. Define*

$$W_i = (X \cup Y, \prec \cup \prec_{S_i} \cup \prec_{T_i}, \sqsubset \cup \sqsubset_{S_i} \cup \sqsubset_{T_i}) \quad (i = 1, 2).$$

If $S_1^\diamond = S_2^\diamond$ and $T_1^\diamond = T_2^\diamond$ then $W_1^\diamond = W_2^\diamond$.

Proof. From $X \cap Y = \prec \cap (Y \times X) = \sqsubset \cap (Y \times X) = \varnothing$ it follows that

(1) $\varrho_{W_i} \cap (Y \times X) = \varnothing$ for $i = 1, 2$.

Moreover, by $S_1^\diamond = S_2^\diamond$ and $T_1^\diamond = T_2^\diamond$, we have

(2) $\varrho_{S_1} = \varrho_{S_2}$ and $\varrho_{T_1} = \varrho_{T_2}$.

(3) $\varrho_{S_1} \circ \prec_{S_1} \circ \varrho_{S_1} = \varrho_{S_2} \circ \prec_{S_2} \circ \varrho_{S_2}$ and $\varrho_{T_1} \circ \prec_{T_1} \circ \varrho_{T_1} = \varrho_{T_2} \circ \prec_{T_2} \circ \varrho_{T_2}$.

From (1) we obtain, for $i = 1, 2$,

$$\varrho_{W_i} \circ \prec_{W_i} \circ \varrho_{W_i} = (\varrho_{S_i} \circ \prec_{S_i} \circ \varrho_{S_i}) \cup (\varrho_{T_i} \circ \prec_{T_i} \circ \varrho_{T_i}) \cup (\varrho_{S_i} \circ \prec_{S_i} \circ \varrho_{S_i} \circ (\prec \cup \sqsubset) \circ \varrho_{T_i}) \cup (\varrho_{S_i} \circ (\prec \cup \sqsubset) \circ \varrho_{T_i} \circ \prec_{T_i} \circ \varrho_{T_i}) \cup (\varrho_{S_i} \circ \prec \circ \varrho_{T_i})$$

$$\varrho_{W_i} = \varrho_{S_i} \cup \varrho_{T_i} \cup (\varrho_{S_i} \circ (\prec \cup \sqsubset) \circ \varrho_{T_i}).$$

This and (2, 3) implies $W_1^\diamond = W_2^\diamond$. ▮

3.4. *Stratified Order Structures and Stratified Posets*

As we already mentioned, the relationship between stratified order structures and stratified posets is very much like that between partial orders and their total extensions.

Let $S = (X, \prec, \sqsubset)$ be a stratified order structure, and let $po = (X, \prec')$ be a stratified poset. Then po is a *stratified (poset) extension* of S, $po \in strat(S)$, if $\prec \subseteq \prec'$ and

$\sqsubset \subseteq \prec' \cup \frown'$.[6] Proposition 3.8 shows that $strat(S)$ is always non-empty.

PROPOSITION 3.8. *If $S = (X, \prec, \sqsubset)$ is a stratified order structure then $can_{(X,\prec)} \in strat(S)$.*

Proof. Let $can_{(X,\prec)} = (X, \prec')$ and $\mho_{(X,\prec')} = B_1 \cdots B_k$. By definition, $\prec \subseteq \prec'$. What we need to show is that $\sqsubset \subseteq \prec' \cup \frown'$. Suppose $a \sqsubset b$ (hence $a \neq b$) and $a \not\prec' b$ and $a \not\frown' b$. Then $b \prec' a$, which means that $a \in B_i$ for some $i \geqslant 2$. Hence, by (*) in Proposition 3.1, there is $c \in B_{i-1}$ such that $c \prec a$. Thus, by $a \sqsubset b$ and C4, $c \prec b$, which yields $c \prec' b$. Hence $b \in B_j$ for some $j \geqslant i$, contradicting $b \prec' a$ and $a \in B_i$. ∎

PROPOSITION 3.9 [16]. *Let $S = (X, \prec, \sqsubset)$ be a stratified order structure and let $a, b \in X$, $a \neq b$.*

(1) *If $a \not\sqsubset b$ then there is $po \in strat(S)$ such that $b \prec_{po} a$.*

(2) *If $a \not\prec b$ then there is $po \in strat(S)$ such that $b \prec_{po} a$ or $b \frown_{po} a$.*

Proof. Follows directly from Propositions 3.6 and 3.8 and the fact that if S' and S'' are two stratified order structures satisfying $S'' \subseteq S'$, then $strat(S') \subseteq strat(S'')$. ∎

Every poset is unambiguously identified by the set of its total order extensions [30]. The same does not hold for stratified order structures. It may even happen that a stratified order structure has no total order extensions (for example, $S = (\{a, b\}, \varnothing, \{(a, b), (b, a)\}))$. To distinguish between different stratified order structures one needs to compare their stratified order extensions.

THEOREM 3.10 [16]. *Let S and S' be stratified order structures.*
Then $S = S'$ if and only if $strat(S) = strat(S')$.

Proof. To prove the right-to-left implication we assume $a \prec_S b$ and $a \not\prec_{S'} b$. Then, by Proposition 3.9(2), there is $po \in strat(S')$ such that $b \prec_{po} a$ or $b \frown_{po} a$ which means $po \notin strat(S)$. Hence $strat(S) \neq strat(S')$. Similarly, using Proposition 3.9(1) we may show that if $a \sqsubset_S b$ and $a \not\sqsubset_{S'} b$ then $strat(S) \neq strat(S')$. Hence $strat(S) = strat(S')$ implies $S \subseteq S'$ and (by symmetry) $S' \subseteq S$. ∎

Every poset can be reconstructed by taking the intersection of its total order extensions [30]. A similar result holds for stratified order structures and their stratified order extensions.

THEOREM 3.11 [16]. *Let $S = (X, \prec, \sqsubset)$ be a stratified order structure. Then*

$$S = \left(X, \bigcap_{po \in strat(S)} \prec_{po}, \bigcap_{po \in strat(S)} \prec_{po} \cup \frown_{po}\right).$$

[6] An equivalent definition of $strat(S)$ is obtained by requiring that if $\mho_{po} = B_1 \cdots B_k$ then $a \in B_i$ and $b \in B_j$ imply that $a \prec b \Rightarrow i < j$ and $a \sqsubset b \Rightarrow i \leqslant j$.

Proof. The theorem is correctly formulated since, by Proposition 3.8, $strat(S) \neq \varnothing$. From the definition of $strat(S)$ it follows that the $\subseteq$ inclusion holds.

Suppose $a, b \in X$, $a \neq b$ and $a \not\prec b$ (or $a \not\sqsubset b$). Then, by Proposition 3.9, there is $po \in strat(S)$ such that $b \prec_{po} a$ or $a \frown_{po} b$ (resp. $b \prec_{po} a$). Hence $a \not\prec_{po} b$ (resp. $a \not\prec_{po} b$ and $a \not\frown_{po} b$), which means that the $\supseteq$ inclusion holds. ∎

A similar theorem holds also for infinite structures [17].

4. GENERALISING MAZURKIEWICZ TRACES

In trace theory [1, 22, 23] partial orders are interpreted as abstract behaviours of concurrent systems such as EN-systems and 1-safe Petri nets. In this section a class of stratified order structures will be introduced to provide a representation of abstract behaviours of EN-systems with inhibitor arcs under the a-priori semantics (see Section 5).

4.1. *Simultaneity and Serialisability*

A *concurrent alphabet* is a triple $\vartheta = (E, sim, ser)$, where E is a non-empty set of events and $ser \subseteq sim \subset E \times E$. We assume that sim is irreflexive and symmetric. Intuitively, if $(e, f) \in sim$ then e and f can occur simultaneously, while $(e, f) \in ser$ means that e and f may occur simultaneously and e may occur before f. We interpret sim and ser as simultaneity and serialisability.

In what follows the concurrent alphabet is fixed. In the examples we use $\vartheta_1 = (E, sim, ser)$, where $E = \{e, f, g, h\}$, $sim = \{(e, f), (f, e), (e, g), (g, e)\}$ and $ser = \{(e, f), (f, e), (e, g)\}$.

In the standard treatment of Mazurkiewicz traces, a concurrent alphabet is defined as $\vartheta = (E, ind)$, where ind is a symmetric and irreflexive *independence* relation on events. Intuitively, independent events can be executed simultaneously and are serialisable. Thus ind corresponds in our presentation to the situation where $sim = ser = ind$ (note that this implies $ser = ser^{-1}$).

4.2. *Step Sequences and Comtraces*

A non-empty finite set $A \subseteq E$ is a *step* if for all distinct $e, f \in A$, $(e, f) \in sim$. For our example concurrent alphabet ϑ_1, the possible steps are $\{e\}$, $\{f\}$, $\{g\}$, $\{h\}$, $\{e, f\}$, and $\{e, g\}$. Finite sequences of steps, called *step sequences*, are meant to represent possible runs of a concurrent system. Abstract histories will be represented by equivalence classes of step sequences.

Let $\approx$ be the relation comprising all pairs (t, u) of step sequences such that

$$t = wAz$$

$$u = wBCz,$$

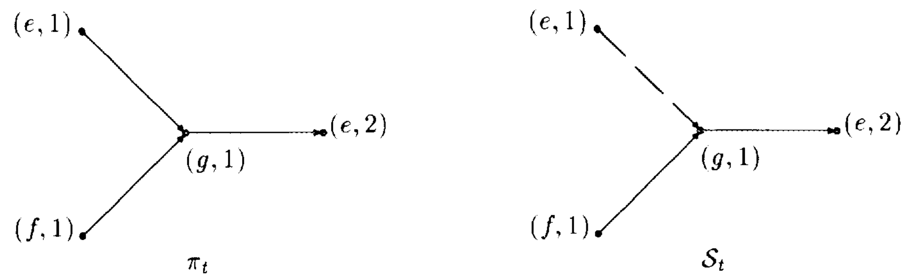

FIG. 3. Poset and stratified order structure generated by a step sequence.

where w, z are step sequences (possibly empty) and A, B, C are steps satisfying

$$B \cap C = \varnothing$$
$$B \cup C = A$$
$$B \times C \subseteq ser.$$

Note that A can only be split into two consecutive steps, B and C, if the events in B are all serialisable with those in C. For the example alphabet ϑ_1,

$$\{f\}\{e,g\} \approx \{f\}\{e\}\{g\}$$
$$\{f\}\{e,g\} \not\approx \{f\}\{g\}\{e\}.$$

The property $\approx$ is supposed to capture is that if $t \approx u$ then t is a valid run of the system if and only if u is. The reflexive symmetric transitive closure of $\approx$ will be denoted $\equiv$ (i.e. $\equiv$ is equal to $(\approx \cup \approx^{-1})^*$) and its equivalence classes will be called *comtraces*.[7] A comtrace containing a given step sequence t will be denoted $[t]$ and interpreted as an abstract history of a concurrent system. For the alphabet ϑ_1,

$$[\{e,f\}\{g\}\{e\}] = \{\{e,f\}\{g\}\{e\}, \{e\}\{f\}\{g\}\{e\}, \{f\}\{e\}\{g\}\{e\}, \{f\}\{e,g\}\{e\}\}.$$

The original notion of Mazurkiewicz trace [22] deals only with sequences of events. It has been generalised in [13] to step sequences which roughly corresponds to assuming that $sim = ser$ in the above definitions.

4.3. *Posets Generated by Step Sequences*

Let $t = A_1 \cdots A_k$ be a step sequence. The set of *event occurrences* in t is defined as

$$occ(t) = \{(e, i) \mid e \in E \wedge 1 \leqslant i \leqslant \#_e(t)\},$$

[7] The problem of deciding whether $t \equiv u$ can be reduced to that of checking whether the stratified order structures induced by t and u (and constructed using the $\diamond$-closure) are equal (cf. Theorem 4.10).

where $\#_e(t) = |\{i \mid e \in A_i\}|$, for all $e \in E$. For $\alpha = (e, i) \in occ(t)$, the *position* within t and *label* are defined respectively by $pos_t(\alpha) = \min\{j \mid \#_e(A_1 \cdots A_j) = i\}$ and $l(\alpha) = e$. For ϑ_1 and $t = \{e, f\}\{g\}\{e\}$ we have $occ(t) = \{(e, 1), (e, 2), (f, 1), (g, 1)\}$, $pos_t(e, 2) = 3$ and $l(f, 1) = f$.

The step sequence t induces a partial order, $\pi_t = (occ(t), \prec)$, where

$$\alpha \prec \beta \Leftrightarrow pos_t(\alpha) < pos_t(\beta).$$

For ϑ_1 and $t = \{e, f\}\{g\}\{e\}$, the Hasse diagram of π_t is shown in Fig. 3.

Directly from the definition of $\approx$ we obtain:

PROPOSITION 4.1. *Let $t = A_1 \cdots A_k$ be a step sequence and $\alpha, \beta \in occ(t)$.*

(1) *π_t is a stratified poset and $\mho_{\pi_t} = pos_t^{-1}(1) \cdots pos_t^{-1}(k)$.*

(2) *Let $\mho_{\pi_t} = B_1 \cdots B_k$. Then $l(B_i) = A_i$ and $|B_i| = |A_i|$, for all i.*

(3) *If u is a step sequence such that $t \approx u$ or $u \approx t$ then*

$$|(pos_t(\alpha) - pos_t(\beta)) - (pos_u(\alpha) - pos_u(\beta))| \leqslant 1.$$

The next result states two "invariant" properties concerning the relative position of two event occurrences in step sequences belonging to the same comtrace.

PROPOSITION 4.2. *Let t be a step sequence and let $\alpha, \beta \in occ(t)$.*

(1) *If $pos_t(\alpha) < pos_t(\beta)$ and $(l(\alpha), l(\beta)) \notin ser$ then $pos_u(\alpha) < pos_u(\beta)$ for all $u \in [t]$.*

(2) *If $pos_t(\alpha) \leqslant pos_t(\beta)$ and $(l(\beta), l(\alpha)) \notin ser$ then $pos_u(\alpha) \leqslant pos_u(\beta)$ for all $u \in [t]$.*

Proof. If (1) does not hold, then there are $u, w \in [t]$ such that $pos_u(\alpha) < pos_u(\beta)$, $pos_w(\alpha) \geqslant pos_w(\beta)$, and $u \approx w$ or $w \approx u$. Hence, by Proposition 4.1(3), $pos_u(\alpha) = pos_u(\beta) - 1$ and $pos_w(\alpha) = pos_w(\beta)$. From the definition of $\approx$ it then follows that $w \approx u$ and $(l(\alpha), l(\beta)) \in ser$, a contradiction.

(2) can be proved in a similar way. ∎

4.4. *Stratified Order Structures Generated by Step Sequences*

In trace theory it is possible to construct, for every sequence of events τ, a partial order whose linearisations correspond to the sequences in the trace comprising τ. In the model we are developing now, a similar construction is possible. This time, however, for a step sequence t we are going to construct a stratified order structure, $\mathscr{S}_t$. Its definition is based on Proposition 4.2 which expresses two basic invariant properties of non-serialisable event occurrences. Part (1) captures a situation that α *always precedes* β, and part (2) captures a situation that α *never follows* β, in the sequences belonging to a comtrace. We turn these into the following definition:

Let t be a step sequence and $\alpha, \beta \in occ(t)$, $\alpha \neq \beta$. Then

$$\alpha \prec_t \beta \Leftrightarrow (l(\alpha), l(\beta)) \notin ser \wedge pos_t(\alpha) < pos_t(\beta)$$

$$\alpha \sqsubset_t \beta \Leftrightarrow (l(\beta), l(\alpha)) \notin ser \wedge pos_t(\alpha) \leqslant pos_t(\beta).$$

The stratified order structure *induced* by t is then defined as $\mathscr{S}_t = (occ(t), \prec_t, \sqsubset_t)^{\diamond}$. Proposition 4.4 shows that $\mathscr{S}_t$ is indeed a stratified order structure. For ϑ_1 and $t = \{e, f\}\{g\}\{e\}$, $\mathscr{S}_t$ is shown in Fig. 3.

PROPOSITION 4.3. *Let* $\mathscr{S}_t = (occ(t), \prec, \sqsubset)$ *and* $\alpha, \beta \in occ(t)$.

(1) *If* $\alpha \prec \beta$ *then* $pos_t(\alpha) < pos_t(\beta)$.

(2) *If* $\alpha \sqsubset \beta$ *then* $pos_t(\alpha) \leqslant pos_t(\beta)$.

(3) *If* $l(\alpha) = l(\beta)$ *and* $pos_t(\alpha) < pos_t(\beta)$ *then* $\alpha \prec \beta$.

Proof. (1, 2) Follow directly from the definition of $\prec_t$, $\sqsubset_t$, and the $\diamond$-closure.

(3) Since ser is irreflexive, $\alpha \prec_t \beta$. Hence $\alpha \prec \beta$. ∎

PROPOSITION 4.4. $\mathscr{S}_t$ *is a stratified order structure and* $\pi_t \in strat(\mathscr{S}_t)$.

Proof. Let $(\alpha, \beta) \in (\prec_t \cup \sqsubset_t)^* \circ \prec_t \circ (\prec_t \cup \sqsubset_t)^*$. From the definition of $\prec_t$ and $\sqsubset_t$, $pos_t(\alpha) < pos_t(\beta)$. Hence, by Proposition 3.3, $\mathscr{S}_t$ is a stratified order structure. $\pi_t \in strat(\mathscr{S}_t)$ follows directly from Proposition 4.3(1, 2). ∎

If $ser = sim$ then $\mathscr{S}_t$ is de facto a poset.

PROPOSITION 4.5. *If* $ser = sim$ *and* $\mathscr{S}_t = (occ(t), \prec, \sqsubset)$ *then* $\prec = \sqsubset = \prec_t^+$.

Proof. By $ser = sim$, $\prec_t = \sqsubset_t$ since ser is symmetric and, furthermore, $a \neq b$ and $(l(\beta), l(\alpha)) \notin ser$ implies $pos_t(\alpha) \neq pos_t(\beta)$.

Hence $\prec = \prec_t^* \circ \prec_t \circ \prec_t^* = \prec_t^+$ and $\sqsubset = \prec_t^* - id_{occ(t)} = \prec_t^+$. ∎

4.5. *Comtraces and Stratified Order Structures*

In this section we prove some basic properties of comtraces and the corresponding stratified order structures. First we show that the stratified extensions of $\mathscr{S}_t$ are all induced by step sequences.

PROPOSITION 4.6. *Let* t *be a step sequence and* $\pi \in strat(\mathscr{S}_t)$. *Then there is a step sequence* u *such that* $\pi = \pi_u$.

Proof. Let $\pi = (occ(t), \prec')$, $\mathscr{S}_t = (occ(t), \prec, \sqsubset)$ and $\mho_\pi = B_1 \cdots B_k$. We will show that $u = l(B_1) \cdots l(B_k)$ is a step sequence such that $\pi = \pi_u$.

Suppose α and β are distinct elements in B_i such that $(l(\alpha), l(\beta)) \notin sim$. Then $(l(\alpha), l(\beta)) \notin ser$ and $(l(\beta), l(\alpha)) \notin ser$. If $pos_t(\alpha) \neq pos_t(\beta)$ then $\alpha \prec_t \beta$ or $\beta \prec_t \alpha$, which implies $\alpha \prec \beta$ or $\beta \prec \alpha$. This and $\pi \in strat(\mathscr{S}_t)$ implies $\alpha \prec' \beta$ or $\beta \prec' \alpha$, contradicting $\alpha, \beta \in B_i$. Hence $pos_t(\alpha) = pos_t(\beta)$, which is impossible, since t is a step sequence. We have thus proved that for all $i \leqslant k$,

(1) $\alpha, \beta \in B_i \wedge \alpha \neq \beta \Rightarrow (l(\alpha), l(\beta)) \in sim$.

Moreover, by Proposition 4.3(3), if $(e, i), (e, i+1) \in occ(t)$ then $(e, i) \prec (e, i+1)$, which implies

(2) If $(e, k_0) \in B_k$ and $(e, m_0) \in B_m$ then: $k_0 < m_0 \Leftrightarrow k < m$.

From (1) it follows that u is a step sequence and (since sim is irreflexive) $|l(B_i)| = |B_i|$ for all i. Hence $occ(u) = occ(t)$ and, by (2), $pos_u^{-1}(i) = B_i$ for all i. Thus, by Proposition 4.1(1), $\mho_{\pi_u} = \mho_\pi$. Hence $\pi_u = \pi$. ∎

Theorem 4.10 below states that two step sequences induce the same stratified order structures if and only if they belong to the same comtrace. To prove this we need three auxiliary results.

LEMMA 4.7. *Let* t *and* u *be step sequences such that* $t \equiv u$. *Then* $\mathscr{S}_t = \mathscr{S}_u$.

Proof. We first show that if $t \approx u$ (or $u \approx t$) then $\prec_t \subseteq \prec_u$ and $\sqsubset_u \subseteq \sqsubset_t$. It is easy to see that if this does not hold then there are α and β such that $pos_t(\alpha) < pos_t(\beta)$, $pos_u(\beta) \leqslant pos_u(\alpha)$ and $(l(\alpha), l(\beta)) \notin ser$. Thus, by Proposition 4.1(3), $pos_u(\alpha) = pos_u(\beta)$. Hence, by Proposition 4.2(2), $pos_t(\alpha) \geqslant pos_t(\beta)$, a contradiction.

From what we have just proved it follows that if $t \approx u$ then $\prec_t \subseteq \prec_u$ and $\sqsubset_u \subseteq \sqsubset_t$, and (by reversing the roles of t and u and taking the "or" part) $\prec_u \subseteq \prec_t$ and $\sqsubset_t \subseteq \sqsubset_u$. Hence $\mathscr{S}_t = \mathscr{S}_u$. Thus the thesis follows since $\equiv$ is the transitive symmetric reflexive closure of $\approx$. ∎

A step sequence $t = A_1 \cdots A_k$ is *canonical* if for all $i \geqslant 2$ there is no step $B \subseteq A_i$ satisfying $A_{i-1} \times B \subseteq ser$ and $B \times (A_i - B) \subseteq ser$. Note that $B = A_i$ is allowed but $B = \varnothing$ is not. If $sim = ser$ then the above definition reduces to that in [5].

PROPOSITION 4.8. *Let t be a canonical step sequence and* $\mathcal{S}_t = (occ(t), \prec, \sqsubset)$. *Then* $\pi_t = can_{(occ(t), \prec)}$.

Proof. Let $t = A_1 \cdots A_k$. By Proposition 4.4, $\pi_t \in strat(\mathcal{S}_t)$, so π_t is an extension of $(occ(t), \prec)$. Let $\mho_{\pi_t} = B_1 \cdots B_k$. By Proposition 3.1, it suffices to show that for every $i \geq 2$ and every $\beta \in B_i$ there is $\alpha \in B_{i-1}$ such that $\alpha \prec \beta$. If this does not hold then

$$B = \{\beta \in B_i \mid (\forall \alpha \in B_{i-1})\, \alpha \not\prec \beta\} \neq \emptyset$$

for some $i \geq 2$. From the definition of $\mathcal{S}_t$, $A_{i-1} \times l(B) \subseteq ser$. Suppose there is $\alpha \in B$ and $\beta \in B_i - B$ such that $(l(\alpha), l(\beta)) \notin ser$. Then $\beta \sqsubset \alpha$. Moreover, by the definition of B, $\gamma \prec \beta$ for some $\gamma \in B_{i-1}$. Hence $\gamma \prec \alpha$, contradicting the definition of B. Thus we have $l(B) \times (A_i - l(B)) \subseteq ser$ and $A_{i-1} \times l(B) \subseteq ser$, contradicting t being canonical. ∎

PROPOSITION 4.9. *For every step sequence t there is a canonical step sequence u such that* $t \equiv u$.

Proof. For every step sequence $u = A_1 \cdots A_k$, let $\mu(u) = 1 \cdot |A_1| + \cdots + k \cdot |A_k|$. There is $u \in [t]$ such that $\mu(u) \leq \mu(w)$ for all $w \in [t]$. Suppose $u = A_1 \cdots A_k$ is not canonical. Then there is $i \geq 2$ and a step $B \subseteq A_i$ such that

$$A_{i-1} \times B \subseteq ser$$
$$B \times (A_i - B) \subseteq ser.$$

If $B = A_i$ then $w \approx u$ and $\mu(w) < \mu(u)$, where

$$w = A_1 \cdots A_{i-2}(A_{i-1} \cup A_i)\, A_{i+1} \cdots A_k.$$

If $B \neq A_i$ then $w \approx z$ and $u \approx z$ and $\mu(w) < \mu(u)$, where

$$z = A_1 \cdots A_{i-2} A_{i-1} B (A_i - B)\, A_{i+1} \cdots A_k$$
$$w = A_1 \cdots A_{i-2}(A_{i-1} \cup B)(A_i - B)\, A_{i+1} \cdots A_k.$$

In either case we obtain a contradiction with the choice of u. ∎

Again, for $sim = ser$ the above proposition corresponds to a result of [5].

THEOREM 4.10. *Let t and u be step sequences. Then* $\mathcal{S}_t = \mathcal{S}_u$ *if and only if* $t \equiv u$.

Proof. By Lemma 4.7, we only need to show the left-to-right implication. From Proposition 4.9 it follows that there are canonical step sequences t', u' such that $t \equiv t'$ and $u \equiv u'$. By Lemma 4.7, $\mathcal{S}_{t'} = \mathcal{S}_{u'}$. This and Proposition 4.8 yields $\pi_{t'} = \pi_{u'}$. Thus $t' = u'$ and, consequently, $t \equiv u$. ∎

We end this section proving two other major results. Theorem 4.12 says that the stratified extensions of $\mathcal{S}_t$ are exactly those generated by step sequences in $[t]$. Theorem 4.13 says that the stratified order structure induced by a comtrace is uniquely identified by any of its stratified extensions.

LEMMA 4.11. *Let t and u be step sequences and* $\pi_u \in strat(\mathcal{S}_t)$. *Then* $\mathcal{S}_u = \mathcal{S}_t$.

Proof. Let $\mathcal{S}_t = (occ(t), \prec, \sqsubset)$, $\pi_u = (occ(u), \prec')$ and $\alpha, \beta \in occ(t) = occ(u)$. We have

$$\alpha \prec_t \beta \Rightarrow \alpha \prec \beta$$
$$\Rightarrow \alpha \prec' \beta$$
$$\Rightarrow pos_u(\alpha) < pos_u(\beta)$$
$$\alpha \prec_t \beta \Rightarrow (l(\alpha), l(\beta)) \notin ser.$$

which means that $\alpha \prec_t \beta$ implies $\alpha \prec_u \beta$.

Suppose $\alpha \sqsubset_t \beta$. Then $(l(\beta), l(\alpha)) \notin ser$ and $\alpha \sqsubset \beta$. The latter implies $(\alpha, \beta) \in \prec' \cup \frown'$. Hence $(l(\beta), l(\alpha)) \notin ser$ and $pos_u(\alpha) \leq pos_u(\beta)$ which yields $\alpha \sqsubset_u \beta$. Consequently, $\prec_t \subseteq \prec_u$ and $\sqsubset_t \subseteq \sqsubset_u$ which means that $\mathcal{S}_t \subseteq \mathcal{S}_u$.

Suppose $\alpha \prec_u \beta$ and $\alpha \not\prec_t \beta$. By $\alpha \prec_u \beta$, $(l(\alpha), l(\beta)) \notin ser$. This and $\alpha \not\prec_t \beta$ means that $pos_t(\alpha) \geq pos_t(\beta)$. Hence $\beta \sqsubset_t \alpha$ (and so $\beta \sqsubset \alpha$), producing a contradiction with $\pi_u \in strat(\mathcal{S}_t)$ and $\alpha \prec' \beta$ (which follows from $\alpha \prec_u \beta$).

Suppose $\alpha \sqsubset_u \beta$ and $\alpha \not\sqsubset_t \beta$. By $\alpha \sqsubset_u \beta$, $(l(\beta), l(\alpha)) \notin ser$. This and $\alpha \not\sqsubset_t \beta$ means that $pos_t(\beta) < pos_t(\alpha)$. Hence $\beta \prec_t \alpha$, producing a contradiction with $\pi_u \in strat(\mathcal{S}_t)$ and $\beta \not\prec' a$ (follows from $\alpha \sqsubset_u \beta$).

Thus $\mathcal{S}_u \subseteq \mathcal{S}_t$. ∎

THEOREM 4.12. *Let t be a step sequence. Then* $strat(\mathcal{S}_t) = \{\pi_u \mid u \in [t]\}$.

Proof. If $u \in [t]$ then, by Theorem 4.10, $\mathcal{S}_u = \mathcal{S}_t$. This and Proposition 4.4 yields $\pi_u \in strat(\mathcal{S}_t)$. Hence the $\supseteq$ inclusion holds.

Suppose $\pi \in strat(\mathcal{S}_t)$. By Proposition 4.6, there is a step sequence u such that $\pi = \pi_u$. Thus, by Lemma 4.11, $\mathcal{S}_u = \mathcal{S}_t$. This and Theorem 4.10 yields $t \equiv u$. Hence the $\subseteq$ inclusion also holds. ∎

THEOREM 4.13. *Let t and u be step sequences such that* $strat(\mathcal{S}_t) \cap strat(\mathcal{S}_u) \neq \emptyset$. *Then* $t \equiv u$.

Proof. Let $\pi \in strat(\mathcal{S}_t) \cap strat(\mathcal{S}_u)$. By Proposition 4.6, there is a step sequence w such that $\pi = \pi_w$. By Lemma 4.11, $\mathcal{S}_u = \mathcal{S}_w = \mathcal{S}_t$. This and Theorem 4.10 yields $t \equiv u$. ∎

Comtraces form a monoid with the identity $[\lambda]$, where λ is the empty step sequence, and the monoidal operation $\odot$ defined by $[t] \odot [u] = [tu]$. That $\odot$ is well defined follows from the following result.

PROPOSITION 4.14. *Let* t_1 *and* u_1 *be step sequences. Then* $[t_1 u_1] = [t_2 u_2]$, *for all* $t_2 \in [t_1]$ *and* $u_2 \in [u_1]$.

Proof. By Theorem 4.10, it suffices to show $\mathscr{S}_{t_1u_1} = \mathscr{S}_{t_2u_2}$. Let $X = occ(t_1) = occ(t_2)$ and $Y = occ(t_1u_1) - occ(t_1) = occ(t_2u_2) - occ(t_2)$. Define for $i = 1, 2$:

$$S_i = (X, \prec_{t_iu_i}|_{X \times X}, \sqsubset_{t_iu_i}|_{X \times X})$$
$$T_i = (Y, \prec_{t_iu_i}|_{Y \times Y}, \sqsubset_{t_iu_i}|_{Y \times Y}).$$

Moreover, let $\prec = \prec_{t_1u_1}|_{X \times Y}$ and $\sqsubset = \sqsubset_{t_1u_1}|_{X \times Y}$. We observe that:

(1) $\prec = \prec_{t_2u_2}|_{X \times Y}$ and $\sqsubset = \sqsubset_{t_2u_2}|_{X \times Y}$.

(2) $X \cap Y = \varnothing$.

(3) For $i = 1, 2$, $\prec_{t_iu_i} = \prec \cup \prec_{t_iu_i}|_{X \times X} \cup \prec_{t_iu_i}|_{Y \times Y}$ and $\sqsubset_{t_iu_i} = \sqsubset \cup \sqsubset_{t_iu_i}|_{X \times X} \cup \sqsubset_{t_iu_i}|_{Y \times Y}$.

By $t_1 \equiv t_2$, $S_1^{\diamond} = S_2^{\diamond}$, and[8] by $u_1 \equiv u_2$, $T_1^{\diamond} = T_2^{\diamond}$. Thus, by (1, 2, 3) and Proposition 3.7, $\mathscr{S}_{t_1u_1} = \mathscr{S}_{t_2u_2}$. ▌

5. ELEMENTARY NET SYSTEMS WITH INHIBITOR ARCS

Traces provide an abstract semantics of 1-safe Petri nets [22] and EN-systems [26]. In this section we will show that comtraces can be used to provide an abstract a-priori semantics for EN-systems with inhibitor arcs. As mentioned in Section 2, posets and Mazurkiewicz traces are too weak for this purpose in the general case.

5.1. *Inhibitor Arcs*

A net with inhibitor arcs is a tuple $N = (S, T, F, I)$ where S, T are finite disjoint sets, $F \subseteq (S \times T) \cup (T \times S)$ and $I \subseteq S \times T$. The meaning and graphical representation of S, T and F are the same as in the standard net theory, whereas I is a set of inhibitor arcs. An *inhibitor arc* $(s, e) \in I$ means that e can be enabled only if s is not marked; in the diagrams, (s, e) is indicated by an edge with a small circle. For every $x \in S \cup T$,

$$x^{\bullet} = \{ y \mid (x, y) \in F \}$$
$${}^{\bullet}x = \{ y \mid (y, x) \in F \}$$
$$x^{\diamond} = \{ y \mid (x, y) \in I \cup I^{-1} \}.$$

The dot-notation extends in the usual way to sets. We assume that for every $x \in T$, both ${}^{\bullet}x$ and $x^{\bullet}$ are non-empty and disjoint. Moreover, they both must have empty intersection with $x^{\diamond}$.[9]

[8] Follows from the fact that $(occ(u_i), \prec_{u_i}, \sqsubset_{u_i})$ and T_i $(i = 1, 2)$ are isomorphic via (the same) mapping $\psi: occ(u_i) \to Y$ defined by $\psi(e, j) = (e, \#_e(t_i) + j)$.

[9] These assumptions can be best explained by looking at the event enabling rule in Section 5.3: ${}^{\bullet}x \cap x^{\bullet} = {}^{\bullet}x \cap x^{\diamond} = \varnothing$ excludes events which could never occur, and if $s \in x^{\bullet} \cap x^{\diamond}$ then the arc $(s, x) \in I$ would be redundant as $(x, s) \in F$ implies that s must be empty for x to be enabled. Note that for the a-posteriori semantics, $s \in x^{\bullet} \cap x^{\diamond}$ would mean that x could never be executed.

5.2. *ENI-Systems*

An *elementary net system with inhibitor arcs* (ENI-system) is a tuple

$$\Xi = (B, E, F, I, c_{in})$$

where (B, E, F, I) is a net with inhibitor arcs, and $c_{in} \subseteq B$ is the *initial case*. Each element of E is called an *event*, and of B a *condition*. In general, any $c \subseteq B$ is a case. In the diagrams, it will be represented by placing tokens inside circles representing the conditions in c.

EN-systems (ENI-systems with $I = \varnothing$) constitute probably the simplest system model of net theory. ENI-systems could be seen as the simplest system model whose non-interleaving a-priori semantics would require relational structures richer than causal partial orders.

EN-systems can be given a trace semantics [26] with the independence relation derived from the structural properties of the net. In Section 5.4 this approach will be generalised to ENI-systems. Figure 4a shows an example ENI-system. We will assume that Ξ is fixed throughout the rest of this section.

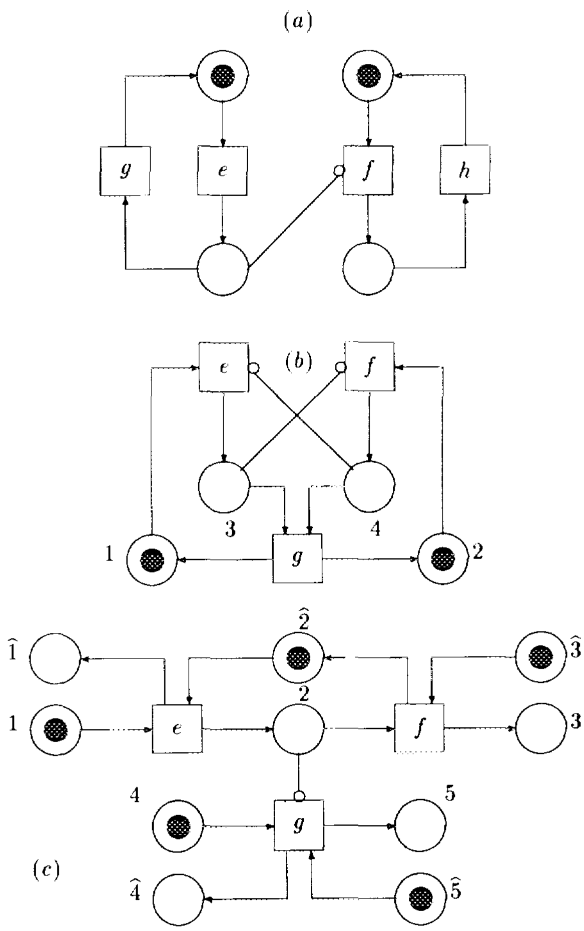

FIG. 4. Elementary net systems with inhibitor arcs.

5.3. *Operational Semantics*

The operational semantics of Ξ is defined via a "token game" which differs from that defined for ordinary EN-systems by insisting that an event can be enabled only if no condition with which it is joined by an inhibitor arc is marked.

We first introduce the interleaving (or firing sequence) semantics of Ξ. An event e is *enabled* at case c if ${}^\bullet e \subseteq c$ and $(e^\bullet \cup e^{\circ}) \cap c = \emptyset$. An enabled e can *occur* leading to a new marking $c' = c - {}^\bullet e \cup e^\bullet$. We denote this by $c[e\rangle\, c'$. A *firing sequence* of Ξ is any sequence of events $e_1 \cdots e_n$ for which there are cases $c_1, ..., c_n$ satisfying

$$c_{in}[e_1\rangle\, c_1[e_2\rangle\, c_2 \cdots [e_n\rangle\, c_n.$$

We generalise the above definition to sequences of sets of events executed simultaneously. Let $U \subseteq E$ be a non-empty set such that for all distinct $e, f \in U$,

$$(e^\bullet \cup {}^\bullet e) \cap (f^\bullet \cup {}^\bullet f) = \emptyset.$$

Then U is *enabled* at case c if ${}^\bullet U \subseteq c$ and $(U^\bullet \cup U^{\circ}) \cap c = \emptyset$. We also denote $c[U\rangle\, c'$, where $c' = c - {}^\bullet U \cup U^\bullet$. A *step sequence* is a sequence of sets $U_1 \cdots U_n$ for which there are cases $c_1, ..., c_n$ satisfying

$$c_{in}[U_1\rangle\, c_1[U_2\rangle\, c_2 \cdots [U_n\rangle\, c_n.$$

We will denote $c_{in}[U_1 \cdots U_n\rangle\, c_n$ and $U_1 \cdots U_n \in steps(\Xi)$.

For the ENI-system of Fig. 4a, *feghfegh* is a firing sequence while *ef* is not. Moreover, $\{e, f\}\{g, h\}\{e, f\}$ $\{g, h\}$ is a step sequence while $\{e\}\{f, g\}$ is not. *Note that there can be cases reachable (from the initial case) under the step sequence firing rule which are not reachable under the firing sequence rule.* For the ENI-system in Fig. 4b, $\{3, 4\}$ is such a case. There is an essential difference between the firing sequence and (a-priori) step sequence semantics of Ξ. For example, the ENI-system of Fig. 4b has only two non-empty firing sequences, e and f, yet infinitely many step sequences generated by the regular expression $(\{e, f\}\ \{g\})^* (\lambda, \{e\}, \{f\}, \{e, f\})$, where λ is the empty step sequence.

5.4. *Comtraces of ENI-Systems*

For EN-systems, traces are constructed by first deriving an independence relation on events, *ind*. The situation is more complicated if we consider ENI-systems. For example, the ENI-system of Fig. 4a can execute e and f simultaneously, or f followed by e. But e followed by f is not allowed. Hence simultaneity does not imply independence. To deal with this problem we will specify two relations: one identifying events which can be executed in one step (simultaneously), the other identifying events which can be executed in a specific order (serialised). Define *sim*, $ser \subseteq E \times E$ as follows:

$$(e, f) \in sim \Leftrightarrow (e^\bullet \cup {}^\bullet e) \cap (f^\bullet \cup {}^\bullet f)$$
$$= \emptyset \wedge (e^{\circ} \cap {}^\bullet f) \cup (f^{\circ} \cap {}^\bullet e) = \emptyset$$
$$(e, f) \in ser \Leftrightarrow (e, f) \in sim \wedge e^\bullet \cap f^{\circ} = \emptyset.$$

Let $\vartheta = (E, sim, ser)$ be the concurrent alphabet with *sim* and *ser* we have just defined. That comtraces provide an abstract semantics of ENI-systems follows from the next result which can be proved directly from the definition of ϑ, the concurrent enabling rule, and the results presented in Section 4.

PROPOSITION 5.1. (1) *All elements of $steps(\Xi)$ are step sequences w.r.t. ϑ.*

(2) *If t, u are step sequences w.r.t. ϑ such that $t \equiv u$ then $t \in steps(\Xi)$ if and only if $u \in steps(\Xi)$.*

(3) *$steps(\Xi)$ can be partitioned into disjoint comtraces.*

This means that there is a basic consistency between step sequences generated by the concurrent alphabet ϑ and step sequences obtained via the operational rule.

5.5. *Processes of ENI-Systems*

From Proposition 5.1(1) it follows that every step sequence t of Ξ induces a stratified order structure $\mathcal{S}_t$ (see Section 4.4) which can be identified with the step sequences in $[t]$—the comtrace containing t. In this section we will show that it is possible to generate $\mathcal{S}_t$ directly from t, by generalising the standard construction of processes (occurrence nets) for Petri nets [3, 26, 29].

In the rest of this paper we assume that every condition $s \in B$ has its *complement*. That is, there is exactly one condition $\hat{s} \in B$ such that $\hat{s}^\bullet = {}^\bullet s$, ${}^\bullet\hat{s} = s^\bullet$ and $|\{s, \hat{s}\} \cap c_{in}| = 1$. Note that $\hat{\hat{s}} = s$.

PROPOSITION 5.2. *For every case c reachable from c_{in} and every $s \in B$, $|\{s, \hat{s}\} \cap c| = 1$.*

Constructing a process of a Petri net amounts to unfolding of the net into an occurrence net respecting local environments of events. To see whether this approach can be adopted for ENI-systems, we consider Ξ of Fig. 4c, and two step sequences, $t_1 = \{g\}\{e\}\{f\}$ and $t_2 = \{e\}\{f\}\{g\}$. By unfolding Ξ, in both cases we obtain the same net shown in Fig. 5. But t_1 and t_2 belong to different comtraces, $[t_2] = \{t_2\}$ and $[t_1] = \{t_1, \{g, e\}\{f\}\}$, and should generate two different occurrence nets. It is therefore necessary to modify the unfolding procedure. The reason for this is that in t_1 event g was enabled because 2 has not yet been marked with a token produced by the occurrence of e, while in t_2 event g was made enabled by the occurrence of f which removed the token from 2. Intuitively, the two occurrences of g resulted

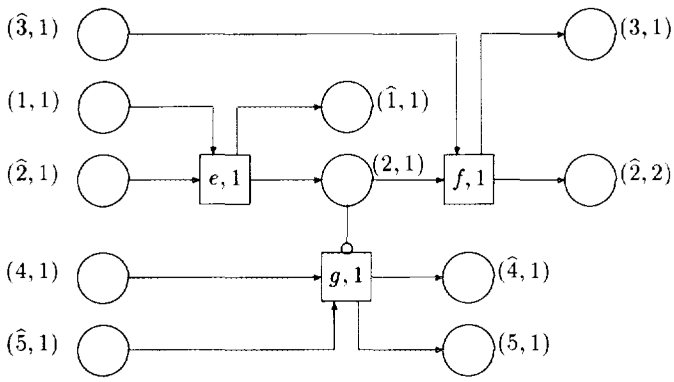

FIG. 5. An attempt to derive the occurrence net of an ENI-system.

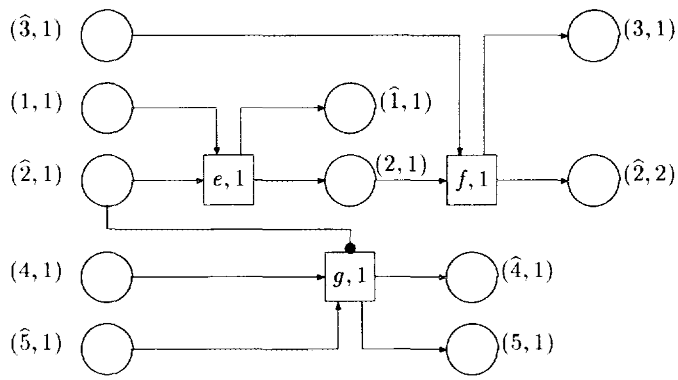

FIG. 6. Process (occurrence net) generated by t_1.

from two different "non-holdings" of 2. In unfolding Ξ, however, we had in both cases only one occurrence of 2, and could not distinguish between these two, clearly different, situations. To solve this problem, the unfolding will be modified by binding the occurrences of g to the holdings of the complement condition $\hat{2}$. That is, g can be enabled only if $\hat{2}$ holds. We then will unfold Ξ into occurrence nets with activator arcs—indicated by black dots at one end. Essentially, an activator arc $(\hat{s}, e)$ will replace inhibitor arc (s, e). As a result, we generate two different processes (occurrence nets) for the step sequences t_1 and t_2, as shown in Figs. 6 and 7.

Let $t = U_1 \cdots U_n$ be a step sequence of Ξ fixed for the rest of this section. We define the *process* generated by t as $\Pi_t = N_n$, where N_n is the last net in the sequence $N_0, ..., N_n$ constructed in the following way:

CONSTRUCTION. Each net $N_k = (S_k, T_k, F_k, A_k)$, $0 \leqslant k \leqslant n$, is a net with activator arcs.[10] The first three components of N_k are the same as in the definition of the net with inhibitor arcs, while $A_k \subseteq S_k \times T_k$ is the set of *activator* arcs. The elements of $S_k \cup T_k$ are of the form (x, i), where $x \in B \cup E$ and $i \geqslant 1$. We will denote $l(x, i) = x$ and $n(x, i) = i$. Moreover, for every $x \in B \cup E$ and $k \leqslant n$, $\#_x^k$ is the number of $\alpha \in S_k \cup T_k$ such that $l(\alpha) = x$.

Step 0. $N_0 = (\{(s, 1) \mid s \in c_{in}\}, \varnothing, \varnothing, \varnothing)$.

Step $k+1$. Given N_k and $U = U_{k+1}$ we define N_{k+1} in the following way:

$$S_{k+1} = S_k \cup \{(s, \#_s^k + 1) \mid s \in U^\bullet\}$$
$$T_{k+1} = T_k \cup \{(e, \#_e^k + 1) \mid e \in U\}$$
$$F_{k+1} = F_k \cup \{((s, \#_s^k), (e, \#_e^k + 1)) \mid e \in U \wedge (s, e) \in F\}$$
$$\cup \{((e, \#_e^k + 1), (s, \#_s^k + 1)) \mid e \in U \wedge (e, s) \in F\}$$
$$A_{k+1} = A_k \cup \{((\hat{s}, \#_{\hat{s}}^k), (e, \#_e^k + 1)) \mid e \in U \wedge (s, e) \in I\}. \ \blacksquare$$

[10] Recall that an activator arc has the meaning "opposite" to that of an inhibitor arc. More precisely, if s and e are joined by an activator arc, then e can only occur if s contains a token, but the firing of e *does not* remove the token from s.

If in the above construction we ignore everything concerning inhibitor and activator arcs, then it reduces to that of a process (occurrence net) generated by the step sequence t for the EN-system underlying Ξ. This implies a number of useful properties, formulated as the next lemma (cf. [3, 29]). Below, for every $0 \leqslant k \leqslant n$, we denote by Max_k the set of all $\alpha \in S_k$ for which there is no $\gamma \in T_k$ such that $(\alpha, \gamma) \in F_k$.

LEMMA 5.3. *Let* $0 \leqslant k \leqslant n$.

(1) *(S_k, T_k, F_k) is a process of the EN-system (B, E, F, c_{in}).*

(2) *l is injective on Max_k and $c_{in}[U_1 \cdots U_k\rangle\ l(Max_k)$.*

(3) *If $\alpha \in Max_k$ and $\beta \in Max_j$, where $k \leqslant j$, then:*

(a) $l(\alpha) \in \{l(\beta), \widehat{l(\beta)}\} \Rightarrow (\alpha, \beta) \in F_n^*$.

(b) $l(\alpha) = l(\beta) \Rightarrow n(\alpha) \leqslant n(\beta)$.

(4) $S_k = \bigcup_{j=0}^k Max_j$. $\blacksquare$

We first observe that the construction is correct, by noting that A_{k+1} is well defined since $e \in U$ and $(s, e) \in I$

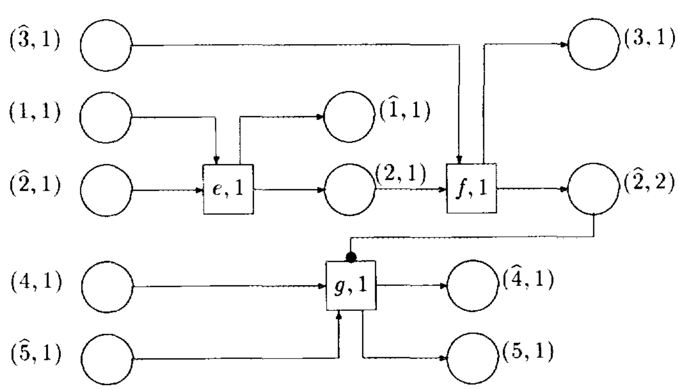

FIG. 7. Process (occurrence net) generated by t_2.

implies $\#^k_{\hat{s}} \geqslant 1$ (this follows from Lemma 5.3(2), e being enabled at $l(Max_k)$, and Proposition 5.2). We also note that

$$T_n = occ(t).$$

LEMMA 5.4. *Let* $0 \leqslant k < n$ *and* $\gamma \in T_{k+1} - T_k$.

(1) $pos_t(\gamma) = k+1$.

(2) $\{\alpha \mid (\alpha, \gamma) \in F_n \cup A_n\} \subseteq Max_k$.

(3) $\{\alpha \mid (\gamma, \alpha) \in F_n\} \subseteq Max_{k+1}$.

(4) $\{l(\alpha) \mid (\alpha, \gamma) \in F_n\} = \{s \mid (s, l(\gamma)) \in F\}$.

(5) $\{l(\alpha) \mid (\gamma, \alpha) \in F_n\} = \{s \mid (l(\gamma), s) \in F\}$.

(6) $\{l(\alpha) \mid (\alpha, \gamma) \in A_n\} = \{s \mid (\hat{s}, l(\gamma)) \in I\}$.

Proof. (1, 4, 5, 6) Follow directly from the construction. (3) and $\{\alpha \mid (\alpha, \gamma) \in F_n\} \subseteq Max_k$ follow from the corresponding properties of the processes of EN-systems (cf. [3, 29]) and Lemma 5.3(1). $\{\alpha \mid (\alpha, \gamma) \in A_n\} \subseteq Max_k$ follows from Lemma 5.3(2, 3b, 4) and Proposition 5.2. ▮

To show that the construction of Π_t is sound, we will prove that $\mathcal{S}_t$ and $\mathcal{S}^{\diamond}$—a stratified order structure in a natural way induced by Π_t—are identical. Let $\mathcal{S} = (T_n, \prec, \sqsubset)$, where (see Fig. 8):

$$\gamma \prec \delta \Leftrightarrow (\exists \alpha \in S_n)(\gamma, \alpha) \in F_n \wedge (\alpha, \delta) \in F_n \cup A_n$$
$$\gamma \sqsubset \delta \Leftrightarrow (\exists \alpha \in S_n)(\alpha, \delta) \in F_n \wedge (\alpha, \gamma) \in A_n.$$

We first prove three auxiliary lemmata.

LEMMA 5.5. *Let* $\gamma, \delta \in T_n$.

(1) $\sqsubset$ *is irreflexive.*

(2) *If* $\gamma \prec \delta$ *then* $pos_t(\gamma) < pos_t(\delta)$.

(3) *If* $\gamma \sqsubset \delta$ *then* $pos_t(\gamma) \leqslant pos_t(\delta)$.

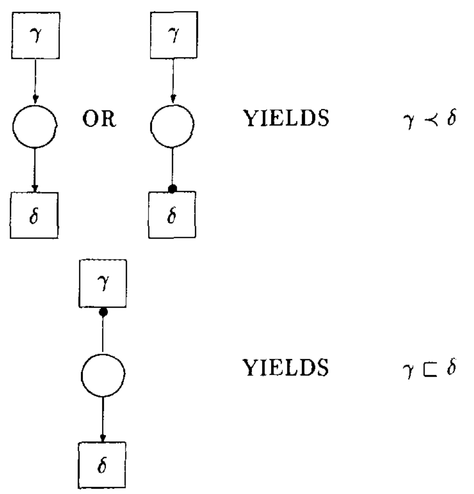

FIG. 8. Definition of $\prec$ and $\sqsubset$ derived for Π_t.

Proof. Let $k = pos_t(\gamma)$ and $m = pos_t(\delta)$.

(1) Follows from $e^{\bullet} \cap {}^{\circ}e = \varnothing$ for all $e \in E$.

(2) Suppose $(\gamma, \alpha) \in F_n$ and $(\alpha, \delta) \in F_n \cup A_n$. By Lemma 5.4(1, 3), $\alpha \in Max_k$ and (by construction) $\alpha \notin Max_i$ for $i < k$. Moreover, from Lemma 5.4(1, 2) it follows that $\alpha \in Max_{m-1}$. Hence $m - 1 \geqslant k$.

(3) Suppose $(\alpha, \delta) \in F_n$ and $(\alpha, \gamma) \in A_n$. By Lemma 5.4(1, 2), $\alpha \in Max_{m-1}$ and (by construction) $\alpha \notin Max_i$ for $i > m-1$. Moreover, again from Lemma 5.4(1, 2), it follows that $\alpha \in Max_{k-1}$. Hence $k - 1 \leqslant m - 1$. ▮

LEMMA 5.6. *Let* $\gamma, \delta \in T_n$.

(1) *If* $pos_t(\gamma) < pos_t(\delta)$ *and there are* $\alpha, \beta \in S_n$ *such that*

$(\alpha, \gamma) \in A_n$ *and* $(\beta, \delta) \in F_n$ *and* $l(\alpha) = \widehat{l(\beta)}$, *or*

$(\gamma, \alpha) \in F_n$ *and* $(\beta, \delta) \in F_n \cup A_n$ *and* $l(\alpha) \in \{l(\beta), \widehat{l(\beta)}\}$

then $(\gamma, \delta) \in \varrho_{\mathcal{S}} \circ \prec \circ \varrho_{\mathcal{S}}$.

(2) *If* $pos_t(\gamma) \leqslant pos_t(\delta)$ *and there are* $\alpha, \beta \in S_n$ *such that*

$(\alpha, \gamma) \in A_n$ *and* $(\beta, \delta) \in F_n$ *and* $l(\alpha) = l(\beta)$

then $(\gamma, \delta) \in \varrho_{\mathcal{S}} - id_{T_n}$.

Proof. Let $k = pos_t(\gamma)$ and $m = pos_t(\delta)$.

(1) Suppose $(\alpha, \gamma) \in A_n$ and $(\beta, \delta) \in F_n$ and $l(\alpha) = \widehat{l(\beta)}$. (Other cases can be dealt with in a similar way.) From Lemma 5.4(1, 2) it follows that $\alpha \in Max_{k-1}$ and $\beta \in Max_{m-1}$. From $k < m$, $l(\alpha) = \widehat{l(\beta)}$ and Lemma 5.3 (3a), we obtain $(\alpha, \beta) \in F_n^*$. This and $\alpha \neq \beta$ yields $(\alpha, \beta) \in F_n^+$. Hence there are $\gamma_1, ..., \gamma_j \in T_n$ $(j \geqslant 1)$ such that $(\alpha, \gamma_1) \in F_n$, $\gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_j$ and $(\gamma_j, \beta) \in F_n$. Thus $\gamma \sqsubset \gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_j \prec \delta$ which (together with $j \geqslant 1$) yields $(\gamma, \delta) \in \varrho_{\mathcal{S}} \circ \prec \circ \varrho_{\mathcal{S}}$.

(2) As in (1), we can show $(\alpha, \beta) \in F_n^*$. If $\alpha = \beta$ then $\gamma \sqsubset \delta$. Hence, by Lemma 5.5(1), $(\gamma, \delta) \in \varrho_{\mathcal{S}} - id_{T_n}$. If $\alpha \neq \beta$ then as in (1) there are $\gamma_1, ..., \gamma_j \in T_n$ $(j \geqslant 1)$ such that $\gamma \sqsubset \gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_j \prec \delta$. Hence $(\gamma, \delta) \in \varrho_{\mathcal{S}}$. Moreover, by $j \geqslant 1$ and Lemma 5.5(2, 3), $k < m$. Thus $(\gamma, \delta) \in \varrho_{\mathcal{S}} - id_{T_n}$.

LEMMA 5.7. *Let* $\gamma, \delta \in T_n$.

(1) *If* $(l(\gamma), l(\delta)) \notin sim$ *then there are* $\alpha, \beta \in S_n$ *such that one of the following holds, where the roles of* γ *and* δ *may be interchanged*:

$(\gamma, \alpha) \in F_n$ *and* $(\beta, \delta) \in F_n$ *and* $l(\alpha) \in \{l(\beta), \widehat{l(\beta)}\}$, *or*

$(\gamma, \alpha) \in F_n$ *and* $(\beta, \delta) \in A_n$ *and* $l(\alpha) = l(\beta)$, *or*

$(\alpha, \gamma) \in A_n$ *and* $(\beta, \delta) \in F_n$ *and* $l(\alpha) = \widehat{l(\beta)}$.

(2) *If* $(l(\gamma), l(\delta)) \in sim - ser$ *then there are* $\alpha, \beta \in S_n$ *such that*

$(\gamma, \alpha) \in F_n$ *and* $(\beta, \delta) \in A_n$ *and* $l(\alpha) = \widehat{l(\beta)}$.

(3) *If* $(l(\delta), l(\gamma)) \in sim - ser$ *then there are* $\alpha, \beta \in S_n$ *such that*

$$(\alpha, \gamma) \in A_n \text{ and } (\beta, \delta) \in F_n \text{ and } l(\alpha) = l(\beta).$$

Proof. Follows from the definition of *sim*, *ser*, and Lemma 5.4 (4, 5, 6). For example, if $(l(\gamma), l(\delta)) \in sim - ser$ then there is $s \in l(\gamma)^{\bullet} \cap l(\delta)^{\circ}$. Thus, by Lemma 5.4(5, 6), there are $\alpha, \beta \in S_n$ such that $l(\alpha) = s$, $l(\beta) = \hat{s}$, $(\gamma, \alpha) \in F_n$ and $(\beta, \delta) \in A_n$. Hence (2) holds. We further observe that $\hat{s} \in {}^{\bullet}l(\gamma)$ which, by Lemma 5.4 (4), means that there is $\alpha' \in S_n$ such that $l(\alpha') = \hat{s}$ and $(\alpha', \gamma) \in F_n$. From this and $(\beta, \delta) \in A_n$ (and after interchanging the roles of γ and δ) it follows that (3) is also satisfied. ∎

We can now prove the desired result.

THEOREM 5.8. $\mathscr{S}^{\diamond} = \mathscr{S}_t$.

Proof. Let $\gamma, \delta \in occ(t) = T_n$, $l(\gamma) = e$ and $l(\delta) = f$.

Suppose $\gamma \prec \delta$. Then, for some $\alpha = (s, i)$, $(\gamma, \alpha) \in F_n$ and $(\alpha, \delta) \in F_n \cup A_n$. If $(\alpha, \delta) \in F_n$ then $s \in e^{\bullet} \cap {}^{\bullet}f$. If $(\alpha, \delta) \in A_n$ then $\hat{s} \in {}^{\bullet}e \cap f^{\circ}$. In either case $(e, f) \notin sim$ which implies $(e, f) \notin ser$. Hence, by Lemma 5.5(2), $\gamma \prec_t \delta$. Similarly, we may show that $\sqsubset \subseteq \sqsubset_t$. This, together with the irreflexivity of $\sqsubset_t$ and Proposition 3.5, means that $\mathscr{S}^{\diamond}$ is a stratified order structure such that $\mathscr{S}^{\diamond} \subseteq \mathscr{S}_t$. To show that the reverse inclusion holds it suffices, again by Proposition 3.5, to demonstrate that $\prec_t \subseteq \varrho_{\mathscr{S}} \circ \prec \circ \varrho_{\mathscr{S}}$ and $\sqsubset_t \subseteq \varrho_{\mathscr{S}} - id_{T_n}$.

Suppose $\gamma \prec_t \delta$. Then $pos_t(\gamma) < pos_t(\delta)$, and $(e, f) \notin sim \vee (e, f) \in sim - ser$. Hence, by Lemmata 5.6(1) and 5.7(1, 2), $(\gamma, \delta) \in \varrho_{\mathscr{S}} \circ \prec \circ \varrho_{\mathscr{S}}$.

Suppose $\gamma \sqsubset_t \delta$. Then $pos_t(\gamma) \leqslant post_t(\delta)$, and $(f, e) \notin sim \vee (f, e) \in sim - ser$. If $(f, e) \notin sim$ then $(e, f) \notin sim$ and $pos_t(\gamma) \neq pos_t(\delta)$, so by Lemmata 5.6(1) and 5.7(1), $(\gamma, \delta) \in \varrho_{\mathscr{S}} \circ \prec \circ \varrho_{\mathscr{S}}$. Moreover, $\gamma \neq \delta$ since $pos_t(\gamma) \neq pos_t(\delta)$. Hence $(\gamma, \delta) \in \varrho_{\mathscr{S}} - id_{T_n}$. If $(f, e) \in sim - ser$ then, by Lemmata 5.6(2) and 5.7(3), $(\gamma, \delta) \in \varrho_{\mathscr{S}} - id_{T_n}$. ∎

The above result justifies the chosen notion of a process (occurrence net) of an ENI-system. For it states that Π_t can be seen as an accurate representation of the abstract history (comtrace) of the ENI-system, $[t]$, to which t belongs (cf. the results in Section 4 and Proposition 5.1).

6. A-POSTERIORI SEMANTICS OF ENI-SYSTEMS

In this section we will define formally the a-posteriori semantics of the ENI-system Ξ, i.e., one in which the simultaneous firing of e and f in Fig. 1 is not allowed. As it will turn out, the results already obtained are general enough to cover this type of semantics as well; one only needs to modify some of the definitions.

Let $\Xi = (B, E, F, I, c_{in})$ be an ENI-system with the a-posteriori semantics. Its interleaving semantics is exactly the same as that defined in Section 5.3. Moreover, a non-empty set of events $U \subseteq E$ is *enabled* at case c if U is enabled at c as defined in Section 5.3 and, in addition, $U^{\bullet} \cap U^{\circ} = \varnothing$.

We then define $c[U\rangle_0 c'$, $[U_1 \cdots U_n\rangle_0$ and $steps_0(\Xi)$ as the corresponding notions in Section 5.3.

Since under the a-posteriori semantics e and f cannot be fired simultaneously if $e^{\circ} \cap f^{\bullet} \neq \varnothing$, we have to re-define the simultaneity relation from Section 5.4.

Define the concurrent alphabet $\vartheta_0 = (E, sim_0, ser_0)$, where[11]

$$sim_0 = ser_0 = ser \cup ser^{-1}.$$

Note that the serialisability relation has become redundant. Clearly, all the results of Section 4 are valid for ϑ_0; in most cases in a considerably simplified form. In particular, from Proposition 4.5 it follows that for every $t \in steps_0(\Xi)$, $\prec_{\mathscr{S}_t} = \sqsubset_{\mathscr{S}_t}$, so comtraces are now nothing but partial orders.

The following equivalent of Proposition 5.1 can be proven directly from the definition of ϑ_0, the simultaneous enabling rule for the a-posteriori semantics, and the results in Section 4.

PROPOSITION 6.1. (1) *$steps_0(\Xi)$ are step sequences w.r.t. ϑ_0.*

(2) *If t, u are step sequences w.r.t. ϑ_0 such that $t \equiv u$ then $t \in steps_0(\Xi)$ if and only if $u \in steps_0(\Xi)$.*

(3) *$steps_0(\Xi)$ can be partitioned into disjoint comtraces.* ∎

Thus there is a basic consistency between the step sequences generated by ϑ_0 and the step sequences obtained via the a-posteriori operational rule.

The construction of processes, i.e. activator occurrence nets for the a-posteriori semantics is *exactly the same* as that described in Section 5.5. This is possible since $steps_0(\Xi) \subseteq steps(\Xi)$. For $t \in steps_0(\Xi)$ the same activator occurrence net Π_t represents the process under the a-priori semantics as well as under the a-posteriori semantics. However, Π_t is in both cases interpreted in a different way. Under the a-priori semantics defined in Section 5, Π_t represents the stratified order structure $\mathscr{S}^{\diamond} = (T_n, \prec, \sqsubset)^{\diamond}$ where $\prec$ and $\sqsubset$ were defined in Fig. 8. Under the a-posteriori semantics, Π_t will represent the stratified order structure $\mathscr{S}_0^{\diamond}$, where $\mathscr{S}_0 = (T_n, \prec \cup \sqsubset, \prec \cup \sqsubset)$. It turns out that this is consistent with the comtrace semantics defined for ϑ_0.

THEOREM 6.2. *If $t \in steps_0(\Xi)$ then $\mathscr{S}_0^{\diamond} = \mathscr{S}_t$ assuming that $\mathscr{S}_t$ is defined for ϑ_0.*

Proof. From Lemma 5.5(2, 3), the definition of sim_0, and Proposition 6.1(1) it follows that:

(1) If $\gamma \sqsubset \delta$ then $pos_t(\gamma) < pos_t(\delta)$.

(2) $\prec \cup \sqsubset$ is irreflexive.

[11] Below *sim* and *ser* are defined as in Section 5.4.

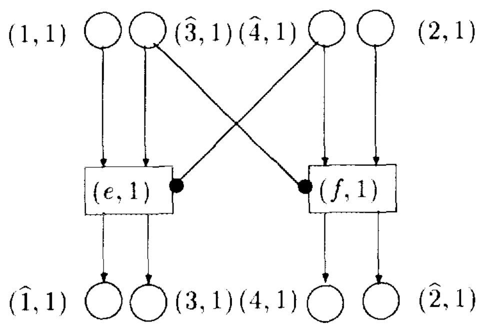

FIG. 9. Process which cannot be generated under a-posteriori semantics.

By Proposition 4.5 and (2), $\prec_{\mathscr{S}_t} = \sqsubset_{\mathscr{S}_t} = \prec_t^+$ and $\prec_{\mathscr{S}_0} = \sqsubset_{\mathscr{S}_0} = (\prec \cup \sqsubset)^+$. Hence it suffices to prove that $(\prec \cup \sqsubset) \subseteq \prec_t$ and $\prec_t \subseteq (\prec \cup \sqsubset)^+$.

Suppose $(\gamma, \delta) \in (\prec \cup \sqsubset)$. Then $(l(\gamma), l(\delta)) \notin ser_0$ which together with (1) yields $\gamma \prec_t \delta$.

Suppose $(\gamma, \delta) \in \prec_t$. Then $pos_t(\gamma) < pos_t(\delta)$ and $(l(\gamma), l(\delta)) \notin ser_0$. The latter implies that $(l(\gamma), l(\delta)) \notin sim$ or $(l(\gamma), l(\delta)) \in sim - ser$ or $(l(\delta), l(\gamma)) \in sim - ser$. Hence, by Lemmata 5.6 and 5.7,

$$(\gamma, \delta) \in ((\prec \cup \sqsubset)^* \circ \prec \circ (\prec \cup \sqsubset)^*) \cup ((\prec \cup \sqsubset)^* - id_{T_n})$$
$$=_{(2)} (\prec \cup \sqsubset)^+. \quad \blacksquare$$

The a-posteriori process semantics presented in this section corresponds to the cc-processes of [24]. We feel that the approach of [24] cannot be easily extended to the a-priori semantics.

It is worth mentioning that there are occurrence activator nets generated by the a-priori semantics that cannot be derived under the a-posteriori semantics. For instance, the occurrence activator net in Fig. 9 represents the process $\Pi_{\{e,f\}}$ generated by the ENI-system from Fig. 4b (after adding the complement conditions) under the a-priori semantics, but this occurrence net cannot be derived under the a-posteriori semantics.

7. CONCLUDING REMARKS

We have introduced the concept of a *comtrace*—an extension of Mazurkiewicz trace—and shown that comtraces correspond to *stratified order structures* in the same way as Mazurkiewicz traces correspond to posets. Next we have shown how both stratified order structures and comtraces can provide a semantical model for inhibitor nets consistent with operational (a-priori) semantics expressed in terms of step sequences. This in general cannot be done by using just posets and Mazurkiewicz traces. Although our model has been built having the a-priori operational rule in mind, it is general and flexible enough to handle the a-posteriori operational rule as well (essentially, as a simplified case of the a-priori semantics). In this way we have developed a semantical model for both a-priori and a-posteriori semantics.

In [31] Vogler introduced "step traces," another generalisation of classical trace theory. There is an important common characteristics shared by our approach and that of [31], namely in both cases it is possible to have steps which are "indivisible." More precisely, dividing such a step leads to a run which is an observation of a different abstract concurrent history of the system. Despite this similarity, comtraces are not the same as step traces. For instance, there is no step trace equivalent to the comtrace $[\{a\}\{b\}] = \{\{a,b\}, \{a\}\{b\}\}$.[12] We conjecture that step traces with steps being sets rather than multisets of events are equivalent to comtraces with a symmetric serialisability relation, *ser*. That step traces and comtraces are fundamentally different models is consistent with the approach developed in [16]; it can be seen that step traces conform to paradigm π_7 of concurrency (in the terminology of [16]), while comtraces to paradigm π_3.[13]

We conjecture that the approach presented in this paper (except for the process construction) can be smoothly extended to 1-safe nets with inhibitor arcs. The process construction can also be extended provided that no side condition[14] is incident to an inhibitor arc. This follows from the fact that our construction relies heavily on being able to construct complement conditions (places). Moreover, it seems that the entire approach can be applied (after minor changes) to 1-safe nets with activator arcs as the process construction could then be done without complement conditions.

Finally, we conjecture that the processes of an ENI-system can be given an axiomatic characterisation, similar to that which exists for the occurrence nets of an EN-system. The main extension required would be to have a condition that the $\diamond$-closure of two relations defined similarly as $\prec$ and $\sqsubset$ in Fig. 8 yields a stratified order structure.

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[12] The comparison in the other direction is more difficult as [31] uses multisets rather than sets to model single steps.

[13] We are grateful to Walter Vogler for pointing this out to us.

[14] $s \in B$ is a side condition if there is an e such that $s \in {}^\bullet e \cap e^\bullet$.

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