

Structure of concurrency

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Abstract

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Noninterleaving models of concurrency assume that behavioural properties of systems can be adequately modelled in terms of causal partial orders. We claim that the structure of concurrency is richer, with causality being only one of the invariants generated by a set of closely related executions or observations. The model we propose supports three levels of abstraction: the observation level, invariant level and system level; and we will proceed from the bottom (observation) level to the top (system) level. This is in contrast to the way other models for concurrency are introduced, as they essentially support two levels of abstraction, the system level and behavioural level (which includes both observations and invariants), with the direction of development going from the system to behavioural level. In this paper we first discuss the notion of an observation of a concurrent behaviour; in particular, we investigate the role played by interval partial orders. We then introduce a general framework for dealing with invariants generated by sets of closely related observations. This leads to the formulation of the notion of a (concurrent) history whose structural properties are subsequently studied.

Contents

0. Introduction	6
1. Motivation	7
2. Observations	9
2.1. Posets and principal posets	11
2.2. Observations and interval orders	14
2.3. Interleaving and step sequences	17

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3. Invariants	17
3.1. Report systems	18
3.2. Invariants and paradigms in RS_{con} : an intuition	19
3.3. Simple report formulas	20
3.4. Signatures	21
4. Histories	22
5. Paradigms	25
6. Report system of concurrent observations	28
6.1. Simple report invariants	28
6.2. Components of simple report invariants	30
6.3. Paradigms and signatures	31
6.4. The paradigms of partial order histories	35
7. Representation theorems	36
7.1. Paradigm π_3	37
7.2. Step sequences within π_3	40
8. Related work	45
9. Systems	47
10. Conclusions	47
Appendix	48
Acknowledgment	51
References	51

0. Introduction

The existing models of concurrency are usually developed on the *system* and *behavioural* levels, and are *top-down* in the sense that the concept of a system is introduced first. The system level is usually based on some notion of an abstract machine [29, 38, 44, 50] or algebraically defined process [15, 34]. The operational concepts on the behavioural level are influenced by the system level and are usually expressed in terms of interleavings [15, 33, 37], step sequences [17, 42, 43], and (labelled) partial orders [4, 29, 31, 40, 41, 47]. It seems that a disadvantage of such an approach is that the behavioural level includes both *single* observations of concurrent histories (interleavings and step sequences), and invariants characterising *sets* of observations (causal partial orders). As a result, it is difficult to develop a fully satisfactory model. For example, the description of invariants other than causality is confusing. We believe that in order to obtain a truly general model of concurrency, the behavioural level can be replaced by the *invariant* and *observation* levels. Moreover, the development should proceed from the observation to the system level. In this way, behavioural notions can be studied in more objective setting, without being influenced by any specific representation of concurrent systems dealt with on the system level.

In this paper we focus on the observation and invariant levels. We define observations as partially ordered sets of event occurrences, where ordering represents *precedence*, and incomparability represents *simultaneity*. We then introduce a class of basic invariants, and define a concurrent history to be the set of all observations

consistent with a set of invariants. After that we discuss a connection between paradigms (or general laws) of concurrency and the invariants. We identify eight basic paradigms, including that usually adopted by different existing models: *two events can be observed as simultaneous if and only if they can be observed in both orders*. Different paradigms admit concurrent histories with different structural properties. As a result, one may choose different invariant representations for concurrent histories. In particular, the above paradigms admits histories which can be represented by causal partial orders. However, for the remaining seven paradigms, causal partial orders either have to be replaced by stronger invariants or augmented. For one of these paradigms, an axiomatic model as well as representation theorems for invariants will be provided.

The existing models for concurrency essentially use only one kind of invariant, usually referred to as *causality*. Even more complex structures, such as pomsets [40], event structures of [49], or concurrent histories in the sense of [6], are in principle based on causal partial orders. Both interleaving and partial order models have been developed to a high degree of sophistication and proved to be successful specification and verification frameworks. However, some aspects of concurrent behaviour are still difficult to tackle. For example, the specification of priorities using partial orders is in some circumstances problematic [5, 17, 22, 26]; in our opinion, mainly because their concurrent behaviour cannot always be defined in terms of causality-based structures. A similar comment applies to inhibitor Petri nets [37] which are virtually admired by practitioners and almost completely rejected by theoreticians. Problems like these follow from a general assumption that concurrent behaviours can always be adequately modelled in terms of causality-based structures. We claim that the structure of concurrency phenomenon is richer, with causality being only one of the invariants generated by a set of closely related observations. An attempt to define other invariants was made in [12, 27, 28], however, with different objectives in mind. We will show how these approaches fit into our approach.

The paper is organised as follows. A motivating example is discussed in the next section. In Section 2 we present the model of observations. Section 3 introduces invariants in a general setting which is independent of any specific notion of observation. Section 4 contains the definition of a history, while Section 5 establishes a link between paradigms and invariants. Section 6 discusses the notions developed in the preceding sections for the observation model from Section 2. In Section 7 a detailed analysis of one of the paradigms is presented. Section 8 briefly describes some related work. A short statement about the system level is provided in Section 9.

1. Motivation

Consider the nets in Fig. 1. (PN_5 and PN_6 employ inhibitor arcs – an inhibitor arc between place p and transition t means that if t is enabled then p must be unmarked

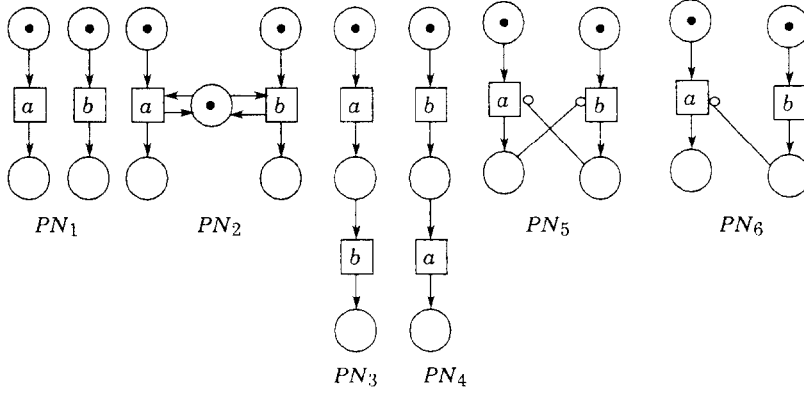


Fig. 1. Motivating example.

[37].) We want to define their semantics so that:

- (1) Each net generates exactly one a, b -history (i.e., one involving both a and b).
- (2) Different nets generate different a, b -histories.
- (3) Histories are defined on the same level of abstraction as the causality relation.

In terms of step sequences, interpreted as executions or observations, the nets generate [20] the following step sequences involving both a and b :

PN_1 generates $\{a\}\{b\}, \{b\}\{a\}$ and $\{a, b\}$,

PN_2 generates $\{a\}\{b\}$ and $\{b\}\{a\}$,

PN_3 generates $\{a\}\{b\}$,

PN_4 generates $\{b\}\{a\}$,

PN_5 generates $\{a, b\}$,

PN_6 generates $\{a\}\{b\}$ and $\{a, b\}$.

Whereas it seems natural to require that each PN_i , for $i \neq 2$, generates just one a, b -history (there is no conflict between a and b), this may not be obvious for PN_2 . To see that it may in some cases be advantageous to allow PN_2 generate only one a, b -history, we consider the following program statement:

$a: x := x + 1 \ \& \ b: x := x + 3.$

Here ‘&’ denotes *commutativity* operator [30] implying that the assignments may be performed in any order but not simultaneously. We think that this could be adequately modelled by PN_2 generating *one* history consisting of two, essentially equivalent, executions $\{a\}\{b\}$ and $\{b\}\{a\}$. Hence, we want each PN_i to generate exactly one a, b -history Δ_i , where

$$\Delta_1 = \{\{a\}\{b\}, \{b\}\{a\}, \{a, b\}\},$$

$$\Delta_2 = \{\{a\}\{b\}, \{b\}\{a\}\},$$

$$\Delta_3 = \{\{a\}\{b\}\},$$

$$\Delta_4 = \{\{b\}\{a\}\},$$

$$\Delta_5 = \{\{a, b\}\},$$

$$\Delta_6 = \{\{a\}\{b\}, \{a, b\}\}.$$

A question which one might now ask is whether the Δ_i 's could be represented in a more structured or compact way using, e.g., the notion of causality. Whereas this can be achieved for Δ_1 (a and b are independent), Δ_3 (a causes b) and Δ_4 (b causes a), no such characterisation is possible for the remaining histories. We may, however, introduce three new relations (invariants): *commutativity* ($\rightleftharpoons$), *synchronisation* ($\leftrightarrow$) and *weak causality* ($\nearrow$) in the following way:

$a \rightleftharpoons b$ iff a precedes b or b precedes a in the step sequences a history comprises.

$a \leftrightarrow b$ iff a is simultaneous with b in the step sequences a history comprises.

$a \nearrow b$ iff a never follows b in the step sequences a history comprises.

Now it is possible to characterise Δ_2 by $a \rightleftharpoons b$, Δ_5 by $a \leftrightarrow b$, and Δ_6 by $a \nearrow b$.

Although it is possible to require that PN_2 generate only one a, b -history, there may also be cases where it would be more appropriate to interpret PN_2 as a net generating two disjoint a, b -histories, $\Delta_3 = \{\{a\}\{b\}\}$ and $\Delta_4 = \{\{b\}\{a\}\}$. A question then arises as to how one might characterise these two different interpretations of the behaviour of PN_2 . In this paper we propose a solution based on the notion of a *paradigm*. A paradigm is a statement about the internal structure of a single history, such as: *if there is a step sequence in which a preceded b , and a step sequence in which b preceded a , then there is also one in which a and b were simultaneous*. If this paradigm were adopted, Δ_2 would no longer be a valid history, and we would have to replace it by Δ_3 and Δ_4 .

Remark. Although we used nets to illustrate the above discussion, our approach is not intended to be tied to any particular model of concurrent systems.

2. Observations

Observation is an abstract model of the execution of a concurrent system. It is a report supplied by an observer who has to fill in a (possibly infinite) matrix with rows and columns indexed by event occurrences. The observer fills in the entire matrix, except the diagonal, using $\rightarrow$ to denote precedence, $\leftarrow$ following, and $\leftrightarrow$ simultaneity. For example, the fact that a was observed simultaneously with b and c , and b preceded c , would be represented as in Table 1.

Table 1
Observer's report

	<i>a</i>	<i>b</i>	<i>c</i>
<i>a</i>		$\leftrightarrow$	$\leftrightarrow$
<i>b</i>	$\leftrightarrow$		$\rightarrow$
<i>c</i>	$\leftrightarrow$	$\leftarrow$	

In the existing literature one can identify basically three kinds of observations: In the interleaving approach [15, 33, 37], observations are sequences of event occurrences. The step sequence approach [17, 42, 43] defines observations as sequences of sets of events observed simultaneously. The third approach advocates the use of interval orders: [18, 36, 47] and, implicitly, [46]; however, (except [18]) usually without providing precise motivation and without adapting the theory of interval orders [8] to the needs of concurrency theory. The partial orders of [4, 29, 41] or pomsets of [40], where ordering represents *causality* and incomparability represents *independence*, cannot, in general, be interpreted as observations. As it was pointed out in [34], causality cannot be observed (by single observers, see [39]). Causal partial orders represent *sets* of closely related observations and belong to the invariant level. In this section we shall define precisely what kind of mathematical objects could be regarded as observations and what properties they possess. We will make the following basic assumptions:

- (A1) The observer can state that one event preceded another event, or that two events occurred simultaneously.
- (A2) The observer can always state whether two events occurred simultaneously, or whether one event preceded another event.

Together with transitivity of the precedence relation, these mean that observations can be represented by *partially ordered sets* of event occurrences, where ordering represents *precedence*, and incomparability represents *simultaneity*. Note that leaving out A2 would essentially amount to the introduction of uncertainty into the model.

Not all partial orders can be interpreted as valid observations. The three additional assumptions are:

- (A3) The observer only perceives a single thread of time.
- (A4) One observes finitely many events during a finite period of time.
- (A5) Events are finite.

A5 means that we exclude nonterminating events. A4 and A5 mean that an event can be preceded or simultaneous only with finitely many events. (Partial orders with this property will be called *initially finite*.) To capture A3 we first note that for any maximal set of simultaneous events there must be a point on the observer's time scale at which all the events in the set have been observed. Then A3 can be expressed by requiring that the time points corresponding to such maximal sets be linearly ordered.

2.1. Posets and principal posets

A *partially ordered set* (poset) is a pair $po = (\text{dom}(po), \xrightarrow{po})$ such that $\text{dom}(po)$ is a nonempty set and $\xrightarrow{po}$ is an irreflexive transitive relation on $\text{dom}(po)$. (We reserve the symbol $<$ to denote the usual ordering in $\mathbb{R}$.) po is *total* if for all distinct a and b , $a \xrightarrow{po} b$ or $b \xrightarrow{po} a$ holds. po is *initially finite* if for every a there is only finitely many b such that $a \xrightarrow{po} b$ does not hold. po is *combinatorial* if $\xrightarrow{po}$ is the transitive closure of the *immediate successor* relation q defined by

$$a q b: \Leftrightarrow a \xrightarrow{po} b \wedge \neg \exists c. a \xrightarrow{po} c \xrightarrow{po} b.$$

We will denote $\leftrightarrow_{po} b$ if a and b are distinct incomparable elements of po , while $Cuts_{po}$ will denote the set of *maximal antichains* [9], i.e., sets C of incomparable elements such that each $x \notin C$ is comparable with at least one element in C . We also define $C_{po} = (Cuts_{po}, \rightsquigarrow_{po})$, where $\rightsquigarrow_{po}$ is a relation on $Cuts_{po}$ such that $B \rightsquigarrow_{po} C$ if $B \neq C$ and there are no $b \in B$ and $c \in C$ satisfying $c \xrightarrow{po} b$. po is *stratified* [9] if $\leftrightarrow_{po} \cup id_{\text{dom}(po)}$ is an equivalence relation. A *discrete representation* of po is any $\Phi: \text{dom}(po) \rightarrow \mathbb{N}$, such that for all $a, b \in \text{dom}(po)$, $a \xrightarrow{po} b \Rightarrow \Phi(a) < \Phi(b)$. The representation is *image-finite* if $\Phi^{-1}(n)$ is finite for all n , and is *exact* if $a \xrightarrow{po} b \Leftrightarrow \Phi(a) < \Phi(b)$.

If po represents an observation then $\xrightarrow{po}$ will be interpreted as *precedence*, and $\leftrightarrow_{po}$ as *simultaneity*. For the poset in Fig. 2(a) we have: $\text{dom}(po) = \{a, b, c, d\}$, $a \xrightarrow{po} b$, $d \leftrightarrow_{po} a$, $Cuts_{po} = \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\}$ and $\{a, c\} \rightsquigarrow_{po} \{b, c\}$ (C_{po} is shown in Fig. 2(b)). We first show that C_{po} is always a poset.

Proposition 2.1. *Let po be a poset.*

- (1) *If $a \xrightarrow{po} b$ then there are $A, B \in Cuts_{po}$ such that $a \in A$, $b \in B$ and $A \rightsquigarrow_{po} B$.*
- (2) *If $A \rightsquigarrow_{po} B$ and $a \in A - B$ then there is $b \in B$ such that $a \xrightarrow{po} b$.*
- (3) *If $A \rightsquigarrow_{po} B$ and $b \in B - A$ then there is $a \in A$ such that $a \xrightarrow{po} b$.*

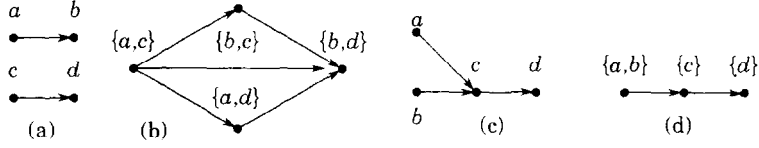


Fig. 2. Posets and principal posets.

- (4) If $A \rightsquigarrow_{po} B \rightsquigarrow_{po} C$ and $a \in A \cap C$ then $a \in B$.
 (5) If $A \rightsquigarrow_{po} B$ then $A - B \neq \emptyset$ and $B - A \neq \emptyset$.

Proof. (1) Let $a \in A \in \text{Cuts}_{po}$ and $C = \{c \mid c \rightarrow_{po} b\}$. Define $D = A - C \cup \{b\}$. Clearly, there is $B \in \text{Cuts}_{po}$ such that $D \subseteq B$. Suppose $\neg A \rightsquigarrow_{po} B$. Since $A \neq B$, we must have $c \rightarrow_{po} d$ for some $c \in B$ and $d \in A$. We obtain $d \in A - B \subseteq A - D = A \cap C$, which yields $d \rightarrow_{po} b$. Hence, $c \rightarrow_{po} d \rightarrow_{po} b$, contradicting $c, b \in B \in \text{Cuts}_{po}$.

(2) If $\neg a \rightarrow_{po} b$ for all $b \in B$ then, by $A \rightsquigarrow_{po} B$ and $a \in A - B$, $a \leftrightarrow_{po} b$ for all $b \in B$. Hence, B is not a maximal antichain, a contradiction.

(3) Similarly as for (2).

(4) If $a \notin B$ then, by (2), $a \rightarrow_{po} b$ for some $b \in B$, contradicting $B \rightsquigarrow_{po} C$.

(5) Follows from $A \neq B$ and the maximality of cuts. $\square$

Proposition 2.2. For every poset po , C_{po} is also a poset.

Proof. Suppose $A \rightsquigarrow_{po} B \rightsquigarrow_{po} C$. By Proposition 2.1(2, 5), $b \rightarrow_{po} c$ for some $c \in C$ and $b \in B$. Hence, $A \neq C$. If $\neg A \rightsquigarrow_{po} C$ then, $c \rightarrow_{po} a$ for some $c \in C$ and $a \in A$. By $B \rightsquigarrow_{po} C$, $a \notin B$. Thus, by Proposition 2.1(2), $a \rightarrow_{po} b$ for some $b \in B$. Hence, $c \rightarrow_{po} b$, contradicting $B \rightsquigarrow_{po} C$. $\square$

C_{po} will be called the *principal poset* of po . It will be used to formalise A3. We first investigate the relationship between posets and their principal posets.

Proposition 2.3. Let po and pr be posets. Then $C_{po} = C_{pr}$ iff $po = pr$.

Proof. It suffices to prove the left-to right implication. Suppose $C_{po} = C_{pr}$ and $a \rightarrow_{po} b$. By Proposition 2.1(1), there are $A, B \in \text{Cuts}_{po}$ such that $a \in A$, $b \in B$ and $A \rightsquigarrow_{po} B$. Clearly, $A \rightsquigarrow_{pr} B$. Hence, $\neg b \rightarrow_{pr} a$. Moreover, $\neg b \leftrightarrow_{pr} a$ since, otherwise, there would be $C \in \text{Cuts}_{pr}$ with $a, b \in C$. Clearly, $C \notin \text{Cuts}_{po}$, contradicting $\text{Cuts}_{po} = \text{Cuts}_{pr}$. Hence, $a \rightarrow_{pr} b$. $\square$

Proposition 2.4. Every initially finite poset has an injective discrete representation.

Proof. Let po be an initially finite poset. For $a \in \text{dom}(po)$, let $\nabla_{a, po} = \{b \in \text{dom}(po) \mid b \rightarrow_{po} a \vee a \leftrightarrow_{po} b\}$. From Szpilrajn-Marczewski extension theorem [45] it

follows that po can be extended to a total order t such that $\text{dom}(t) = \text{dom}(po)$ and $\rightarrow_{po} \subseteq \rightarrow_t$. The latter implies $V_{a,t} \subseteq V_{a,po}$ for all a . Hence, t is initially finite. Define $\Phi(a) = \text{card}(V_{a,t})$. Clearly, Φ is injective since t is total. Moreover,

$$a \rightarrow_{po} b \Rightarrow a \rightarrow_t b \Rightarrow V_{a,t} \subsetneq V_{b,t} \Rightarrow \Phi(a) < \Phi(b). \quad \square$$

Corollary 2.5. *Every initially finite poset is countable and combinatorial.*

The implication in Proposition 2.4 cannot be reversed (take $po = (\mathbb{N}, \emptyset)$ and $\Phi(i) = i$).

Proposition 2.6. *If a poset po is initially finite then C_{po} is also initially finite.*

Proof. Let $C \in \text{Cuts}_{po}$. We first observe that $E = \bigcup_{c \in C} V_{c,po}$ is finite, since po is initially finite. Suppose $V_{C,C_{po}}$ is infinite. If $B \in V_{C,C_{po}}$ then $b \rightarrow_{po} c$ for some $b \in B$ and $c \in C$. Hence, $B \cap E \neq \emptyset$. Consequently, since E is finite and $V_{C,C_{po}}$ is infinite, there is $e \in E$ which belongs to infinitely many cuts in $V_{C,C_{po}}$. Hence, $\{d \mid e \leftrightarrow_{po} d\}$ must be infinite, contradicting the initial finiteness of po . $\square$

The implication in Proposition 2.6 cannot be reversed (take $po = (\mathbb{N}, \emptyset)$).

Proposition 2.7. *If all cuts of a poset po are finite, and $\{c \mid a \rightarrow_{po} c \rightarrow_{po} b\}$ is finite for all a and b , then C_{po} is combinatorial.*

Proof. It suffices to show that $\{C \mid A \rightsquigarrow_{po} C \rightsquigarrow_{po} B\}$ is finite for all $A, B \in \text{Cuts}_{po}$. Suppose $A \rightsquigarrow_{po} C \rightsquigarrow_{po} B$. If $c \in C - (A \cup B)$ then, by Proposition 2.1(2, 3), $a \rightarrow_{po} c \rightarrow_{po} b$ for some $a \in A$ and $b \in B$. Hence, $C \subseteq D$, where $D = A \cup B \cup \{c \mid \exists a \in A \exists b \in B. a \rightarrow_{po} c \rightarrow_{po} b\}$. Clearly, D is finite. Hence, there is only finitely many C satisfying $A \rightsquigarrow_{po} C \rightsquigarrow_{po} B$. $\square$

Corollary 2.8. *If po is initially finite poset then C_{po} is combinatorial.*

We end this section proving that the principal order is total iff the original poset does not contain the four-element poset of Fig. 2(a).

Proposition 2.9. *Let po be a poset. Then the following are equivalent.*

(1) *There are no $a, b, c, d \in \text{dom}(po)$ such that*

$$a \rightarrow_{po} b, c \rightarrow_{po} d \text{ and } c \leftrightarrow_{po} b \leftrightarrow_{po} d \leftrightarrow_{po} a \leftrightarrow_{po} c.$$

(2) *C_{po} is total.*

(3) *For all $a, b, c, d \in \text{dom}(po)$, $a \rightarrow_{po} b \wedge c \rightarrow_{po} d \Rightarrow a \rightarrow_{po} d \vee c \rightarrow_{po} b$.*

Proof. (2) $\Leftrightarrow$ (3): This is proved below as Theorem 2.12.

(3) $\Rightarrow$ (1): Obvious.

(1) $\Rightarrow$ (3): If $\text{card}(\{a, b, c, d\}) < 4$ then (3) is always satisfied. Suppose $a \xrightarrow{po} b \wedge c \xrightarrow{po} d$ and $\text{card}(\{a, b, c, d\}) = 4$. From (1) it follows that

$$Z = (\{a, b\} \times \{c, d\} \cup \{c, d\} \times \{a, b\}) \cap \xrightarrow{po} \neq \emptyset.$$

Now, by taking any $(x, y) \in Z$ one may easily show that $a \xrightarrow{po} d$ or $c \xrightarrow{po} b$. $\square$

2.2. Observations and interval orders

Let Ev be the set of event occurrences. The definition of posets representing observations can now be formulated as follows:

An observation, $o \in Obs$, is an initially finite poset such that $\text{dom}(o) \subseteq Ev$ and C_o is total.

Note that the finiteness properties of the observation, A4 and A5, are guaranteed by the poset's initial finiteness, while the assumption about the single thread of time, A3, is captured by total ordering on all the snapshots (maximal antichains).

We now look closer at the structural properties of observations. Directly from Propositions 2.3 and 2.6 and Corollary 2.8, we obtain that the principal poset of an observation o can be represented as

$$A_1 \xrightarrow{o} A_2 \xrightarrow{o} \cdots \xrightarrow{o} A_n, \text{ or} \\ A_1 \xrightarrow{o} A_2 \xrightarrow{o} \cdots \xrightarrow{o} A_i \xrightarrow{o} A_{i+1} \xrightarrow{o} \cdots$$

We call $(A_i)_{i \in \{1, \dots, n\}} = (A_1, A_2, \dots, A_n)$ or $(A_i)_{i \in \mathbb{N}} = (A_1, A_2, \dots)$ the *cut-sequence* of o .

Proposition 2.10. *Let $(A_i)_{i \in J}$ be the cut-sequence of an observation o .*

- (1) *If $i, i+1 \in J$ then $a \xrightarrow{po} b$ for all $a \in A_i - A_{i+1}$ and all $b \in A_{i+1} - A_i$.*
- (2) *$\{i \mid a \in A_i\}$ is finite for all $a \in \text{dom}(o)$.*

Proof. (1) Suppose $\neg a \xrightarrow{o} b$ for some $a \in A_i - A_{i+1}$ and $b \in A_{i+1} - A_i$. This and $A_i \xrightarrow{o} A_{i+1}$ yields $a \leftrightarrow_o b$. Hence, there is m such that $a, b \in A_m$. Clearly, $i \neq m \neq i+1$. By Proposition 2.1(4), $m < i$ implies $b \in A_i$, while $m > i+1$ implies $a \in A_{i+1}$. In either case we obtain a contradiction.

- (2) Follows from the initial finiteness of o . $\square$

Thus, an event always belongs to a finite set of contiguous snapshots. This suggests that events may be characterised by *intervals* on the observer's time scale. There already exists a theory of interval partial orders [7, 8] developed within the measurement theory. We will use some of the notions and results obtained there to characterise observations. The name of interval order follows from [7]; its origin can be traced back to Wiener's 1914 paper [48], where interval orders were used to analyse

temporal events. Abraham et al. [1] claim that such a concept was also known to Russell. In this section we first recall a fundamental result of Fishburn [7], followed by a series of results leading to a representation theorem for observations.

A poset po is an *interval order* [48] if $a \xrightarrow{po} b$ and $c \xrightarrow{po} d$ implies that $a \xrightarrow{po} d$ or $c \xrightarrow{po} b$ holds, i.e., if its graph does not contain a subgraph isomorphic to the poset of Fig. 2(a) (see Proposition 2.9).

An *interval representation* of a poset po is a pair of mappings $\delta = (\Phi, \Psi)$ and a total order $(X, \angle)$ such that $\Phi, \Psi: \text{dom}(po) \rightarrow X$ and for all a and b ,

$$\Phi(a) \angle \Psi(a)$$

$$a \xrightarrow{po} b \Leftrightarrow \Psi(a) \angle \Phi(b).$$

That is, with each a can be associated an interval $J(a) = \{x \mid \Phi(a) \angle x \angle \Psi(a) \vee x = \Phi(a) \vee x = \Psi(a)\}$ such that $a \xrightarrow{po} b$ iff $J(a)$ is to the left of $J(b)$. The interval representation δ is *injective* if Φ is injective, *real* if $X = \mathbb{R}$, and *discrete* if both Φ and Ψ are integer-valued functions.

Theorem 2.11 (Real representation of interval orders, Fishburn [7]). *Let po be a poset such that there is countably many equivalence classes of*

$$eq = \{(a, b) \mid \forall c. (c \xrightarrow{po} a \Leftrightarrow c \xrightarrow{po} b) \wedge (a \xrightarrow{po} c \Leftrightarrow b \xrightarrow{po} c)\}.$$

Then po is an interval order iff it has a real interval representation.

We obtain a general result linking the theory of interval orders with our model of observations.

Theorem 2.12 (Principal posets and interval orders). *A poset po is an interval order iff C_{po} is a total order.*

Note: The theorem is a direct consequence of Theorem 1 and Corollary 2 in Chapter 3 of [8]. In terms of interval graphs similar results were established in [13, 11]. [1] claims this result was known to Russell and Wiener. We present our simple proof to make the presentation self-contained.

Proof. $\Leftarrow$: Suppose $a \xrightarrow{po} b, c \xrightarrow{po} d, \neg a \xrightarrow{po} d$ and $\neg c \xrightarrow{po} b$. We have $a \leftrightarrow_{po} d$ since, otherwise, $c \xrightarrow{po} d \xrightarrow{po} a \xrightarrow{po} b$, contradicting $\neg c \xrightarrow{po} b$. Similarly, $c \leftrightarrow_{po} b$. Hence, there are $A, B \in \text{Cuts}_{po}$ such that $a, d \in A$ and $b, c \in B$. Clearly, $A \neq B$. Moreover, $\neg A \rightsquigarrow_{po} B$ since $c \xrightarrow{po} d$, and $\neg B \rightsquigarrow_{po} A$ since $a \xrightarrow{po} b$. Hence, C_{po} is not total.

$\Rightarrow$: By Proposition 2.2, C_{po} is a poset. Suppose $A, B \in \text{Cuts}_{po}$ are such that $A \neq B$, $\neg A \rightsquigarrow_{po} B$ and $\neg B \rightsquigarrow_{po} A$. Then, $a \xrightarrow{po} b$ for some $b \in A$ and $a \in B$, and $c \xrightarrow{po} d$ for some $c \in A$ and $d \in B$. By $a, d \in B$ and $c, b \in A$, $\neg a \xrightarrow{po} d$ and $\neg c \xrightarrow{po} b$. Hence, po is not an interval order. $\square$

By Theorem 2.12, the poset of Fig. 2(c) is an interval order; its principal order is shown in Fig. 2(d). We have obtained an alternative definition of observable posets:

Observation is an initially finite interval order of event occurrences.

The representation theorem for interval orders (Theorem 2.11) does not take into account the initial finiteness of observations. It can be strengthened (Theorem 2.16) to provide a better characterisation of the way events are observed.

Proposition 2.13. *A poset with an interval representation is an interval order.*

Proof. Let $\hat{c}=(\Phi, \Psi)$ and $(X, \angle)$ be an interval representation of po . Suppose $a \xrightarrow{po} b, c \xrightarrow{po} d, \neg a \xrightarrow{po} d$ and $\neg c \xrightarrow{po} b$. By $\Psi(a) \angle \Phi(b)$ and $\Phi(d) \angle \Psi(a) \vee \Phi(d) = \Psi(a)$, we have $\Phi(d) \angle \Phi(b)$. By $\Psi(c) \angle \Phi(d)$ and $\Phi(b) \angle \Psi(c) \vee \Phi(b) = \Psi(c)$, $\Phi(b) \angle \Phi(d)$. Thus we obtained a contradiction. $\square$

Theorem 2.14 (Injective real representation of interval orders). *A countable poset is an interval order iff it has a real injective interval representation.*

Theorem 2.14 is proved in the Appendix without using Fishburn's result (Theorem 2.11). Moreover, the latter is a direct consequence of the former (the proof below is simpler and uses a different technique than those in [7, 8]).

New proof of Theorem 2.11. If po is countable then Theorem 2.14 is a stronger version of Theorem 2.11. Let po be uncountable and $\text{dom}(po)/_{eq}$ countable. Let $pr = (\text{dom}(po)/_{eq}, R)$, where $([a]_{eq}, [b]_{eq}) \in R \Leftrightarrow a \xrightarrow{po} b$. One can easily see that pr is a well-defined countable interval order. From Theorem 2.14 it follows that there is an injective real interval representation $\hat{c}_0 = (\Phi_0, \Psi_0)$ of pr . Let $\Phi, \Psi : \text{dom}(po) \rightarrow \mathbb{R}$ be defined by: $\Phi(a) = \Phi_0([a]_{eq})$ and $\Psi(a) = \Psi_0([a]_{eq})$. Clearly, $\hat{c} = (\Phi, \Psi)$ is a (noninjective) real interval representation of po . $\square$

Theorem 2.14 can be strengthened if we essentially assume that po is combinatorial.

Lemma 2.15. *Let po be a combinatorial interval order such that for all $a, b \in \text{dom}(po)$,*

- (1) $\{c \mid a \xleftrightarrow{po} c\}$ *is finite.*
- (2) $\{c \mid a \xrightarrow{po} c \xrightarrow{po} b\}$ *is finite.*

Then po has an injective discrete interval representation.

Proof. See Appendix. $\square$

We now can prove the main representation theorem for observations.

Theorem 2.16 (Injective discrete representation of observations). *A poset po of event occurrences is an observation iff it has an injective and discrete interval representation $\hat{c} = (\Phi, \Psi)$ such that $\Phi(a) > 0$ for all a .*

Proof. $\Rightarrow$: From Corollary 2.5 and Theorem 2.12 it follows that po is a combinatorial interval order. Moreover, po is initially finite. Hence, by Lemma 2.15, po has an injective and discrete interval representation $\hat{\partial}=(\Phi, \Psi)$. Moreover, using the initial finiteness of po , we can find $\hat{\partial}=(\Phi, \Psi)$ such that $\Phi(a)>0$ for all a .

$\Leftarrow$: Since $\hat{\partial}$ is injective and $\Phi(\text{dom}(po)) \subseteq \mathbb{N}$, po is initially finite. Moreover, by Proposition 2.13 and Theorem 2.12, C_{po} is total. Hence, po is an observation. $\square$

That is, events involved in an observation can be interpreted as intervals on the discrete time scale. We also conclude that in our model the discrete time scale and dense time scale are equally expressive.

2.3. Interleaving and step sequences

The interleaving and step sequences are part of the model. An *interleaving sequence*, $po \in \text{Obs}_{\text{int}}$, is an initially finite total order such that $\text{dom}(po) \subseteq \text{Ev}$. A *step sequence*, $po \in \text{Obs}_{\text{step}}$, is an initially finite stratified poset such that $\text{dom}(po) \subseteq \text{Ev}$. Clearly, $\text{Obs}_{\text{int}} \subsetneq \text{Obs}_{\text{step}} \subsetneq \text{Obs}$. The representation theorems for the interleaving and step sequences have very simple form.

Proposition 2.17. *Let po be a poset and $\text{dom}(po) \subseteq \text{Ev}$.*

- (1) *po is an interleaving sequence iff it has an exact injective discrete representation.*
- (2) *po is a step sequence iff it has an exact image-finite discrete representation.*

Unlike [47] (and implicitly [46, 36]), we have not arbitrarily assumed that the interval orders should model observations. We have introduced a general notion of observation based on some natural assumptions, A1–A5, about the way events are recorded by the observer. As a consequence, we defined observation as an initially finite poset whose principal order is total. Theorem 2.12 says that this is equivalent to being an initially finite interval order. The classical Fishburn representation theorem which usually provides the motivation for the use of interval orders assumes the dense observer's time scale, even if the orders are combinatorial. We have shown that for initially finite interval orders there is an equivalent injective interval representation using discrete time scale (Theorem 2.16). We have also strengthened Fishburn's characterisation of countable posets by proving the existence of injective representations (Theorem 2.14).

3. Invariants

There are many reasons why describing a concurrent system solely in terms of the observations it may generate can be unsatisfactory. In fact, most of the arguments made in favour of causality-based structures (see [4]) can also support the introduction of the new invariants. To define them, we will focus on the relationship between

the events involved in the observations of the same concurrent history. When dealing with a single observation (as defined in the previous section), we distinguished three forms of relationship between two events, a and b , namely: a before b , a after b , and a simultaneous with b . Given a set of observation Δ and two events a and b in its event-domain, one might ask what was the relative order of the two events in all the observations belonging to Δ . This time the question cannot be answered as easily as in the case of a single observation. For example, there may be some observations in which a occurred before b , some in which a occurred simultaneously with b , but none in which b occurred before a . (We will later characterise such a situation using an invariant, denoted by Δ and defined by: $a \Delta b \Leftrightarrow \forall o \in \Delta. a \xrightarrow{o} b \vee a \leftrightarrow_o b$.) In this section we will investigate how precedence and simultaneity can be lifted from the level of single observation to the level of sets of observations.

3.1. Report systems

To provide a formal framework for dealing with invariants generated by sets of related observations, we first introduce the notion of a report system.

Let Σ_0 be a set of objects (e.g., event occurrences). A relational system $\mu = (\Sigma, r_1, \dots, r_k)$, where $k \geq 2$, is a *report* over Σ_0 if $\Sigma \subseteq \Sigma_0$ and $r_1, \dots, r_k$ form a partition of $\Sigma \times \Sigma - id_\Sigma$. We denote $r_{i,\mu} = r_i$ ($i = 1, \dots, k$) and $\text{dom}(\mu) = \Sigma$. For $(a, b) \in \Sigma \times \Sigma - id_\Sigma$, we denote by $\text{index}(a, b, \mu)$ the $l \leq k$ for which $a r_{l,\mu} b$ holds. A *report system* over Σ_0 is any nonempty set RS of reports over Σ_0 such that if $(\Sigma, r_1, \dots, r_k) \in RS$ and $(\Sigma', s_1, \dots, s_l) \in RS$ then $k = l$.

Let RS be a report system fixed until the end of Section 5, and k be the number of the relations in its reports.

The report system of concurrent observations, RS_{con} , is defined over the set of event occurrences and comprises all reports (Σ, r_1, r_2, r_3) such that there is an observation $o \in \text{Obs}$ satisfying $\text{dom}(o) = \Sigma$, $\xrightarrow{o} = r_1$, $\xleftarrow{o} = r_2$ and $\leftrightarrow_o = r_3$. That is, reports in RS_{con} are just different representations of observations (see Section 6).

There are two reasons why we have introduced the general notion of a report system, instead of directly dealing with observations. Firstly, the general approach can be easily adapted if, for instance, one needs to introduce a relation representing observer's uncertainty about the relative order of events. The new report system would then contain reports $(\Sigma, r_1, r_2, r_3, r_4)$, with r_4 representing *uncertainty*. Similarly, one could use a model similar to Allen structures [2] or allow reports to be produced by teams of observers as in [39]. Secondly, *many of the properties of invariants are independent of the specific representation chosen for observations*, and it seems important to be able to separate them from those properties which follow from the specific properties of interval orders.

The first approximation of the notion of a history is introduced as follows: A *report set* over RS is a nonempty set Δ of reports over RS with a common domain, denoted by $\text{dom}(\Delta)$. We denote this by $\Delta \in RSet(RS)$.

Throughout the rest of this paper, we will assume that RS is *nondegenerated*, meaning that for every $l \leq k$ there is $\mu \in RS$ such that $r_{l,\mu} \neq \emptyset$. Clearly, RS_{con} is a nondegenerated report system.

3.2. Invariants of paradigms in RS_{con} : an intuition

Consider a report set $\Delta \subseteq RS_{con}$. In this case a *simple relational invariant* of Δ , $I \in SRI(\Delta)$, is a relation on $\text{dom}(\Delta)$ defined by

$$(a, b) \in I \Leftrightarrow a \neq b \wedge \forall o \in \Delta. \Phi(a, b, o),$$

where $\Phi(a, b, o)$ is a formula defined by the grammar:

$$\Phi := \text{true} \mid \text{false} \mid a \xrightarrow{o} b \mid a \xleftarrow{o} b \mid a \leftrightarrow_o b \mid \neg \Phi \mid \Phi \vee \Phi \mid \Phi \wedge \Phi.$$

Some of the basic terms of the above grammar are redundant, e.g., $a \xleftarrow{o} b$ is equivalent to $\neg(a \xrightarrow{o} b \vee a \leftrightarrow_o b)$. However, this does not cause any problems, while simplifies the discussion in general case.

Let $\xrightarrow{\Delta}, \xleftarrow{\Delta}, \leftrightarrow_{\Delta}, \rightleftharpoons_{\Delta}, \nearrow_{\Delta}, \nwarrow_{\Delta}$ be relations on $\text{dom}(\Delta)$ defined as follows:

$$a \xrightarrow{\Delta} b \Leftrightarrow a \neq b \wedge (\forall o \in \Delta. a \xrightarrow{o} b),$$

$$a \xleftarrow{\Delta} b \Leftrightarrow a \neq b \wedge (\forall o \in \Delta. a \xleftarrow{o} b),$$

$$a \leftrightarrow_{\Delta} b \Leftrightarrow a \neq b \wedge (\forall o \in \Delta. a \leftrightarrow_o b),$$

$$a \rightleftharpoons_{\Delta} b \Leftrightarrow a \neq b \wedge (\forall o \in \Delta. a \xrightarrow{o} b \vee a \xleftarrow{o} b),$$

$$a \nearrow_{\Delta} b \Leftrightarrow a \neq b \wedge (\forall o \in \Delta. a \xrightarrow{o} b \vee a \leftrightarrow_o b),$$

$$a \nwarrow_{\Delta} b \Leftrightarrow a \neq b \wedge (\forall o \in \Delta. a \xleftarrow{o} b \vee a \leftrightarrow_o b).$$

$\xrightarrow{\Delta}$ and $\xleftarrow{\Delta}$ are called *causalities*, $\rightleftharpoons_{\Delta}$ *commutativity*, $\leftrightarrow_{\Delta}$ *synchronisation*, while $\nearrow_{\Delta}$ and $\nwarrow_{\Delta}$ *weak causalities*. We will use $\rightarrow, \leftarrow, \leftrightarrow, \rightleftharpoons, \nearrow$ and $\nwarrow$ to denote mappings which for $\Delta \in RSet(RS_{con})$ return, respectively, $\xrightarrow{\Delta}, \xleftarrow{\Delta}, \leftrightarrow_{\Delta}, \rightleftharpoons_{\Delta}, \nearrow_{\Delta}$ and $\nwarrow_{\Delta}$. We shall call these mappings *invariants*, and denote their set by SRI . It can be shown that the following holds (the proof will be presented for the general case):

$$SRI(\Delta) = \{\emptyset, \xrightarrow{\Delta}, \xleftarrow{\Delta}, \leftrightarrow_{\Delta}, \rightleftharpoons_{\Delta}, \nearrow_{\Delta}, \nwarrow_{\Delta}, \text{dom}(\Delta) \times \text{dom}(\Delta) - id_{\text{dom}(\Delta)}\}.$$

By symmetry, we can consider only four nontrivial invariants: $\xrightarrow{\Delta}, \xleftarrow{\Delta}, \rightleftharpoons_{\Delta}$ and $\nearrow_{\Delta}$. Note that $\xrightarrow{\Delta} = \nearrow_{\Delta} \cap \rightleftharpoons_{\Delta}$ and $\xleftarrow{\Delta} = \nwarrow_{\Delta} \cap \rightleftharpoons_{\Delta}$ which means that each invariant in $SRI(\Delta)$ can be derived from $\rightleftharpoons_{\Delta}$ and $\nearrow_{\Delta}$.

The approach to concurrency based on the concept of causality requires that for every history Δ and all $a, b \in \text{dom}(\Delta)$, the following rule (*paradigm*) holds:

$$(\exists o \in \Delta. a \leftrightarrow_o b) \Leftrightarrow (\exists o \in \Delta. a \rightarrow_o b) \wedge (\exists o \in \Delta. a \leftarrow_o b).$$

Paradigms will be used to characterise the internal structure of histories. In Section 6 we will analyse RS_{con} in detail.

3.3. Simple report formulas

Let Δ be a report set over RS and $\Sigma = \text{dom}(\Delta)$. In general, a binary *invariant* of Δ might be defined as a relation $I \subseteq \Sigma \times \Sigma$ characterised by a formula: $\forall (\beta, \gamma) \in I \forall x \in \Delta. \Phi(\beta, \gamma, x)$, where β, γ are variables ranging over Σ , x is a variable ranging over RS , and $\Phi(\beta, \gamma, x)$ is a formula built using the $\beta r_{i,x} \gamma$ terms, quantifiers and standard logical connectives and constants. For example, causality could be characterised by $\forall (\beta, \gamma) \in I \forall x \in \Delta. \beta \xrightarrow{x} \gamma$. In this paper we are interested in most basic invariants, generalising the notions of precedence and simultaneity, characterised by quantifier-free formulas Φ . The *simple report formulas*, $\Phi \in SRF$, are defined as follows:

$$\Phi := \text{true} \mid \text{false} \mid \beta r_{1,x} \gamma \mid \cdots \mid \beta r_{k,x} \gamma \mid \neg \Phi \mid \Phi \vee \Phi \mid \Phi \wedge \Phi.$$

Two formulas, $\Phi(\beta, \gamma, x)$ and $\Phi_0(\beta, \gamma, x)$, are *equivalent*, $\Phi \simeq \Phi_0$, if for all $\mu \in RS$ and all distinct $a, b \in \text{dom}(\mu)$, $\Phi(a, b, \mu) \Leftrightarrow \Phi_0(a, b, \mu)$. (The evaluation of simple report formulas follows the standard rules [35].) Equivalent simple report formulas can be substituted for each other.

Notation 3.1. Let B be the set of sequences $\sigma = (\sigma_1, \dots, \sigma_k)$ such that $\sigma_i \in \{\text{true}, \text{false}\}$. We apply the logical $\neg$, $\vee$ and $\wedge$ operations to be elements of B componentwise. We will usually denote *true* by 1 and *false* by 0.

Theorem 3.2. For every $\sigma = (\sigma_1, \dots, \sigma_k) \in B$, let $\Phi_\sigma = \xi_1 \vee \cdots \vee \xi_k$, where $\xi_i = \sigma_i \wedge \beta r_{i,x} \gamma$. Then (upto $\simeq$): $SRF = \{\Phi_\sigma \mid \sigma \in B\}$. Moreover, $\Phi_\sigma \simeq \Phi_\delta \Leftrightarrow \sigma = \delta$.

Proof. See Appendix. $\square$

For the report set of concurrent observations RS_{con} we have (upto $\simeq$):

$$SRF = \{ \text{false}, a \rightarrow_o b, a \leftarrow_o b, a \leftrightarrow_o b, a \rightarrow_o b \vee a \leftarrow_o b, a \rightarrow_o b \vee a \leftrightarrow_o b, \\ a \leftarrow_o b \vee a \leftrightarrow_o b, \text{true} \}.$$

We now introduce the invariant relations characterised by simple report formulas. A relation $I \subseteq \Sigma \times \Sigma$ is a *simple report invariant* of Δ , denoted by $I \in SRI(\Delta)$, if there is $\Phi_\sigma \in SRF$ such that

$$I = \{ (a, b) \in \Sigma \times \Sigma \mid a \neq b \wedge \forall \mu \in \Delta. \Phi_\sigma(a, b, \mu) \}.$$

In what follows, I will be noted by $I_\sigma(\Delta)$. Moreover, we will use I_σ to denote the mapping, called *invariant*, which for every $\Delta \in RSet(RS)$ returns $I_\sigma(\Delta)$. The set of all such mappings will be denoted by SRI . Note that

$$\mathcal{I} = \{(a, b) \in \Sigma \times \Sigma \mid a \neq b \wedge \forall o \in \Delta. a \xrightarrow{o} b \vee a \xleftarrow{o} b\}.$$

is an example of a simple report invariant in RS_{con} .

Proposition 3.3. (1) $SRI(\Delta) = \{I_\sigma(\Delta) \mid \sigma \in B\}$.

(2) If $\sigma \neq \delta$ then there is $\Delta_0 \in RSet(RS)$ such that $I_\sigma(\Delta_0) \neq I_\delta(\Delta_0)$.

Proof. (1) Follows directly from Theorem 3.2.

(2) Let $\sigma = (\sigma_1, \dots, \sigma_k)$ and $\delta = (\delta_1, \dots, \delta_k)$. Without loss of generality, we assume that $\sigma_1 = 1$ and $\delta_1 = 0$. Since RS is nondegenerated, there is $\mu \in RS$ such that $r_{1,\mu} \neq \emptyset$. Let $(a, b) \in r_{1,\mu}$. Define $\Delta_0 = \{\mu\}$. Clearly, $\Delta_0 \in RSet(RS)$. Furthermore, we have $(a, b) \in I_\sigma(\Delta_0)$ and $(a, b) \notin I_\delta(\Delta_0)$. $\square$

For $i = 1, \dots, k$, let $R_i(\Delta)$ and $\mathfrak{R}_i(\Delta)$ be simple report invariants defined as follows:

$$R_i(\Delta) = \{(a, b) \in \Sigma \times \Sigma \mid a \neq b \wedge \forall \mu \in \Delta. a r_{i,\mu} b\},$$

$$\mathfrak{R}_i(\Delta) = \{(a, b) \in \Sigma \times \Sigma \mid a \neq b \wedge \forall \mu \in \Delta. \neg a r_{i,\mu} b\}.$$

$R_i(\Delta)$ is called an *evidence* (it says that something has happened according to all reports in Δ), and $\mathfrak{R}_i(\Delta)$ is called an *alibi* (it says that something has not been reported). For RS_{con} , $\xrightarrow{\Delta}, \xleftarrow{\Delta}, \xleftrightarrow{\Delta}$ are evidences and $\nrightarrow_{\Delta}, \nleftarrow_{\Delta}, \nleftrightarrow_{\Delta}$ are alibis. It is possible to express each simple report invariant as an intersection of alibis.

Proposition 3.4. If $\sigma = (\sigma_1, \dots, \sigma_k) \in B$ and $\{i_1, \dots, i_s\} = \{i \mid \sigma_i = 0\} \neq \emptyset$ then $I_\sigma(\Delta) = \mathfrak{R}_{i_1}(\Delta) \cap \dots \cap \mathfrak{R}_{i_s}(\Delta)$.

3.4. Signatures

Although the set of simple report invariants comprises 2^k relations, we do not really need all of them since they are not independent. We will now address the problem of finding a set of invariants from which all the relations in $SRI(\Delta)$ can be derived.

A *signature* of a nonempty set $\Delta \subseteq RSet(RS)$ is a set of invariants $S \subseteq SRI$ such that, for all $\Delta, \Delta_0 \in \Delta$, if $\text{dom}(\Delta) = \text{dom}(\Delta_0)$ then

$$\forall I \in S. I(\Delta) = I(\Delta_0) \Rightarrow \forall I \in SRI. I(\Delta) = I(\Delta_0).$$

S is *universal* if $\Delta = RSet(RS)$. For RS_{con} , $\{\nearrow, \rightrightarrows\}$ is a universal signature. Clearly, SRI is always a universal signature. In general, the smaller Δ is, the fewer and simpler invariants one needs to obtain a signature.

Consider RS_{con} and two observations, o_1 and o_2 , shown in Fig. 7. Then $S = \{\rightarrow\}$ is a signature for $A = \{\{o_1\}, \{o_2\}\}$, while $\{\leftrightarrow\}$ is not ($\leftrightarrow = \emptyset = \leftrightarrow$). We further observe that S can be regarded as ‘smaller’ than two other signatures of A , $\{\rightarrow, \leftrightarrow\}$ and $\{\nearrow\}$. (For the latter, this is motivated by the fact that $\rightarrow \subseteq \nearrow$ holds, for all Δ .)

A signature comprises invariants which for every $\Delta \in A$ provide enough information to construct $SRI(\Delta)$. It is, therefore, natural to always look for a ‘minimal’ signature. For distinct $I, J \in SRI$, let $I \blacktriangleright J$ if $I(\Delta) \subseteq J(\Delta)$, for all $\Delta \in RSet(RS)$. That is $I \blacktriangleright J$ means that the size of $I(\Delta)$ never exceeds that of $J(\Delta)$. For RS_{con} , we have: $\rightarrow \blacktriangleright \nearrow$, $\rightarrow \blacktriangleright \leftrightarrow$ and $\leftrightarrow \blacktriangleright \nearrow$.

A signature S of $A \subseteq RSet(RS)$ is *minimal* if the following hold:

No proper subset of S is a signature of A .

If $I \blacktriangleright J$ and $J \in S$ then $S - \{J\} \cup \{I\}$ is not a signature of A .

For A above $\{\rightarrow\}$ is minimal signature, while $\{\rightarrow, \leftrightarrow\}$ and $\{\nearrow\}$ are not.

Theorem 3.5. (Existence of minimal signature). *For every nonempty $A \subseteq RSet(RS)$ there is a minimal signature.*

Proof. Let $\Gamma : SRI \rightarrow \mathbb{N}$ be any mapping such that $I \blacktriangleright J \Rightarrow \Gamma(I) < \Gamma(J)$. For a signature S , let $\Gamma(S) = \sum_{I \in S} \Gamma(I)$. Clearly, if S is not minimal then there is a signature S' such that $\Gamma(S') < \Gamma(S)$ (see the last definition). Thus, since the number of the signatures of A is finite, there is at least one minimal signature. $\square$

Finding minimal signature can be a nontrivial problem. However, it is always possible to find one comprising no more than k invariants:

Theorem 3.6. $\{\mathcal{A}_1, \dots, \mathcal{A}_k\}$ is a universal signature.

Proof. Follows directly from Proposition 3.4. $\square$

4. Histories

A report set Δ was the first approximation of the notion of a history; it has been assumed that the reports in Δ have the same domain. What we also need is some notion of completeness for Δ which would be based on the invariant properties introduced in the previous section.

Let $\Delta \in RSet(RS)$ and $S \subseteq SRI$. The S -closure of Δ , denoted by $\Delta^{\langle S \rangle}$, comprises all $\mu \in RS$ such that $\text{dom}(\mu) = \text{dom}(\Delta)$, and for all $I_\sigma \in S$,

$$(a, b) \in I_\sigma(\Delta) \Rightarrow \Phi_\sigma(a, b, \mu).$$

Consider $\Delta = \{o_1, o_3, o_8\} \in RSet(RS_{con})$ and $S = \{\rightrightarrows, \nearrow\}$, defined in Fig. 3. Then $o \in \Delta^{<S>}$ iff $\text{dom}(o) = \{a, b, c, d, e\}$ and

$$\forall x, y \in \text{dom}(o). (x \xrightarrow{\Delta} y \Rightarrow x \xrightarrow{o} y \vee x \xleftarrow{o} y) \wedge (x \xleftarrow{\Delta} y \Rightarrow x \xrightarrow{o} y \vee x \xleftarrow{o} y)$$

where $\nearrow$ and $\xleftarrow{\Delta}$ are as in Fig. 3. One may check that

$$\Delta^{<\nearrow, \rightrightarrows>} = \Delta_0 = \{o_1, \dots, o_{10}\} = \Delta_0^{<\nearrow, \rightrightarrows>}.$$

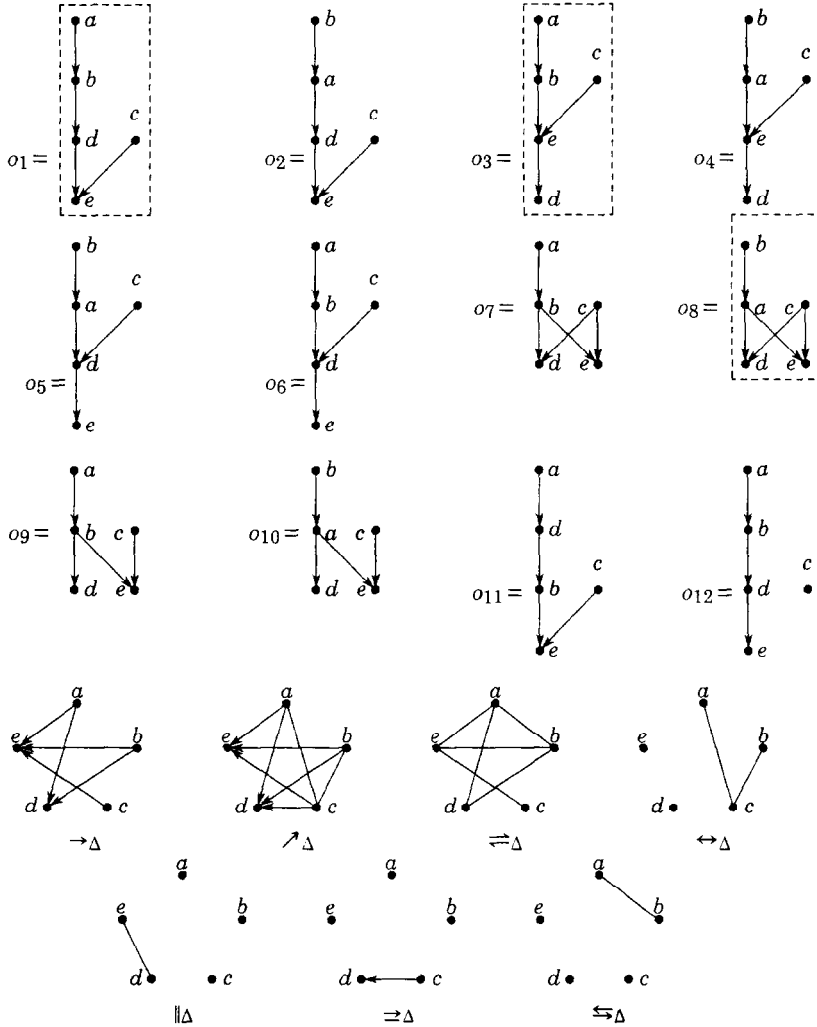


Fig. 3. Invariant closure and components (symmetric relationship is represented by undirected arcs): $\Delta = \{o_1, o_3, o_8\}$ and $\Delta_0 = \{o_1, \dots, o_{10}\}$, $S = \{\rightrightarrows, \nearrow\}$, $\Delta^{<S>} = \Delta^{<SRI>} = \Delta_0 = \Delta_0^{<S>} = \Delta_0^{<SRI>}$ and $CSRI(\Delta) = \{\emptyset, \rightarrow_{\Delta}, \leftarrow_{\Delta}, \nearrow_{\Delta}, \nwarrow_{\Delta}, \rightrightarrows_{\Delta}, \leftarrow_{\Delta}, \parallel_{\Delta}\}$.

Similarly, if $\text{dom}(o) = \{a, b, c, d, e\}$ then

$$o \in \Delta^{\langle \Rightarrow \rangle} \Leftrightarrow \forall x, y \in \text{dom}(o). (x \xrightarrow[\Delta]{\Rightarrow} y \Rightarrow x \xrightarrow{o} y \vee x \xleftarrow{o} y),$$

$$o \in \Delta^{\langle \nearrow \rangle} \Leftrightarrow \forall x, y \in \text{dom}(o). (x \nearrow y \Rightarrow x \xrightarrow{o} y \vee x \xleftarrow{o} y).$$

Note that $o_{11} \in \Delta^{\langle \Rightarrow \rangle} - \Delta^{\langle \nearrow \rangle}$ and $o_{12} \in \Delta^{\langle \nearrow \rangle} - \Delta^{\langle \Rightarrow \rangle}$. For example, $o_{11} \notin \Delta^{\langle \nearrow \rangle}$ since $d \xrightarrow{o_{11}} b$ and $b \not\nearrow d$, and $o_{12} \notin \Delta^{\langle \Rightarrow \rangle}$ because $c \xleftarrow{o_{12}} e$ but $c \not\xrightarrow[\Delta]{\Rightarrow} e$.

Proposition 4.1 (Basic properties of S-closure). *Let $\Delta, \Delta_0 \in \text{RSet}(RS)$ be such that $\text{dom}(\Delta) = \text{dom}(\Delta_0)$. Moreover, let $S \subseteq S_0 \subseteq \text{SRI}$.*

- (1) $\Delta \subseteq \Delta^{\langle S \rangle}$.
- (2) $\Delta^{\langle S_0 \rangle} \subseteq \Delta^{\langle S \rangle}$.
- (3) $(\forall I \in S. I(\Delta) = I(\Delta_0)) \Rightarrow \Delta^{\langle S \rangle} = \Delta_0^{\langle S \rangle}$.
- (4) $I(\Delta) = I(\Delta^{\langle S \rangle})$ for all $I \in S$.
- (5) $(\Delta^{\langle S \rangle})^{\langle S \rangle} = \Delta^{\langle S \rangle}$.

Proof. (1)–(3): Obvious.

(4) Let $I_\sigma \in S$. We have the following:

$$\begin{aligned} I_\sigma(\Delta^{\langle S \rangle}) &= \{(a, b) \in \Sigma \times \Sigma \mid a \neq b \wedge \forall \mu \in \Delta^{\langle S \rangle}. \Phi_\sigma(a, b, \mu)\} \\ &=_{(1)} \{(a, b) \in \Sigma \times \Sigma \mid a \neq b \wedge (\forall \mu \in \Delta. \Phi_\sigma(a, b, \mu)) \wedge (\forall \mu \in \Delta^{\langle S \rangle}. \Phi_\sigma(a, b, \mu))\}. \end{aligned}$$

From the definition of S-closure it follows that

$$\begin{aligned} &((a, b) \in \Sigma \times \Sigma \wedge a \neq b \wedge \forall \mu \in \Delta^{\langle S \rangle}. \Phi_\sigma(a, b, \mu)) \\ &\Leftrightarrow (a, b) \in I_\sigma(\Delta) \Leftrightarrow \forall \mu \in \Delta. \Phi_\sigma(a, b, \mu). \end{aligned}$$

Hence,

$$I_\sigma(\Delta^{\langle S \rangle}) = \{(a, b) \in \Sigma \times \Sigma \mid a \neq b \wedge \forall \mu \in \Delta. \Phi_\sigma(a, b, \mu)\} = I_\sigma(\Delta).$$

(5) Follows directly from (3) and (4). $\square$

Proposition 4.2. (Closure by universal signature). *If $\Delta \in \text{RSet}(RS)$ and S is a universal signature then $\Delta^{\langle S \rangle} = \Delta^{\langle \text{SRI} \rangle}$.*

Proof. By Proposition 4.1(2), $\Delta^{\langle \text{SRI} \rangle} \subseteq \Delta^{\langle S \rangle}$. To show the reverse inclusion we first observe that, by Proposition 4.1(4), $\forall I \in S. I(\Delta) = I(\Delta^{\langle S \rangle})$. Hence, since S is a universal signature, $\forall I \in \text{SRI}. I(\Delta^{\langle S \rangle})$. Consequently, by Proposition 4.1(3), $\Delta^{\langle \text{SRI} \rangle} = (\Delta^{\langle S \rangle})^{\langle \text{SRI} \rangle}$. Thus, by Proposition 4.1(2), $\Delta^{\langle S \rangle} \subseteq (\Delta^{\langle S \rangle})^{\langle \text{SRI} \rangle}$. Hence, $\Delta^{\langle S \rangle} \subseteq \Delta^{\langle \text{SRI} \rangle}$. $\square$

We now may introduce formally the central notion of our model:

*A history over the report system RS , $\Delta \in \text{Hist}(RS)$,
is a nonempty report set Δ such that $\Delta = \Delta^{\langle \text{SRI} \rangle}$.*

Remark. The term “history” has been used by many authors, e.g., [6, 16, 24, 32], to denote different concepts in the area of concurrency. We added yet another notion to that list, but we feel that it captures best the meaning of the last definition.

In other words, every history is a report set which can be fully described by the invariants it generates. For example, in Fig. 3, Δ is not a history, while Δ_0 is ($\{\nearrow, \rightrightarrows\}$ is a universal signature and $\Delta_0^{\langle \nearrow, \rightrightarrows \rangle} = \Delta_0$). As a direct consequence of Proposition 4.2 and Theorem 3.6 we obtain the following:

Proposition 4.3. *A nonempty report set Δ is a history iff*

$$\Delta = \{\mu \in RS \mid \text{dom}(\mu) = \text{dom}(\Delta) \wedge \forall i \leq k. (a, b) \in \mathfrak{I}_i(\Delta) \Rightarrow \neg a r_{i,\mu} b\}.$$

Proposition 4.4 (Identification of history by signature). *Let S be a signature of $\Delta \subseteq RSet(RS)$. If $\Delta, \Delta_0 \in \mathcal{A}$ are two histories with the same domain then*

$$(\forall I \in S. I(\Delta) = I(\Delta_0)) \Rightarrow \Delta = \Delta_0.$$

Proof. From the definition of the signature, it follows that $\forall I \in SRI. I(\Delta) = I(\Delta_0)$. Hence, by Proposition 4.1(3), $\Delta^{\langle SRI \rangle} = \Delta_0^{\langle SRI \rangle}$. This and $\Delta, \Delta_0 \in \text{Hist}(RS)$ yields $\Delta = \Delta_0$. $\square$

The last result implies that if Δ is a history then the following can be identified:

Δ – set of observations,

$\{I_\sigma(\Delta) \mid \sigma \in B\}$ – all invariants,

$\mathfrak{I}_1(\Delta), \mathfrak{I}_2(\Delta), \dots, \mathfrak{I}_k(\Delta)$ – all alibis,

$I_{i_1}(\Delta), \dots, I_{i_l}(\Delta), \Delta$ – some invariants and a family of report sets,

where $\{I_{i_1}, \dots, I_{i_l}\}$ is a signature of Δ and $\Delta \in \mathcal{A}$. For example, the history Δ_0 in Fig. 3 can be identified with $\{\nearrow, \rightrightarrows\}$, where $\nearrow$ and $\rightrightarrows$ are shown in Fig. 3.

5. Paradigms

In this section we consider structural properties of a single history. Suppose Δ is a history over RS_{con} , $o \in \Delta$ and $a \leftrightarrow_o b$. The classical approach based on causality relation would now imply that there be two additional observations in Δ , one in which a precedes b , and one in which b precedes a . So far our model does not provide any means to ensure that Δ does include the two additional observations. What we need is the ability to express rules relating different observations of the same history, such as:

$$(\exists o \in \Delta. a \leftrightarrow_o b) \Leftrightarrow (\exists o \in \Delta. a \rightarrow_o b) \wedge (\exists o \in \Delta. a \leftarrow_o b).$$

We will call such rules, capturing the structural properties of histories, *paradigms* of the report system. They can be used to project the structural properties of systems described on the system level onto the behaviours (histories) dealt with on the invariant level; different paradigms will essentially correspond to different types of constructs used on the system level. The *paradigms*, $\omega \in \text{Par}$, are defined by

$$\omega := \text{true} \mid \text{false} \mid \Psi_1 \mid \dots \mid \Psi_k \mid \neg \omega \mid \omega \vee \omega \mid \omega \wedge \omega \mid \omega \Rightarrow \omega,$$

where each $\Psi_i = \exists x. \beta r_{i,x} \gamma$ is called a *simple trait*. It is a formula stating that a given relationship $r_{i,x}$ has been observed. The evaluation of $\omega \in \text{Par}$ follows the standard rules [35]. A history $\Delta \in \text{Hist}(\text{RS})$ satisfies a paradigm $\omega(\beta, \gamma) \in \text{Par}$ if for all $a, b \in \text{dom}(\Delta)$,

$$a \neq b \Rightarrow \omega_\Delta(a, b),$$

where the index in ω_Δ means that x ranges over Δ . We denote this by $\Delta \in \text{Par}(\omega)$. Two paradigms, ω and ω_0 , are *equivalent*, denoted by $\omega \sim \omega_0$, if $\text{Par}(\omega) = \text{Par}(\omega_0)$.

Before formulating a characterisation theorem for paradigms, we discuss the relationship between paradigms and the *components* of simple report invariants.

Let $\Delta \in \text{RSet}(\text{RS})$, $\Sigma = \text{dom}(\Delta)$ and $\sigma = (\sigma_1, \dots, \sigma_k) \in B$. Define

$$C_\sigma(\Delta) = \{(a, b) \in \Sigma \times \Sigma - \text{id}_\Sigma \mid \forall i \leq k. ((\exists \mu \in \Delta. a r_{i,\mu} b) \Leftrightarrow \sigma_i = 1)\}.$$

The set $\text{CSRI}(\Delta) = \{C_\sigma(\Delta) \mid \sigma \in B\}$ is called the set of *components* of simple report invariants (see Fig. 3). Each component can be obtained from the sets in $\text{SRI}(\Delta)$ using the standard set-theoretic operations, and each set which can be obtained in this way is the union of some of the components of $\text{CSRI}(\Delta)$.

For Δ of Fig. 3, we have $\parallel_\Delta = C_{111}(\Delta)$ and $\leftrightarrow_\Delta = C_{001}(\Delta)$, where

$$\begin{aligned} a \parallel_\Delta b &\Leftrightarrow \exists o \in \Delta. a \rightarrow_o b \wedge \exists o \in \Delta. a \leftarrow_o b \wedge \exists o \in \Delta. a \leftrightarrow_o b \\ a \leftrightarrow_\Delta b &\Leftrightarrow \neg \exists o \in \Delta. a \rightarrow_o b \wedge \neg \exists o \in \Delta. a \leftarrow_o b \wedge \exists o \in \Delta. a \leftrightarrow_o b. \end{aligned}$$

Proposition 5.1. *Let $\Delta \in \text{Hist}(\text{RS})$.*

- (1) $C_\sigma(\Delta) \cap C_\theta(\Delta) = \emptyset$ for $\sigma \neq \theta$.
- (2) $\bigcup_{\sigma \in B} C_\sigma(\Delta) = \text{dom}(\Delta) \times \text{dom}(\Delta) - \text{id}_{\text{dom}(\Delta)}$.
- (3) $C_{00\dots 0}(\Delta) = \emptyset$. $\square$

Lemma 5.2. *For every $\omega(\beta, \gamma) \in \text{Par}$ there are $\sigma^1, \dots, \sigma^l$ ($l \geq 1$) such that if $\Delta \in \text{Hist}(\text{RS})$ and $a, b \in \text{dom}(\Delta)$, $a \neq b$, then the following holds.*

$$\omega_\Delta(a, b) \Leftrightarrow (a, b) \notin C_{\sigma^1}(\Delta) \cup \dots \cup C_{\sigma^l}(\Delta).$$

Proof. For $\omega = \text{true}$ we have

$$\omega_\Delta(a, b) \Leftrightarrow (a, b) \notin \emptyset \Leftrightarrow_{\text{Prop. 5.1(3)}} (a, b) \notin C_{00\dots 0}(\Delta).$$

For $\omega = \text{false}$ we have

$$\begin{aligned}\omega_A(a, b) &\Leftrightarrow (a, b) \notin \text{dom}(\Delta) \times \text{dom}(\Delta) - \text{id}_{\text{dom}(\Delta)} \\ &\Leftrightarrow_{\text{Prop. 5.1(2)}} (a, b) \notin \bigcup_{\sigma \in B} C_\sigma(\Delta).\end{aligned}$$

For $\omega = \Psi_i$, where $1 \leq i \leq k$, we have

$$\omega_A(a, b) \Leftrightarrow (a, b) \notin \bigcup_{\sigma_i=0} C_{(\sigma_1, \dots, \sigma_k)}(\Delta).$$

Suppose now that ω and δ are such that the following hold.

$$\begin{aligned}\omega_A(a, b) &\Leftrightarrow (a, b) \notin C_{\sigma^1}(\Delta) \cup \dots \cup C_{\sigma^l}(\Delta), \\ \delta_A(a, b) &\Leftrightarrow (a, b) \notin C_{\theta^1}(\Delta) \cup \dots \cup C_{\theta^m}(\Delta).\end{aligned}$$

We need to show that the Lemma holds for $\neg\omega$ and $\omega \wedge \delta$ (then it would, of course, hold also for $\omega \vee \delta$ and $\omega \Rightarrow \delta$). For $\neg\omega$ we have

$$\begin{aligned}(\neg\omega)_A(a, b) &\Leftrightarrow \text{not } \omega_A(a, b) \Leftrightarrow (a, b) \in C_{\sigma^1}(\Delta) \cup \dots \cup C_{\sigma^l}(\Delta) \\ &\Leftrightarrow_{\text{Prop. 5.1(1, 2, 3)}} (a, b) \in C_{00\dots 0}(\Delta) \cup \bigcup_{\sigma \notin \{\sigma^1, \dots, \sigma^l\}} C_\sigma(\Delta).\end{aligned}$$

For $\omega \wedge \delta$ we have

$$\begin{aligned}(\omega \wedge \delta)_A(a, b) &\Leftrightarrow \omega_A(a, b) \text{ and } \delta_A(a, b) \\ &\Leftrightarrow (a, b) \notin C_{\sigma^1}(\Delta) \cup \dots \cup C_{\sigma^l}(\Delta) \text{ and } \\ &\quad (a, b) \notin C_{\theta^1}(\Delta) \cup \dots \cup C_{\theta^m}(\Delta) \\ &\Leftrightarrow (a, b) \notin C_{\sigma^1}(\Delta) \cup \dots \cup C_{\sigma^l}(\Delta) \cup C_{\theta^1}(\Delta) \cup \dots \cup C_{\theta^m}(\Delta). \quad \square\end{aligned}$$

Next we obtain a characterisation of paradigms in terms of empty components.

Theorem 5.3. *For every $\omega \in \text{Par}$ there are $\sigma^1, \dots, \sigma^l \in B$ ($l \geq 1$) such that*

$$\text{Par}(\omega) = \{ \Delta \in \text{Hist}(RS) \mid C_{\sigma^1}(\Delta) \cup \dots \cup C_{\sigma^l}(\Delta) = \emptyset \}.$$

Conversely, if $\sigma^1, \dots, \sigma^l \in B$ ($l \geq 1$) then there is $\omega \in \text{Par}$ such that the above holds.

Proof. The first part follows directly from Lemma 5.2. The second part follows from the fact that for every $\sigma \in B$ there is $\kappa_\sigma \in \text{Par}$ such that $\text{Par}(\kappa_\sigma) = \{ \Delta \in \text{Hist}(RS) \mid C_\sigma(\Delta) = \emptyset \}$ (see Proposition 5.4 below). Hence, for $\omega = \kappa_{\sigma^1} \wedge \dots \wedge \kappa_{\sigma^l}$ we have

$$\begin{aligned}\text{Par}(\omega) &= \text{Par}(\kappa_{\sigma^1} \wedge \dots \wedge \kappa_{\sigma^l}) = \text{Par}(\kappa_{\sigma^1}) \cap \dots \cap \text{Par}(\kappa_{\sigma^l}) \\ &= \{ \Delta \in \text{Hist}(RS) \mid C_{\sigma^1}(\Delta) \cup \dots \cup C_{\sigma^l}(\Delta) = \emptyset \}. \quad \square\end{aligned}$$

Theorem 5.3 establishes a link between the paradigms of report systems and the components of simple report invariants. To obtain an alternative characterisation of paradigms, we proceed as follows: Let $\sigma = (\sigma_1, \dots, \sigma_k) \in B$, $\{i_1, \dots, i_p\} = \{i \mid \sigma_i = 1\}$ and $\{j_1, \dots, j_q\} = \{j \mid \sigma_j = 0\}$. A *simple report law*, $\kappa_\sigma \in \text{SRL}$, is defined as

$$\text{true} \wedge \Psi_{i_1} \wedge \dots \wedge \Psi_{i_p} \Rightarrow \text{false} \vee \Psi_{j_1} \vee \dots \vee \Psi_{j_q}.$$

Proposition 5.4. *For all $\Delta \in \text{Hist}(RS)$ and $\sigma \in B$, $\Delta \in \text{Par}(\kappa_\sigma) \Leftrightarrow C_\sigma(\Delta) = \emptyset$.*

Proof. Let $\kappa_\sigma = (\text{true} \wedge \Psi_{i_1} \wedge \dots \wedge \Psi_{i_p} \Rightarrow \text{false} \vee \Psi_{j_1} \vee \dots \vee \Psi_{j_q})$. We have the following (below a, b range over $\text{dom}(\Delta) \times \text{dom}(\Delta) - \text{id}_{\text{dom}(\Delta)}$):

$$\begin{aligned}
& \Delta \in \text{Par}(\kappa_\sigma) \\
& \Leftrightarrow \forall a, b. (\text{true} \wedge \exists x. a r_{i_1, x} b \wedge \dots \wedge \exists x. a r_{i_p, x} b) \\
& \quad \Rightarrow (\text{false} \vee \exists x. a r_{j_1, x} b \vee \dots \vee \exists x. a r_{j_q, x} b) \\
& \Leftrightarrow \forall a, b. \neg (\text{true} \wedge \exists x. a r_{i_1, x} b \wedge \dots \wedge \exists x. a r_{i_p, x} b) \\
& \quad \vee \text{false} \vee \exists x. a r_{j_1, x} b \vee \dots \vee \exists x. a r_{j_q, x} b \\
& \Leftrightarrow \forall a, b. \neg (\text{true} \wedge \exists x. a r_{i_1, x} b \wedge \dots \wedge \exists x. a r_{i_p, x} b \\
& \quad \wedge \text{true} \wedge \neg \exists x. a r_{j_1, x} b \wedge \dots \wedge \neg \exists x. a r_{j_q, x} b) \\
& \Leftrightarrow \forall a, b. (a, b) \notin C_\sigma(\Delta) \Leftrightarrow C_\sigma(\Delta) = \emptyset. \quad \square
\end{aligned}$$

By joining Theorem 5.3 and Proposition 5.4, we obtain the main characterisation theorem for paradigms of report systems.

Theorem 5.5. *Paradigms are conjunctions of simple report laws*

$$\text{Par} = \{ \kappa_{\sigma^1} \wedge \dots \wedge \kappa_{\sigma^l} \mid l \geq 1 \wedge \sigma^1, \dots, \sigma^l \in B \}.$$

Note: The above equality holds up to $\sim$.

6. Report system of concurrent observations

We now will use the results from the previous sections to analyse the report system of concurrent observations. RS_{con} comprises reports $\mu = (\Sigma, r_1, r_2, r_3)$ for which there is an observation $o \in \text{Obs}$ such that $\text{dom}(o) = \Sigma$, $\xrightarrow{o} = r_1$, $\xleftarrow{o} = r_2$ and $\xleftrightarrow{o} = r_3$. We identify μ with the observation o .

6.1. Simple report invariants

Let $\Delta \in \text{Hist}(RS_{\text{con}})$ be a history, fixed until the end of Section 6.3. Moreover, let $\Sigma = \text{dom}(\Delta)$ and $\Omega = \Sigma \times \Sigma - \text{id}_\Sigma$. Recall that although there are eight simple report invariants in $\text{SRI}(\Delta)$, it is sufficient only to consider four: $\xrightarrow{\Delta}$, $\xleftarrow{\Delta}$, $\xleftrightarrow{\Delta}$ and $\nearrow_{\Delta}$. The first two can be interpreted, respectively, as *causality* and *synchronisation*. The third invariant, $\xleftrightarrow{\Delta}$, can be interpreted as *commutativity* since $a \xleftrightarrow{\Delta} b$ implies that there is no observation $o \in \Delta$ for which $a \xleftarrow{o} b$. The last invariant, $\nearrow_{\Delta}$, can be interpreted as *weak causality*, as $a \nearrow_{\Delta} b$ implies $a \xrightarrow{o} b \vee a \xleftarrow{o} b$ for all $o \in \Delta$. We now prove a number of properties of simple report invariants.

Proposition 6.1. (1) $a \rightarrow_{\Delta} b \Rightarrow a \not\rightarrow_{\Delta} b \wedge \neg b \not\rightarrow_{\Delta} a$.

(2) $a \rightarrow_{\Delta} b \not\rightarrow_{\Delta} c \vee a \not\rightarrow_{\Delta} b \rightarrow_{\Delta} c \Rightarrow a \not\rightarrow_{\Delta} c$.

Proof. (1) Obvious.

(2) Suppose $a \rightarrow_{\Delta} b \not\rightarrow_{\Delta} c$ and $\neg a \not\rightarrow_{\Delta} c$. We first observe that $a \neq c$ since, otherwise, we would have $a \rightarrow_{\Delta} b$ and $b \not\rightarrow_{\Delta} a$. Hence, there is $o \in \Delta$ such that $c \rightarrow_o a$. Thus, by $a \rightarrow_{\Delta} b$, $c \rightarrow_o b$. On the other hand, $b \not\rightarrow_{\Delta} c$ implies $b \rightarrow_o c$ or $b \leftrightarrow_o c$, a contradiction. Hence, $a \rightarrow_{\Delta} b \not\rightarrow_{\Delta} c \Rightarrow a \not\rightarrow_{\Delta} c$. The second part can be shown in a similar way. $\square$

Proposition 6.2. (1) $a \rightarrow_{\Delta} b \not\rightarrow_{\Delta} c \rightarrow_{\Delta} d \Rightarrow a \rightarrow_{\Delta} d$.

(2) $a \not\rightarrow_{\Delta} b \rightarrow_{\Delta} c \not\rightarrow_{\Delta} d \Rightarrow a \not\rightarrow_{\Delta} d \vee a = d$.

Proof. (1) Suppose $a \rightarrow_{\Delta} b \not\rightarrow_{\Delta} c \rightarrow_{\Delta} d$ and $\neg a \rightarrow_{\Delta} d$. We first observe that $a \neq d$ since, otherwise, we would have $c \rightarrow_{\Delta} a \rightarrow_{\Delta} b$ and $b \not\rightarrow_{\Delta} c$. Thus, from $\neg a \rightarrow_{\Delta} d$ it follows that there is $o \in \Delta$ such that $\neg a \rightarrow_o d$. We also have $a \rightarrow_o b, c \rightarrow_o d$ and $\neg c \rightarrow_o b$, a contradiction with the definition of an interval order.

(2) Suppose $a \not\rightarrow_{\Delta} b \rightarrow_{\Delta} c \not\rightarrow_{\Delta} d, a \neq d$ and $\neg a \not\rightarrow_{\Delta} d$. From $\neg a \not\rightarrow_{\Delta} d$ it follows that there is $o \in \Delta$ such that $d \rightarrow_o a$. We also have $b \rightarrow_o c, \neg b \rightarrow_o a$ and $\neg d \rightarrow_o c$, a contradiction with the definition of an interval order. $\square$

Proposition 6.3. Let $\Delta \subseteq Obs_{step}$.

(1) $a \not\rightarrow_{\Delta} b \not\rightarrow_{\Delta} c \Rightarrow a \not\rightarrow_{\Delta} c \vee a = c$.

(2) $a \not\rightarrow_{\Delta} b \rightarrow_{\Delta} c \vee a \rightarrow_{\Delta} b \not\rightarrow_{\Delta} c \Rightarrow a \rightarrow_{\Delta} c$.

Proof. (1) Suppose $a \not\rightarrow_{\Delta} b \not\rightarrow_{\Delta} c, a \neq c$ and $\neg a \not\rightarrow_{\Delta} c$. From $\neg a \not\rightarrow_{\Delta} c$ it follows that there is $o \in \Delta$ such that $c \rightarrow_o a$. By $a \not\rightarrow_{\Delta} b, a \rightarrow_o b \vee a \leftrightarrow_o b$. If $a \rightarrow_o b$ then $c \rightarrow_o b$. If $a \leftrightarrow_o b$ then, because o is a step sequence, also $c \rightarrow_o b$. Hence, in both cases there is a contradiction with $\neg a \not\rightarrow_{\Delta} c$.

(2) Suppose $a \rightarrow_{\Delta} b \not\rightarrow_{\Delta} c$. From Proposition 6.1(2) we have $a \not\rightarrow_{\Delta} c$. Suppose $\neg a \rightarrow_{\Delta} c$. Then there is $o \in \Delta$ such that $a \leftrightarrow_o c$. By $a \rightarrow_{\Delta} b, a \rightarrow_o b$. But because o is a step sequence, this means that $c \rightarrow_o b$, a contradiction with $b \not\rightarrow_{\Delta} c$. The second part can be shown in a similar way. $\square$

In Section 7 we show that sometimes the assertions from the above three propositions can be used as axioms for minimal signatures.

Proposition 6.4. (1) If $\Delta \subseteq Obs_{step}$ then $a \leftrightarrow_{\Delta} b \leftrightarrow_{\Delta} c \Rightarrow (a \leftrightarrow_{\Delta} c \vee a = c)$.

(2) $a \leftrightarrow_{\Delta} b \leftrightarrow_{\Delta} c \leftrightarrow_{\Delta} a \leftrightarrow_{\Delta} d \leftrightarrow_{\Delta} b \Rightarrow (c = d \vee c \leftrightarrow_{\Delta} d)$.

Proof. (1) From Proposition 6.3(1) and $\leftrightarrow_{\Delta} = \not\rightarrow_{\Delta} \cap \rightarrow_{\Delta}$.

(2) Suppose that $\neg(c = d \vee c \leftrightarrow_{\Delta} d)$. Without loss of generality, we may assume that there is $o \in \Delta$ such that $c \rightarrow_o d$. We may also assume that $a \rightarrow_o b$. Thus, by $a \leftrightarrow_o d$ and $b \leftrightarrow_o c$, we obtain a contradiction with the definition of interval poset. $\square$

6.2. Components of simple report invariants

The relationship between the components of simple report invariants is illustrated in Fig. 4 (see also Fig. 3). Note that we use the following notation:

$$\begin{aligned} C_{100}(\Delta) &= \rightarrow_{\Delta}, & C_{010}(\Delta) &= \leftarrow_{\Delta}, & C_{001}(\Delta) &= \leftrightarrow_{\Delta}, & C_{111}(\Delta) &= \parallel_{\Delta}, \\ C_{110}(\Delta) &= \overleftarrow{\rightarrow}_{\Delta}, & C_{101}(\Delta) &= \overrightarrow{\leftarrow}_{\Delta}, & C_{011}(\Delta) &= \overleftrightarrow{\leftarrow}_{\Delta}. \end{aligned}$$

We do not need $C_{000}(\Delta)$ as it is always empty (Proposition 5.1(3)). By symmetry we only discuss five components: $\rightarrow_{\Delta}$, $\parallel_{\Delta}$, $\overleftarrow{\rightarrow}_{\Delta}$, $\overleftrightarrow{\leftarrow}_{\Delta}$, and $\overrightarrow{\leftarrow}_{\Delta}$. The first component (and also an invariant), $\rightarrow_{\Delta}$, is a well-known *causality*. The next component, $\parallel_{\Delta}$, should be interpreted as *concurrency* (two events can be observed simultaneously and in both orders); it is supported by the so-called true concurrency models. The third component, $\overleftarrow{\rightarrow}_{\Delta}$, represents *interleaving* (two events can be observed in both orders, but not simultaneously). Interleaving is used, e.g., in models that are based on sequences of event occurrences. The fourth component (and also an invariant), $\overleftrightarrow{\leftarrow}_{\Delta}$, can be interpreted as *synchronisation*. It is used in its implicit form to model ‘handshake’ communication. The fifth component, $\overrightarrow{\leftarrow}_{\Delta}$, is not, to our knowledge, supported by any of the existing models. It captures *disabling* of one event by another event, and was first discussed in [17] and [22], from where we took a priority system represented by the net in Fig. 5 (b has a higher priority than c). In the initial state c can occur simultaneously with a , or c can be executed first and then a . In both cases the priority constraint is satisfied. However, it is not possible for a to precede c since the execution of a makes event b enabled, *disabling* c . Hence, the system generates a concurrent history Δ such that $c \overrightarrow{\leftarrow}_{\Delta} a$. Note that in [5] it was observed that whether $\{a, c\}$ should be allowed as a valid observation is intrinsically related to whether or not one can

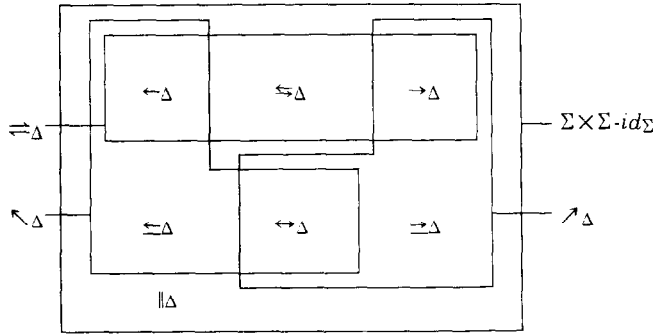


Fig. 4. Components and simple report invariants in RS_{con} .

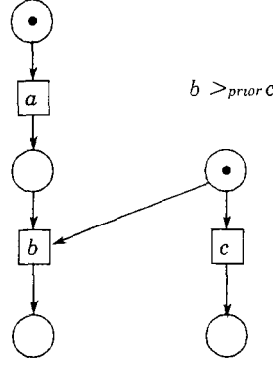


Fig. 5. Priority system.

regard a as an event taking some time. Essentially, if a is instantaneous (takes zero time) then $\{a, c\}$ should not be allowed, and a partial order semantics can be constructed along the lines described in [5]. If, however, a cannot be regarded as instantaneous (possibly because a is itself a compound event) then $\{a, c\}$ should be allowed. As [13] point out, a proper treatment of priorities in real-time systems usually requires considering noninstantaneous events. Note that, for the six histories discussed in Section 1, we have the following:

$$a \parallel_{A_1} b, \quad a \xleftrightarrow{A_2} b, \quad a \xrightarrow{A_3} b, \quad b \xrightarrow{A_4} a, \quad a \xleftrightarrow{A_5} b, \quad a \xrightarrow{A_6} b.$$

6.3. Paradigms and signatures

In the terminology introduced in Section 5, we have

$$\Psi_1(\beta, \gamma) = \exists o. \beta \xrightarrow{o} \gamma,$$

$$\Psi_2(\beta, \gamma) = \exists o. \beta \xleftarrow{o} \gamma,$$

$$\Psi_3(\beta, \gamma) = \exists o. \beta \xleftrightarrow{o} \gamma.$$

Some of the paradigms of RS_{con} are equivalent, which reduces the number of cases we consider. There are $2^3 = 8$ simple report laws; however, only five of them are independent, namely,

$$\omega_1 = \Psi_3 \Rightarrow \Psi_1 \vee \Psi_2,$$

$$\omega_2 = \Psi_1 \wedge \Psi_2 \Rightarrow \Psi_3,$$

$$\omega_3 = \Psi_1 \wedge \Psi_3 \Rightarrow \Psi_2,$$

$$\omega_4 = \Psi_1 \Rightarrow \Psi_2 \vee \Psi_3,$$

$$\omega_5 = \Psi_1 \wedge \Psi_2 \wedge \Psi_3 \Rightarrow \text{false}.$$

From Proposition 5.4 we obtain the following.

Proposition 6.5. *Let $\Delta \in \text{Hist}(RS_{con})$.*

- (1) $\Delta \in \text{Par}(\omega_1) \Leftrightarrow \overleftarrow{\Delta} = \emptyset,$
- (2) $\Delta \in \text{Par}(\omega_2) \Leftrightarrow \overleftrightarrow{\Delta} = \emptyset,$
- (3) $\Delta \in \text{Par}(\omega_3) \Leftrightarrow \overrightarrow{\Delta} = \overleftarrow{\Delta} = \emptyset,$
- (4) $\Delta \in \text{Par}(\omega_4) \Leftrightarrow \overrightarrow{\Delta} = \overleftarrow{\Delta} = \emptyset,$
- (5) $\Delta \in \text{Par}(\omega_5) \Leftrightarrow \parallel_{\Delta} = \emptyset.$

From Theorem 5.5, it follows that there are $2^5 = 32$ possible paradigms for RS_{con} . But the nature of problems considered in concurrency theory are such that two of the simple report laws may be rejected. The first rejected law is ω_4 , which excludes the sequential composition construct. For a similar reason, we reject ω_5 since it excludes systems consisting of completely independent components. Hence, we have $2^3 = 8$ paradigms to consider:

$$\begin{aligned} \pi_1 &= \text{true}, & \pi_2 &= \omega_1, & \pi_3 &= \omega_2, & \pi_4 &= \omega_3, & \pi_5 &= \omega_1 \wedge \omega_2, \\ \pi_6 &= \omega_1 \wedge \omega_3, & \pi_7 &= \omega_2 \wedge \omega_3, & \pi_8 &= \omega_1 \wedge \omega_2 \wedge \omega_3 \end{aligned}$$

The connection between the eight paradigms and simple report invariants is established below.

Theorem 6.6. *Let $\Delta \in \text{Hist}(RS_{con})$.*

- (1) $\Delta \in \text{Par}(\pi_1),$
- (2) $\Delta \in \text{Par}(\pi_2) \Leftrightarrow \overleftarrow{\Delta} = \emptyset,$
- (3) $\Delta \in \text{Par}(\pi_3) \Leftrightarrow \overleftrightarrow{\Delta} = \emptyset,$
- (4) $\Delta \in \text{Par}(\pi_4) \Leftrightarrow \overrightarrow{\Delta} = \emptyset,$
- (5) $\Delta \in \text{Par}(\pi_5) \Leftrightarrow \overleftarrow{\Delta} = \overrightarrow{\Delta} = \emptyset,$
- (6) $\Delta \in \text{Par}(\pi_6) \Leftrightarrow \overleftarrow{\Delta} = \overrightarrow{\Delta} = \emptyset,$
- (7) $\Delta \in \text{Par}(\pi_7) \Leftrightarrow \overleftrightarrow{\Delta} = \overrightarrow{\Delta} = \emptyset,$
- (8) $\Delta \in \text{Par}(\pi_8) \Leftrightarrow \overleftarrow{\Delta} = \overleftrightarrow{\Delta} = \overrightarrow{\Delta} = \emptyset.$

Proof. Follows directly from Proposition 6.5. $\square$

We obtained a hierarchy of the fundamental paradigms of concurrency shown in Fig. 6. Paradigm π_1 simply admits all concurrent histories. The most restrictive paradigm, π_8 , admits concurrent histories Δ such that

$$\exists o \in \Delta. a \leftrightarrow_o b \Leftrightarrow (\exists o \in \Delta. a \rightarrow_o b) \wedge (\exists o \in \Delta. b \rightarrow_o a).$$

It is adopted by several models, including [4, 29, 41, 44, 49, 50]. Paradigm π_6 essentially says that simultaneity can only be observed if events are independent

$$\exists o \in \Delta. a \leftrightarrow_o b \Rightarrow (\exists o \in \Delta. a \rightarrow_o b) \wedge (\exists o \in \Delta. b \rightarrow_o a).$$

Complementary to π_6 is paradigm π_3 , as it says that the existence of observations in both orders implies a possibility of observing simultaneously

$$(\exists o \in \Delta. a \rightarrow_o b) \wedge (\exists o \in \Delta. b \rightarrow_o a) \Rightarrow \exists o \in \Delta. a \leftrightarrow_o b.$$

The remaining paradigms have less elegant representation in terms of simple report laws. Table 2 shows the components each paradigm excludes. We end this section deriving minimal signatures of the eight fundamental paradigms.

Theorem 6.7 (Minimal signatures for paradigms). (1) $\{\Leftarrow, \nearrow\}$ is a minimal signature for $\text{Par}(\pi_1)$, $\text{Par}(\pi_2)$ and $\text{Par}(\pi_4)$.

(2) $\{\rightarrow, \Leftarrow\}$ is a minimal signature for $\text{Par}(\pi_6)$.

(3) $\{\rightarrow, \nearrow\}$ is a minimal signature for $\text{Par}(\pi_3)$ and $\text{Par}(\pi_5)$.

(4) $\{\nearrow\}$ is a minimal signature for $\text{Par}(\pi_7)$.

(5) $\{\rightarrow\}$ is a minimal signature for $\text{Par}(\pi_8)$.

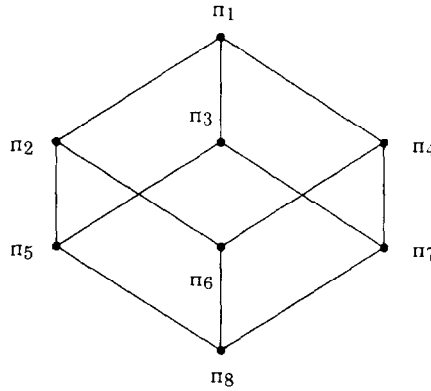


Fig. 6. Hierarchy of fundamental paradigms.

Table 2
Paradigms, components and signatures

Paradigm	Empty components	Minimal signatures
π_1		$\{\nearrow, \rightrightarrows\}$
π_2	$\leftrightarrow_{\Delta}$	$\{\nearrow, \rightrightarrows\}$
π_3	$\xrightarrow{\Delta}$	$\{\rightarrow, \nearrow\}$
π_4	$\xrightarrow{\Delta}$	$\{\nearrow, \rightrightarrows\}$
π_5	$\leftrightarrow_{\Delta} \xrightarrow{\Delta}$	$\{\rightarrow, \nearrow\}$
π_6	$\leftrightarrow_{\Delta} \xrightarrow{\Delta}$	$\{\rightrightarrows, \rightarrow\}$
π_7	$\xrightarrow{\Delta} \xrightarrow{\Delta}$	$\{\nearrow\}$
π_8	$\leftrightarrow_{\Delta} \xrightarrow{\Delta} \xrightarrow{\Delta}$	$\{\rightarrow\}$

Proof. Let $(MSig_i)$ Sig_i denote the set of (minimal) signatures of $Par(\pi_i)$. We first recall that $\{\rightrightarrows, \nearrow\}$ is a universal signature. Moreover,

$$\{\rightarrow, \rightrightarrows\} \in Sig_6 \text{ since } \Delta \in Par(\pi_6) \text{ implies } \nearrow_{\Delta} = \rightarrow_{\Delta},$$

$$\{\rightarrow, \nearrow\} \in Sig_3 \cap Sig_5 \text{ since } \Delta \in Par(\pi_3) \text{ implies } \rightrightarrows_{\Delta} = \rightarrow_{\Delta} \cup \leftarrow_{\Delta},$$

$$\{\nearrow\} \in Sig_7 \text{ since } \Delta \in Par(\pi_7) \text{ implies } \rightrightarrows_{\Delta} = \rightarrow_{\Delta} \cup \leftarrow_{\Delta} \text{ and } \rightarrow_{\Delta} = \nearrow_{\Delta} - (\nearrow_{\Delta} \cap (\nearrow_{\Delta})^{-1}),$$

$$\{\rightarrow\} \in Sig_8 \text{ since } \Delta \in Par(\pi_8) \text{ implies } \rightrightarrows_{\Delta} = \rightarrow_{\Delta} \cup \leftarrow_{\Delta} \text{ and } \nearrow_{\Delta} = \rightarrow_{\Delta}.$$

To show the minimality of the signatures, we proceed as follows. Let o_1, o_2 and o_3 be observations shown in Fig. 7, and $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5$ be histories defined by

$$\Delta_1 = \{o_1, o_2, o_3\}, \Delta_2 = \{o_1, o_2\}, \Delta_3 = \{o_1\},$$

$$\Delta_4 = \{o_3\}, \quad \Delta_5 = \{o_1, o_3\}.$$

Note that $\Delta_1, \Delta_3 \in Par(\pi_8)$, $\Delta_2 \in Par(\pi_6)$, $\Delta_4 \in Par(\pi_7)$ and $\Delta_5 \in Par(\pi_5)$.

Fact 1. $\{\rightarrow, \leftrightarrow, \rightrightarrows\} \notin Sig_5$ since $\Delta_1, \Delta_5 \in Par(\pi_5)$ and

$$\rightarrow_{\Delta_1} = \rightarrow_{\Delta_5} = \emptyset, \leftrightarrow_{\Delta_1} = \leftrightarrow_{\Delta_5} = \emptyset \text{ and } \rightrightarrows_{\Delta_1} = \rightrightarrows_{\Delta_5} = \emptyset.$$

Fact 2. $\{\rightarrow, \nearrow\} \notin Sig_6$ since $\Delta_1, \Delta_2 \in Par(\pi_6)$, $\rightarrow_{\Delta_1} = \rightarrow_{\Delta_2} = \emptyset$ and $\nearrow_{\Delta_1} = \nearrow_{\Delta_2} = \emptyset$.

Fact 3. $\{\leftrightarrow, \rightrightarrows\} \notin Sig_6$ since $\Delta_2, \Delta_3 \in Par(\pi_6)$, and

$$\leftrightarrow_{\Delta_2} = \leftrightarrow_{\Delta_3} = \emptyset \text{ and } \rightrightarrows_{\Delta_2} = \rightrightarrows_{\Delta_3} = \{(a, b), (b, a)\}.$$

Fact 4. $\{\nearrow, \leftrightarrow\} \notin Sig_5$ since $\Delta_3, \Delta_5 \in Par(\pi_5)$ and

$$\leftrightarrow_{\Delta_3} = \leftrightarrow_{\Delta_5} = \emptyset \text{ and } \nearrow_{\Delta_3} = \nearrow_{\Delta_5} = \{(a, b)\}.$$

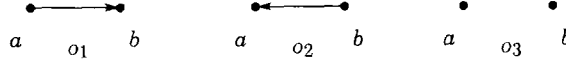


Fig. 7.

Fact 5. $\{\rightarrow, \Leftarrow\} \notin \text{Sig}_4$ since $\Delta_1, \Delta_4 \in \text{Par}(\pi_4)$ and

$$\xrightarrow{\Delta_1} = \xrightarrow{\Delta_4} = \emptyset \quad \text{and} \quad \xleftarrow{\Delta_1} = \xleftarrow{\Delta_4} = \emptyset.$$

To show that $\{\Leftarrow, \nearrow\} \in \text{MSig}_1 \cap \text{MSig}_2$, it suffices to show that none of $\{\Leftarrow\}$, $\{\nearrow\}$, $\{\nearrow, \rightarrow\}$, $\{\Leftarrow, \rightarrow\}$ and $\{\Leftarrow, \leftrightarrow\}$ is a signature of $\text{Par}(\pi_1)$ or $\text{Par}(\pi_2)$. For $\{\Leftarrow\}$, $\{\Leftarrow, \rightarrow\}$ and $\{\Leftarrow, \leftrightarrow\}$ this follows from Fact 1 and $\text{Par}(\pi_5) \subseteq \text{Par}(\pi_2) \subseteq \text{Par}(\pi_1)$. For $\{\nearrow\}$ and $\{\nearrow, \rightarrow\}$, this follows from Fact 2 and $\text{Par}(\pi_6) \subseteq \text{Par}(\pi_2) \subseteq \text{Par}(\pi_1)$.

Similarly, $\{\Leftarrow, \nearrow\} \in \text{MSig}_4$ since none of $\{\Leftarrow\}$, $\{\nearrow\}$, $\{\nearrow, \rightarrow\}$, $\{\Leftarrow, \rightarrow\}$ and $\{\Leftarrow, \leftrightarrow\}$ is a signature of $\text{Par}(\pi_4)$. For $\{\Leftarrow\}$, $\{\nearrow\}$, $\{\nearrow, \rightarrow\}$ and $\{\Leftarrow, \leftrightarrow\}$ this follows from Facts 2 and 3, and $\text{Par}(\pi_6) \subseteq \text{Par}(\pi_4)$. For $\{\rightarrow, \Leftarrow\}$ this follows from Fact 5.

To show that $\{\Leftarrow, \rightarrow\} \in \text{MSig}_6$, it suffices to show neither $\{\Leftarrow\}$ nor $\{\rightarrow\}$ is a signature. The former follows from Fact 3. The latter follows from Fact 2.

To show that $\{\rightarrow, \nearrow\} \in \text{MSig}_3 \cap \text{MSig}_5$, it suffices to show that none of $\{\rightarrow\}$, $\{\nearrow\}$ and $\{\rightarrow, \leftrightarrow\}$ is a signature of $\text{Par}(\pi_3)$ or $\text{Par}(\pi_5)$. For $\{\rightarrow\}$ and $\{\rightarrow, \leftrightarrow\}$ this follows from Fact 1 and $\text{Par}(\pi_5) \subseteq \text{Par}(\pi_3)$. For $\{\nearrow\}$ this follows from Fact 4 and $\text{Par}(\pi_5) \subseteq \text{Par}(\pi_3)$.

To show that $\{\nearrow\} \in \text{MSig}_7$, we observe that neither $\{\rightarrow\}$ nor $\{\leftrightarrow\}$ is a signature of $\text{Par}(\pi_7)$. The former follows from $\Delta_1, \Delta_4 \in \text{Par}(\pi_7)$ and $\xrightarrow{\Delta_1} = \xrightarrow{\Delta_4} = \emptyset$; the latter from $\Delta_1, \Delta_3 \in \text{Par}(\pi_7)$ and $\xleftarrow{\Delta_1} = \xleftarrow{\Delta_3} = \emptyset$.

$\{\rightarrow\}$ is obviously a minimal signature. $\square$

In the most general case, π_1 , the explicit causality invariant is not needed (in fact, there is no universal minimal signature containing $\rightarrow$). We also observe that no paradigm requires a signature comprising more than two invariants (see Table 2). Note that if π_8 holds then causality, $\xrightarrow{\Delta}$, is the only invariant needed, and this fact is a *theorem* in our approach.

6.4. The paradigm of partial order histories

Paradigm π_8 deserves our special attention as it is usually adopted by concurrency models. We now show that for the histories in π_8 it is enough to keep record only of the sequential observations.

A *base* of a history Δ is a pair, $\Delta_0 \subseteq \Delta$ and $S \subseteq \text{SRI}$, such that $\Delta_0^{<S>} = \Delta$. It provides a complete description of a history in terms of a (smaller) set of observations and a suitable set of invariants.

Theorem 6.8 (Histories under π_8 can be represented by interleavings). *If $\Delta \in \text{Par}(\pi_8)$ then $\Delta_{\text{Int}} = \Delta \cap \text{Obs}_{\text{Int}}$ and $\{\rightarrow\}$ form a base of Δ .*

Proof. It suffices to show that $\xrightarrow{\Delta_{\text{Int}}} = \xrightarrow{\Delta}$ since, due to Theorem 6.7(5), $\Delta^{\langle \rightarrow \rangle} = \Delta$. For every $o \in \Delta$, let $\Delta(o) = \{r \in \Delta_{\text{Int}} \mid \xrightarrow{o} \subseteq \xrightarrow{r}\}$. From the extension theorem [45] it follows that, for every $o \in \Delta$, $\Delta(o) \neq \emptyset$ and $\xrightarrow{o} = \bigcap_{r \in \Delta(o)} \xrightarrow{r}$. Furthermore, $\Delta_{\text{Int}} = \bigcup_{o \in \Delta} \Delta(o)$. Thus,

$$\begin{aligned} a \xrightarrow{\Delta} b &\Leftrightarrow \forall o \in \Delta. a \xrightarrow{o} b \Leftrightarrow \forall o \in \Delta \forall r \in \Delta(o). a \xrightarrow{r} b \\ &\Leftrightarrow \forall r \in \Delta_{\text{Int}}. a \xrightarrow{r} b \Leftrightarrow a \xrightarrow{\Delta_{\text{Int}}} b. \quad \square \end{aligned}$$

For π_8 it is possible to adequately represent a history by taking its interleaved observations. This was exactly the idea behind the Mazurkiewicz traces [31, 32] and the interleaving set temporal logic [23]; within our framework, Theorem 6.8 provides a justification of that approach. However, it cannot be extended to any other paradigm introduced in Section 6.3.

7. Representation theorems

We now consider axiomatic models for minimal signatures under paradigm π_3 .

7.1. Paradigm π_3

Paradigm π_3 is general enough to model priority systems and inhibitor nets [20]; from Theorem 6.7(3) it follows that $\{\rightarrow, \nearrow\}$ is its minimal signature. It turns out that it can be axiomatised in terms of relational structures that we call *weak composets* (combined posets). A *weak composet* is a triple

$$wc = (\text{dom}(wc), \xrightarrow{wc}, \nearrow_{wc})$$

such that $\text{dom}(wc)$ is a set of event occurrences and $\xrightarrow{wc}, \nearrow_{wc}$ and binary relations on $\text{dom}(wc)$ satisfying the following:

(WC1) $(\text{dom}(wc), \xrightarrow{wc})$ is a poset, $\nearrow_{wc}$ is irreflexive.

(WC2) $a \xrightarrow{wc} b \Rightarrow a \nearrow_{wc} b \wedge \neg b \nearrow_{wc} a$.

(WC3) $a \xrightarrow{wc} b \nearrow_{wc} c \vee a \nearrow_{wc} b \xrightarrow{wc} c \Rightarrow a \nearrow_{wc} c$.

(WC4) $a \xrightarrow{wc} b \nearrow_{wc} c \xrightarrow{wc} d \Rightarrow a \xrightarrow{wc} d$.

(WC5) $a \nearrow_{wc} b \xrightarrow{wc} c \nearrow_{wc} d \Rightarrow a \nearrow_{wc} d \vee a = d$.

Relational structures similar to weak composets were introduced and subsequently analysed in [1, 3, 27, 28], however, with different objectives in mind. Conditions

WC1–WC4 were used in [27], WC5 in [3]. (Note that [3, 27] required that $\nearrow_{wc}$ be reflexive, but this is a minor technical detail.) Directly from Propositions 6.1 and 6.2 we obtain the following.

Corollary 7.1. *For every $\Delta \in RSet(RS_{con})$, $(\text{dom}(\Delta), \xrightarrow{\Delta}, \nearrow_{\Delta})$ is a weak composet.*

An interval order po is an *interval extension* of a weak composet wc , denoted by $po \in \text{intervals}(wc)$, if $\text{dom}(po) = \text{dom}(wc)$, $\xrightarrow{wc} \subseteq \xrightarrow{po}$ and $\nearrow_{wc} \subseteq \xrightarrow{po} \cup \leftrightarrow_{po}$.

Theorem 7.2. (First representation theorem for weak composets [1, Theorem 2.10]). *Let wc be a weak composet. Then, there is a partial order $(X, \angle)$ and*

$$\Phi, \Psi : \text{dom}(wc) \rightarrow X$$

such that for all distinct a and b in $\text{dom}(wc)$ the following hold:

$$\Phi(a) \angle \Psi(a),$$

$$a \xrightarrow{wc} b \Leftrightarrow \Psi(a) \angle \Phi(b),$$

$$a \nearrow_{wc} b \Leftrightarrow \Phi(a) \angle \Psi(b) \vee \Phi(a) = \Psi(b). \quad \square$$

Proposition 7.3. (Existence of an interval extension for weak composets). *For every weak composet wc , $\text{intervals}(wc) \neq \emptyset$.*

Proof. Let $(X, \angle_1)$ be any total extension of $(X, \angle)$ from Theorem 7.2. Define $po = (\text{dom}(wc), \xrightarrow{po})$ where $a \xrightarrow{po} b \Leftrightarrow \Psi(a) \angle_1 \Phi(b)$. By Proposition 2.13, po is an interval order. Moreover, for all distinct a and b ,

$$a \xrightarrow{wc} b \Rightarrow \Psi(a) \angle \Phi(b) \Rightarrow \Psi(a) \angle_1 \Phi(b) \Rightarrow a \xrightarrow{po} b,$$

$$a \nearrow_{wc} b \Rightarrow \Phi(a) \angle \Psi(b) \vee \Phi(a) = \Psi(b) \Rightarrow \Phi(a) \angle_1 \Psi(b) \vee \Phi(a) = \Psi(b)$$

$$\Rightarrow \neg \Psi(b) \angle_1 \Phi(a) \Rightarrow \neg b \xrightarrow{po} a \Rightarrow a \xrightarrow{po} b \vee a \leftrightarrow_{po} b.$$

Hence, $po \in \text{intervals}(wc)$. $\square$

We shall show that every weak composet is unambiguously identified by the set of its interval extensions, in the same way as every poset is unambiguously identified by the set of its total extensions [45].

Lemma 7.4. *If po is a poset and $a \leftrightarrow_{po} b$ then there is a total order to such that $\text{dom}(to) = \text{dom}(po)$, $\xrightarrow{po} \subseteq \xrightarrow{to}$ and $a \xrightarrow{to} b$.*

Proof. Let $Y = \{a\} \cup \{y \mid y \xrightarrow{po} a\}$, $Z = \{b\} \cup \{z \mid b \xrightarrow{po} z\}$ and $po' = (\text{dom}(po), \angle)$, where

$$\angle = \xrightarrow{po} \cup Y \times Z.$$

We observe that $Y \cap Z = \emptyset$ since $a \leftrightarrow b$. Hence, $\angle$ is irreflexive. Moreover, if $x \angle y \angle z$ then $x \xrightarrow{po} y$ or $y \xrightarrow{po} z$. Suppose $x \angle y \xrightarrow{po} z$ and $\neg x \xrightarrow{po} y$. Then $x \in Y$ and $y, z \in Z$. Thus, $x \angle z$. Similarly, if $x \xrightarrow{po} y \angle z$ and $\neg y \xrightarrow{po} z$ then $x \angle z$. Thus, $\angle$ is also transitive. Hence, po' is a partial order, $a \angle b$ and $\xrightarrow{po} \subseteq \angle$. Let to be any total extension of po' . Then $\xrightarrow{po} \subseteq \xrightarrow{to}$ and $a \xrightarrow{to} b$. $\square$

Lemma 7.5. *Let wc be a weak composet and a, b be distinct elements in its domain. Then*

- (1) *If $\neg a \xrightarrow{wc} b$ then there is $po \in \text{intervals}(wc)$ such that $b \xrightarrow{po} a$ or $b \leftrightarrow_{po} a$.*
- (2) *If $\neg a \nearrow_{wc} b$ then there is $po \in \text{intervals}(wc)$ such that $b \xrightarrow{po} a$.*

Note: This lemma is basically equivalent to Theorem 2.11 in [1]; however, the proof below is much simpler.

Proof. Take $(X, \angle)$ from Theorem 7.2. Suppose $x, y \in X$ are such that $x \leftrightarrow_{\angle} y$. From Lemma 7.4 it follows that there is a total order $(X, \angle_1)$ such that $x \angle_1 y$ and $\angle \subseteq \angle_1$. Define $po_{xy} = (\text{dom}(wc), \xrightarrow{po_{xy}})$, where $c \xrightarrow{po_{xy}} d \Leftrightarrow \Psi(c) \angle_1 \Phi(d)$. By Proposition 2.13 (and proceeding similarly as in the proof of Proposition 7.3), one may show that $po_{xy} \in \text{intervals}(wc)$.

(1) Suppose $\neg a \xrightarrow{wc} b$. We may assume $\neg b \nearrow_{wc} a$; otherwise, every element of $\text{intervals}(wc) \neq \emptyset$ satisfies the required property. Let $x = \Phi(b)$ and $y = \Psi(a)$. We have

$$\begin{aligned} \neg a \xrightarrow{wc} b &\Rightarrow \neg \Psi(a) \angle \Phi(b), \\ \neg b \nearrow_{wc} a &\Rightarrow \neg \Phi(b) \angle \Psi(a) \wedge \Phi(b) \neq \Psi(a). \end{aligned}$$

Hence, $x \leftrightarrow_{\angle} y$ and $po_{xy} \in \text{intervals}(wc)$. Moreover,

$$\begin{aligned} x \angle_1 y &\Rightarrow \Phi(b) \angle_1 \Psi(a) \Rightarrow \neg \Psi(a) \angle_1 \Phi(b) \Rightarrow \neg a \xrightarrow{po_{xy}} b \\ &\Rightarrow b \xrightarrow{po_{xy}} a \vee b \leftrightarrow_{po_{xy}} a. \end{aligned}$$

(2) Suppose $\neg a \nearrow_{wc} b$. We may assume $\neg b \xrightarrow{wc} a$; otherwise, every element of $\text{intervals}(wc) \neq \emptyset$ satisfies the required property. Let $x = \Psi(b)$ and $y = \Phi(a)$. Similarly as before, we obtain $x \leftrightarrow_{\angle} y$ and $po_{xy} \in \text{intervals}(wc)$. In this case,

$$x \angle_1 y \Rightarrow \Psi(b) \angle_1 \Phi(a) \Rightarrow b \xrightarrow{po_{xy}} a. \quad \square$$

Theorem 7.6. *Let wc_1 and wc_2 be weak composets. Then*

$$wc_1 = wc_2 \Leftrightarrow \text{intervals}(wc_1) = \text{intervals}(wc_2).$$

Proof. It suffices to show that $\text{intervals}(wc_1) \subseteq \text{intervals}(wc_2)$ implies $wc_2 \subseteq wc_1$ (i.e., $\text{dom}(wc_2) \subseteq \text{dom}(wc_1)$, $\overrightarrow{wc_2} \subseteq \overrightarrow{wc_1}$ and $\nearrow_{wc_2} \subseteq \nearrow_{wc_1}$). From Proposition 7.3, it follows that $\text{dom}(wc_1) = \text{dom}(wc_2)$. If $a \xrightarrow{wc_2} b$ and $\neg a \xrightarrow{wc_1} b$ then, by Lemma 7.5(1), there is $po \in \text{intervals}(wc_1)$ such that $b \xrightarrow{po} a$ or $b \leftrightarrow_{po} a$. Clearly, $po \notin \text{intervals}(wc_2)$, a contradiction. Hence, $\overrightarrow{wc_2} \subseteq \overrightarrow{wc_1}$. Similarly, by using Lemma 7.5(2), we show $\nearrow_{wc_2} \subseteq \nearrow_{wc_1}$. $\square$

Let X be a nonempty set of interval posets with a common domain Σ . The *combined intersection* of X is the relational structure

$$\bigcap_c(X) = (\Sigma, \overrightarrow{X}, \nearrow_X) = \left(\Sigma, \bigcap_{po \in X} \overrightarrow{po}, \bigcap_{po \in X} (\overrightarrow{po} \cup \leftrightarrow_{po}) \right).$$

Proposition 7.7. *Combined intersection is always a weak composet.*

Proof. Similar as for Proposition 6.1 and 6.2. $\square$

A fundamental result of [45] says that by intersecting all total extensions of a partial order one obtains the original partial order. A similar result holds for weak composets.

Theorem 7.8 (Second representation theorem for weak composets). *Let wc be a weak composet. Then $wc = \bigcap_c(\text{intervals}(wc))$.*

Proof. Let $X = \text{intervals}(wc)$. Clearly, $wc \subseteq \bigcap_c(X)$. If $a \xrightarrow{X} b$ and $\neg a \xrightarrow{wc} b$ then, by Lemma 7.5(1), there is $po \in X$ such that $b \xrightarrow{po} a$ or $b \leftrightarrow_{po} a$, a contradiction with $a \xrightarrow{X} b$. Hence, $\overrightarrow{wc} = \overrightarrow{X}$. To show $\nearrow_X = \nearrow_{wc}$, we use Lemma 7.5(2). $\square$

A poset po is an *observation extension* of a weak composet wc , $po \in \text{obs}(wc)$, if $po \in \text{intervals}(wc)$ and po is initially finite. Note that $\text{obs}(wc)$ can be interpreted as a report set over RS_{con} .

Lemma 7.9. *Let wc be a finite weak composet. Then:*

- (1) $\text{obs}(wc) = \text{intervals}(wc)$.
- (2) $\text{obs}(wc) = \text{obs}(wc)^{\langle \rightarrow, \nearrow \rangle}$.

Note: In (2) symbols $\rightarrow$ and $\nearrow$ denote invariants as defined in Section 6, i.e., they are mappings which for every report set A return, respectively, $\rightarrow_A$ and $\nearrow_A$. In particular, for $\text{obs}(wc)$ they return $\overrightarrow{\text{obs}(wc)}$ and $\nearrow_{\text{obs}(wc)}$.

Proof. (1) Finite interval orders are observations.

(2) By the definition of $\text{obs}(wc)$, for every observation o with $\text{dom}(o) = \text{dom}(wc)$:

$$o \in \text{obs}(wc) \Leftrightarrow \forall a, b \in \text{dom}(wc). (a \xrightarrow{wc} b \Rightarrow a \xrightarrow{o} b) \wedge (a \nearrow_{wc} b \Rightarrow a \xrightarrow{o} b \vee a \leftrightarrow_o b).$$

By the definition of S-closure, for every observation o , with $\text{dom}(o) = \text{dom}(wc)$ (below $\Delta = \text{obs}(wc)$),

$$o \in \Delta^{\langle \rightarrow, \nearrow \rangle} \Leftrightarrow \forall a, b \in \text{dom}(\Delta). (a \rightarrow b \Rightarrow a \xrightarrow{o} b) \wedge (a \nearrow b \Rightarrow a \xrightarrow{o} b \vee a \leftrightarrow_o b).$$

By Theorem 7.8 and (1), $\xrightarrow{wc} = \xrightarrow{\text{obs}(wc)}$ and $\nearrow_{wc} = \nearrow_{\text{obs}(wc)}$, so (2) holds. $\square$

We now can formulate the main result of this section.

Theorem 7.10 (Axiomatisation of finite concurrent histories in π_3). (1) If $\Delta \in \text{Par}(\pi_3)$ and $\text{dom}(\Delta)$ is finite then there is a finite weak composet wc such that $\Delta = \text{obs}(wc)$.
 (2) If wc is a finite weak composet then $\text{obs}(wc) \in \text{Par}(\pi_3)$.

Proof. (1) Define $wc = (\text{dom}(\Delta), \xrightarrow{\Delta}, \nearrow_{\Delta})$. By Corollary 7.1, wc is a finite weak composet. From Theorem 7.8 and Lemma 7.9(1), it follows that $\xrightarrow{wc} = \xrightarrow{\text{obs}(wc)}$ and $\nearrow_{wc} = \nearrow_{\text{obs}(wc)}$. Hence, $\xrightarrow{\Delta} = \xrightarrow{\text{obs}(wc)}$ and $\nearrow_{\Delta} = \nearrow_{\text{obs}(wc)}$. By (2), $\text{obs}(wc) \in \text{Par}(\pi_3)$. Thus, by Theorem 6.7(3), Proposition 4.4, $\xrightarrow{\Delta} = \xrightarrow{\text{obs}(wc)}$ and $\nearrow_{\Delta} = \nearrow_{\text{obs}(wc)}$, we get $\Delta = \text{obs}(wc)$.

(2) By Lemma 7.9(2), $\text{obs}(wc) = \text{obs}(wc)^{\langle \rightarrow, \nearrow \rangle}$. This and Proposition 4.1(1, 2) yields $\text{obs}(wc) \in \text{Hist}(RS_{\text{con}})$. By Theorem 6.6(3), it now suffices to show that $\xleftrightarrow{\text{obs}(wc)} = \emptyset$. Suppose $a \xleftrightarrow{\text{obs}(wc)} b$. Then, by Theorem 7.8, $\neg a \nearrow_{wc} b$ and $\neg b \nearrow_{wc} a$. Let $(X, \angle)$ be as in Theorem 7.2. We have $\Phi(a) \xleftrightarrow{\angle} \Phi(b) \xleftrightarrow{\angle} \Psi(a)$. Using a construction similar to that in Lemma 7.4, one can show that there is a total extension $(X, \angle_1)$ of $(X, \angle)$ such that $\Phi(a) \angle_1 \Phi(b) \angle_1 \Psi(a)$. Define $po = (\text{dom}(po), \xrightarrow{po})$, where $c \xrightarrow{po} d \Leftrightarrow \Psi(c) \angle_1 \Phi(d)$. By proceeding similarly as in Proposition 7.3, one may show that $po \in \text{intervals}(wc) = \text{obs}(wc)$ and $a \xleftrightarrow{po} b$. This, however, contradicts $a \xleftrightarrow{\text{obs}(wc)} b$. $\square$

The last theorem provides an axiomatisation of finite concurrent histories conforming to paradigm π_3 : Every finite weak composet of event occurrences may be interpreted as a representation of a history in π_3 . In other words, in this case histories can be represented by finite weak composets (in the same way as the histories in π_3 can be represented by causal partial orders). If π_3 does not hold, then $\{\rightarrow, \nearrow\}$ may no longer be a signature and $\text{obs}(wc)$ may not be a concurrent history.

7.2. Step sequences within π_3

We now assume that π_3 holds and that all observations are step sequences. In this case we replace weak composets by composets. A *composets* is a triple

$$co = (\text{dom}(co), \xrightarrow{co}, \nearrow_{co})$$

such that $\text{dom}(co)$ is a set of event occurrences and $\xrightarrow{co}, \nearrow_{co}$ are binary relations on $\text{dom}(co)$ satisfying the following:

- (C1) $(\text{dom}(co), \xrightarrow{co})$ is a poset, $\nearrow_{co}$ is irreflexive.

$$(C2) \quad a \xrightarrow{co} b \Rightarrow a \nearrow_{co} b \wedge \neg b \nearrow_{co} a$$

$$(C3) \quad a \nearrow_{co} b \nearrow_{co} c \Rightarrow a \nearrow_{co} c \vee a = c$$

$$(C4) \quad a \xrightarrow{co} b \nearrow_{co} c \vee a \nearrow_{co} b \xrightarrow{co} c \Rightarrow a \xrightarrow{co} c.$$

Composets have been used to model concurrent behaviours in [12, 19, 20]. [20] provides a detailed analysis of finite composets.

Proposition 7.11. *Every composet is a weak composet.*

Theorem 7.12 (First representation theorem for composets). *Let co be a composet.*

(1) *There is a partial order $(X, \angle)$ and $\Phi: \text{dom}(co) \rightarrow X$ such that for all distinct a and b in $\text{dom}(co)$ the following hold:*

$$a \xrightarrow{co} b \Rightarrow \Phi(a) \angle \Phi(b),$$

$$a \nearrow_{co} b \Leftrightarrow \Phi(a) \angle \Phi(b) \vee \Phi(a) = \Phi(b).$$

(2) *There is a partial order $(X, \angle)$ and $\Phi: \text{dom}(co) \rightarrow X$ such that for all distinct a and b in $\text{dom}(co)$ the following hold:*

$$a \xrightarrow{co} b \Leftrightarrow \Phi(a) \angle \Phi(b),$$

$$a \nearrow_{co} b \Leftarrow \Phi(a) \angle \Phi(b) \vee \Phi(a) = \Phi(b).$$

Proof. The proof of (1) is just a modification of a well-known result of E. Schröder (1890) characterising *pre-order* relations. (Axiom C3 says that $\nearrow_{co} \cup \text{id}_X$ is a pre-order [9, 25].)

Define $a \equiv b \Leftrightarrow (a \nearrow_{co} b \wedge b \nearrow_{co} a) \vee a = b$. By C3, $\equiv$ is an equivalence relation on $\text{dom}(co)$. Let $[a]$ denote the equivalence class of $\equiv$ containing a , and $X = \text{dom}(co)/\equiv$.

(1) Define $[a] \angle [b] \Leftrightarrow (a \nearrow_{co} b \wedge \neg b \nearrow_{co} a)$. By C3, $\angle$ is a well-defined irreflexive relation. The transitivity of $\angle$ also follows from C3. Hence, $(X, \angle)$ is a partial order. For all distinct a and b we have

$$a \xrightarrow{co} b \Rightarrow a \nearrow_{co} b \wedge \neg b \nearrow_{co} a \Rightarrow [a] \angle [b].$$

$$[a] \angle [b] \vee [a] = [b] \Leftrightarrow (a \nearrow_{co} b \wedge \neg b \nearrow_{co} a)$$

$$\vee (a \nearrow_{co} b \wedge b \nearrow_{co} a) \Leftrightarrow a \nearrow_{co} b.$$

Hence, we can define $\Phi(a) = [a]$ for all a .

(2) Define $[a] \angle [b] \Leftrightarrow a \xrightarrow{co} b$. Suppose $[a] \angle [b]$, $c \in [a]$ and $d \in [b]$. Then

$$[a] \angle [b] \wedge c \in [a] \wedge d \in [b] \Rightarrow a \xrightarrow{co} b \wedge (a = c \vee c \nearrow_{co} a)$$

$$\wedge (b = d \vee b \nearrow_{co} d) \xrightarrow{C4} c \xrightarrow{co} d.$$

Hence, $\angle$ is well-defined, reflexive ($a \xrightarrow{co} b \Rightarrow \neg b \nearrow_{co} a \Rightarrow [a] \neq [b]$) and transitive (by C1). Thus, $(X, \angle)$ is a partial order. For all distinct a and b , we have

$$[a] \angle [b] \vee [a] = [b] \Rightarrow a \xrightarrow{co} b \vee (a \nearrow_{co} b \wedge b \nearrow_{co} a) \Rightarrow a \nearrow_{co} b.$$

Hence, we can define $\Phi(a)=[a]$ for all a . $\square$

A stratified poset po is a *stratified extension* of a composet co , $po \in \text{strat}(co)$, if $\text{dom}(po) = \text{dom}(co)$, $\xrightarrow{co} \subseteq \xrightarrow{po}$ and $\nearrow_{co} \subseteq \xrightarrow{po} \cup \leftrightarrow_{po}$.

Proposition 7.13 (Existence of stratified extension for composets). *For every nonempty composet co , $\text{strat}(co) \neq \emptyset$.*

Proof. Let $(X, \angle_1)$ be a total extension of $(X, \angle)$ from Theorem 7.12(1). Define $po = (\text{dom}(co), \xrightarrow{po})$, where $a \xrightarrow{po} b \Leftrightarrow \Phi(a) \angle_1 \Phi(b)$. Clearly, po is a stratified poset. Moreover, for all distinct a and b , we have

$$\begin{aligned} a \xrightarrow{co} b &\Rightarrow \Phi(a) \angle \Phi(b) \Rightarrow \Phi(a) \angle_1 \Phi(b) \Rightarrow a \xrightarrow{po} b, \\ a \nearrow_{co} b &\Rightarrow \Phi(a) \angle \Phi(b) \vee \Phi(a) = \Phi(b) \Rightarrow \Phi(a) \angle_1 \Phi(b) \vee \Phi(a) = \Phi(b) \\ &\Rightarrow \neg \Phi(b) \angle_1 \Phi(a) \Rightarrow \neg b \xrightarrow{po} a \Rightarrow a \xrightarrow{po} b \vee a \leftrightarrow_{po} b. \end{aligned}$$

Hence, $po \in \text{strat}(co)$. $\square$

Lemma 7.14. *Let co be a composet and a, b be distinct elements in its domain. Then*

- (1) *If $\neg a \xrightarrow{co} b$ then there is $po \in \text{strat}(co)$ such that $b \xrightarrow{po} a$ or $b \leftrightarrow_{po} a$.*
- (2) *If $\neg a \nearrow_{co} b$ then there is $po \in \text{strat}(co)$ such that $b \xrightarrow{po} a$.*

Proof. Let $Y = \{b\} \cup \{c \mid c \nearrow_{co} b\}$, $W = \{c \mid c \xrightarrow{co} b\}$, $Z = \{a\} \cup \{c \mid a \nearrow_{co} c\}$ and $V = \{c \mid a \xrightarrow{co} c\}$.

- (1) By $\neg a \xrightarrow{co} b$, we have $V \cap Y = \emptyset = W \cap Z$. Define $co_1 = (\text{dom}(co), \xrightarrow{co_1}, \nearrow_{co_1})$, where

$$\xrightarrow{co_1} = \xrightarrow{co} \cup W \times Z \cup Y \times V \text{ and } \nearrow_{co_1} = \nearrow_{co} \cup Y \times Z - \text{id}_{Y \cap Z}.$$

Using a straightforward yet tedious argument it can be shown [21] that co_1 is a composet and $b \nearrow_{co_1} a$. By Proposition 7.13, there is $po \in \text{strat}(co_1) \subseteq \text{strat}(co)$ such that $b \xrightarrow{po} a \vee b \leftrightarrow_{po} a$.

- (2) By $\neg a \nearrow_{co} b$, we have $Z \cap Y = \emptyset$. Define $co_2 = (\text{dom}(co), \xrightarrow{co_2}, \nearrow_{co_2})$, where

$$\xrightarrow{co_2} = \xrightarrow{co} \cup Y \times Z \text{ and } \nearrow_{co_2} = \nearrow_{co} \cup Y \times Z.$$

It can be shown [21] that co_2 is a composet and $b \xrightarrow{co_2} a$. By Proposition 7.13, there is $po \in strat(co_2) \subseteq strat(co)$ such that $b \xrightarrow{po} a$. $\square$

We now can show that the relationship between step sequences and composets is exactly the same as that between interval orders and weak composets.

Theorem 7.15. (1) *Let co_1 and co_2 be composets. Then*

$$co_1 = co_2 \Leftrightarrow strat(co_1) = strat(co_2).$$

(2) *For every nonempty set of stratified posets Δ with a common domain, $\bigcap_c(\Delta)$ is a composet.*

(3) *Let co be a composet. Then $co = \bigcap_c(strat(co))$.*

Proof. ((1) is shown similarly as Theorem 7.6 using Proposition 7.13 and Lemma 7.14; (2) as Proposition 6.3; while (3) as Theorem 7.8 using Lemma 7.14.) $\square$

For the finite case Theorem 7.15 was independently proved in [21].

A poset po is a *step sequence extension* of a composet co , $po \in steps(co)$, if $po \in strat(co)$ and po is initially finite. Note that $steps(co)$ can be interpreted as a report set over RS_{con} .

Lemma 7.16. *Assume that RS_{con} comprises only step sequences. Let co be a finite composet. Then*

$$strat(co) = steps(co) = steps(co)^{\langle \rightarrow, \nearrow \rangle}.$$

Proof. Similarly as Lemma 7.9 using Theorem 7.15. $\square$

The main result of this section reads as follows.

Theorem 7.17 (Axiomatisation of finite concurrent histories in π_3 with step sequence observations). *Assume that RS_{con} comprises only step sequences.*

(1) *If $\Delta \in Par(\pi_3)$ and $dom(\Delta)$ is finite then there is a finite composet co such that $\Delta = steps(co)$.*

(2) *If co is a finite composet then $steps(co) \in Par(\pi_3)$.*

Proof. (1) Similarly as Theorem 7.10(1), using Theorem 7.15 and Lemma 7.16.

(2) By Lemma 7.16, $steps(co) = steps(co)^{\langle \rightarrow, \nearrow \rangle}$. This and Proposition 4.1(1, 2) yields $steps(co) \in Hist(RS_{con})$. By Theorem 6.6(3), it now suffices to show that $\stackrel{\rightarrow}{\subseteq}_{steps(co)} = \emptyset$. Suppose $a \stackrel{\rightarrow}{\subseteq}_{steps(co)} b$. Then by Theorem 7.15, $\neg a \nearrow_{co} b$ and $\neg b \nearrow_{co} a$. Let

$(X, \angle)$ be as in Theorem 7.12(1). We have $\Phi(a) \leftrightarrow_{\angle} \Phi(b)$. Let

$$Y = \{ \Phi(a) \} \cup \{ y \mid \Phi(a) \angle y \} \text{ and } W = \{ \Phi(a) \} \cup \{ w \mid w \angle \Phi(a) \},$$

$$V = \{ v \mid \Phi(b) \angle v \} \text{ and } Z = \{ z \mid z \angle \Phi(b) \},$$

$$X_1 = X - \{ \Phi(b) \} \text{ and } \angle_1 = (\angle \cap X_1 \times X_1) \cup Z \times Y \cup W \times V.$$

As in Lemma 7.4, it can be shown $(X_1, \angle_1)$ is a poset. Define $\Phi_1 : \text{dom}(wc) \rightarrow X_1$ by $\Phi_1(c) = \Phi(c)$, for all $c \neq b$, and $\Phi_1(b) = \Phi(a)$. Let $(X_1, \angle_2)$ be any total extension of $(X_1, \angle_1)$. Define $po = (\text{dom}(co), \xrightarrow{po})$, where $c \xrightarrow{po} d \Leftrightarrow \Phi_1(c) \angle_2 \Phi_1(d)$. Proceeding as in Proposition 7.13, one may show $po \in \text{steps}(co)$ and $a \xrightarrow{po} b$, contradicting $a \xleftrightarrow[\text{steps}(co)]{} b$. $\square$

Theorem 7.17 provides an axiomatisation of finite concurrent histories conforming to paradigm π_3 under the assumption that all observations are step sequences. If π_3 does not hold, then $\{\rightarrow, \nearrow\}$ is no longer a signature and $\text{steps}(co)$ may not be interpreted as a concurrent history of step sequence observations.

The results of this section could be interpreted in three ways. One is to treat them as an extension of Szpilrajn-Marczewski result [45] that each poset is uniquely represented by the set of its total extensions. Theorem 7.15 states that each composet is uniquely represented by the set of its stratified extensions, while Theorem 7.8 together with Theorem 7.6 and Proposition 7.7, say that each weak composet is uniquely represented by the set of its interval extensions.

Theorems 7.10 and 7.17 provide the second, major, interpretation for the finite case: When paradigm π_3 is enforced, finite weak composets are signatures of concurrent histories. Under additional assumption that all observations are step sequences, finite composets become signatures of concurrent histories.

The third way of interpreting the results of this section is to assume relativistic real time observers. In our approach observations are just observer reports about instances of a concurrent behaviour. In principle, we identify observations with executions and next identify equivalent executions creating what we call a concurrent history. Thus our observation is an abstraction of an execution. However, we may also consider the following situation: There is one system execution, physically many observers, and Einstein-Minkowski space-time is assumed. (This is exactly the situation considered in [1, 27, 28].) Each observer's local time is linear, but the time structure generated by all observers is a partial order. Theorem 7.2 (a major result of [1]) says that weak composets can be used to model this kind of system execution provided that observers can observe and report time intervals. If they can use only time points then, by Theorem 7.12, composets seem to be a good model of system executions.

8. Related work

The idea of using structures based on interval orders on the observation level has been advocated in [46] (implicitly) [19], [36] and [47]. In [46], van Glabbeek and Vaandrager introduced the concept of real-time consistency and then defined real-time consistent bisimulation (ST-bisimulation). The intuition behind ST-bisimulation is that when observing a system run we see actions starting and finishing, i.e., the execution of an action corresponds to some time interval, and the order of the actions is exactly that of their time intervals. This is exactly an application of Fishburn's representation theorem (Theorem 2.11) as the definition. Van Glabbeek and Vaandrager in [46] did not define or use interval orders, they expressed this intuition in terms of ST-bisimulation, using Petri nets as a general framework. Nielsen et al. [36] studied the use of (labelled) partial orders as denotational model for process algebras. They started with the step sequence model, and next, by change of atomicity, ended up with the interval order model. They used Wiener's definition ($a \rightarrow b \wedge c \rightarrow d \Rightarrow a \rightarrow d \vee c \rightarrow b$), calling it *P_{or}-property* and seemed to be unaware of earlier results concerning this concept. They did not mention work due to Wiener, Fishburn or others. Janicki and Koutny's work [18] is an early version of the results presented here in Section 2. As in this paper, the motivation was that if a poset is an observation, then its principal poset is total. Theorem 2.12 says that this is equivalent to being an interval order. In [47], Vogler started with a similar motivation as [46], i.e. Fishburn's representation theorem (Theorem 2.11) provided a required intuition. He next defined failure semantics based on interval orders for Petri nets. He used both Fishburn's theorem and Wiener's definition in his work.

From the formal point of view, interval orders can be defined in three ways. One way is to use Wiener's definition (as in [36]); the second is to use Fishburn's representation theorem (intuition in [46, 47]); the third possibility is to use the concept of principal order and Theorem 2.12, as in [18] and this paper. None of [46, 36, 47] provides detailed analysis of the interval orders themselves. Fishburn [7, 8] does, but he always assumes dense time and almost neglects the relationship between interval orders and their principal orders (Theorem 2.12). We consider this relationship very important, as it provides the basic intuition in our definition of observation. We analysed both discrete and dense time, provided representation theorems in both cases, and showed this representation is injective (Theorems 2.14, and 2.16).

In [39] Plotkin and Pratt analysed the situation whereby observers work as a team. Each observer alone can only observe sequences of events, but they can communicate among themselves and subsequently provide a joint statement on their observations. In our framework this means that k observers provide a single report. Plotkin and Pratt in [39] show that the resolving power of a finite team of observers increases strictly with k , and that they can see more complex posets (in fact, pomsets) than interval orders, as the axiom A3 of our definition of observation is no longer valid for teams of observers. The use of such observers would change some results of Section 6. It would not change the analysis of the paradigms, but, e.g., Propositions 6.2 and

6.4(2) would not hold. Most of the results of Section 7 also assume interval order observations. Nevertheless, the observations of [39] can still be modelled as report systems, so they fit into our general framework.

On the invariant level structures similar to composets and weak composets can be found in [1, 3, 12, 19, 20, 27, 28]. In [12], Gaifman and Pratt defined behaviours as structures (called *prosets*—preorder specification sets) of the form: $(\Sigma, <, \equiv, \leq)$, where Σ is a multiset of events, $<, \equiv, \leq$ are relations interpreted as *precedence*, *simultaneity* and *not later than*, and $\equiv$ is defined as $a \equiv b \Leftrightarrow a \leq b \wedge b \leq a$. The axioms for $(\Sigma, <, \leq)$ are essentially the same as C1–C4 for composets (we restrict ourselves to sets, but the extension to multisets is quite straightforward) with $<$ corresponding to $\xrightarrow{co}$, and $\leq$ to $\nearrow_{co} \cup id_{\text{dom}(co)}$. Hence, the results of the entire Section 7.2 hold for the prosets as well. Gaifman and Pratt [12] defined and used prosets, but have not analysed their structure.

In [27, 28] Lamport provides a model for system execution using Einstein’s concept of time–space relationship. He argues that the relativistic view is relevant whenever signal propagation is not negligibly small compared with the execution time of individual operations. He defines a system as a set of operation executions where each operation execution consists of a nonempty set of space–time events. Lamport [27, 28] defines the relations $\rightarrow$ and $\dashrightarrow$ on the set of operation executions as follows:

$$A \rightarrow B \Leftrightarrow \forall a \in A \forall b \in B. a < b,$$

$$A \dashrightarrow B \Leftrightarrow \exists a \in A \exists b \in B. a < b \vee a = b,$$

where A and B are operation executions, a and b are space–time events, and $<$ is the (irreflexive) order in Minkowski space. One may verify that $\rightarrow$ and $\dashrightarrow$ $\dashv id_x$ satisfy the axioms WC1–WC5 for weak composets. Lamport next argues that in computer science we may ignore the space–time events that constitute operation executions, and defines system execution as a structure $(\Sigma, \rightarrow, \dashrightarrow)$, where Σ is a set of operation executions and $\rightarrow, \dashrightarrow$ $\dashv id_x$ satisfy WC1–WC4. He advocates the use of this concept on various levels of abstraction. The structure $(\Sigma, \rightarrow, \dashrightarrow)$, with $\rightarrow, \dashrightarrow$ $\dashv id_x$ satisfying WC1–WC4, is frequently called *Lamport structure* [1]. The axioms corresponding to WC1–WC5 were proposed (in Lamport’s framework) in [3]. Hence, the results of Section 7.1 can also be used in that model. The main result of Abraham et al. [1] plays a central role in obtaining the main results in Section 7.1 (Section 7.2 does not need it). Due to its roots, Lamport’s model is often used to analyse the global time assumptions [1]. In the framework of $\rightarrow, \dashrightarrow$, the global time axioms is stated as: $A \rightarrow B \Leftrightarrow \neg B \dashrightarrow A$. Our observations are just observer reports, they do not mention time explicitly, different observers may observe different instances of the same concurrent history in disjoint time intervals. Global time axiom implies Newton model of time and in our approach all observers observing in the same physical time. So they all must observe the same, i.e. $\exists o. a \xrightarrow{o} b \Rightarrow \forall o. a \xrightarrow{o} b$, which clearly implies: $a \xrightarrow{\Delta} b \Leftrightarrow \neg b \nearrow_{\Delta} a$. In [20] finite composets were analysed in

a style similar to the middle part of Section 7.2. Janicki and Koutny's [19] is an early version of the results presented in Section 6.

9. Systems

The development of the system level is still in an initial phase, however, some nontrivial results do already exist. To some extent, the results of Gaifman and Pratt [12] can be seen as an example of such a development. In [12] the composet-like structures are used to analyse such concepts as: fairness, input event, the location of a process, etc. Another more direct example is Janicki and Koutny's [20] where a formal semantics for inhibitor nets is defined and analysed. Janicki and Koutny's [20] shows that the composets provide an *invariant semantics* for inhibitor nets and that such a semantics is in full agreement with the *operational semantics* defined in terms of step sequences. It also shows that composets can be generated by inhibitor nets just by generalising the standard construction of processes for Petri nets. We believe that the structural complexity of the behaviours generated by concurrent systems depends on the kind of the operators the system uses. If only sequential operators and parallel composition are involved, then causal partial orders suffice to describe concurrent histories. However, if other operators, e.g. priority or commutativity, are allowed, we need more complex structures, e.g., composets or weak composets.

10. Conclusions

In this paper we presented first steps of the development of a new approach to modelling concurrent systems. We started our discussion on the observation level and introduced a general notion of an observation of a concurrent history. We have obtained representation theorems for the general observations and also for some more restricted classes of observations. We then introduced the notion of a report system of concurrent observations, and investigated the invariant properties of sets of related observations. We have identified and interpreted a class of fundamental invariants of concurrent histories. We have also established a connection between the paradigms of concurrency and the invariants of concurrent histories. A direct consequence of Table 2 is that depending on the paradigm, a minimal invariant representation of concurrent histories will in most cases be different. As one of the referees has pointed out, by selecting minimal signature for a paradigm, one can help choosing most adequate algebraic framework before specifying a concurrent system. Finally, we provided an axiomatisation of minimal signature for one of the paradigms.

Appendix

Lemma A.1. *Countable total orders have real injective interval representations.*

Note: In fact, this is an immediate consequence of Theorem 2.11, as for total orders eq reduces to identity. However, the result can independently be proved by induction (the simple proof below is due to Franek [10]).

Proof. It suffices to show that if po and pr are finite total orders, with $\xrightarrow{po} \subseteq \xrightarrow{pr}$ and $\text{dom}(pr) = \text{dom}(po) \cup \{a\}$, such that $\partial = (\Phi, \Psi)$ is a real injective interval representation of po , then one can define $\Phi(a)$ and $\Psi(a)$ in such a way that the extended ∂ is a real injective interval representation for pr . To show this we observe that since pr is total and finite, there is an interval $(x, y) \subseteq \mathbb{R}$ such that $\Psi(b) \leq x$ for all $b \in \text{dom}(po)$ satisfying $b \xrightarrow{pr} a$, and $y \leq \Phi(c)$ for all $c \in \text{dom}(po)$ satisfying $a \xrightarrow{pr} c$. It now suffices to define $\Phi(a) = x + \varepsilon$ and $\Psi(a) = y - \varepsilon$, where $0 < \varepsilon < \frac{1}{2}(y - x)$. $\square$

Lemma A.2. *Let $\succsim$ be a relation on the domain of a poset po defined by*

$$a \succsim b \iff a \xrightarrow{po} b \vee (\exists c. c \xrightarrow{po} b \wedge b \xleftrightarrow{po} a \xleftrightarrow{po} c).$$

Then po is an interval order iff $(\text{dom}(po), \succsim)$ is a poset.

Note: The left-to-right implication is equivalent to the first part of Theorem 2 in Section 2 of [8]. We provide a proof to make the presentation self-contained.

Proof. $\Leftarrow$: Suppose $a \xrightarrow{po} b$, $c \xrightarrow{po} d$, $\neg a \xrightarrow{po} d$ and $\neg c \xrightarrow{po} b$. Then $b \succsim d$ and $d \succsim b$. Hence, $(\text{dom}(po), \succsim)$ is not a poset.

$\Rightarrow$: We only need to show the transitivity of $\succsim$. Suppose $a \succsim b \succsim c$ and $\neg a \xrightarrow{po} c$. We consider three cases.

Case 1: $a \xrightarrow{po} b \xleftrightarrow{po} c$. Then $d \xrightarrow{po} c$ and $b \xleftrightarrow{po} d$ for some d . We have $\neg a \xrightarrow{po} c$ and $\neg d \xrightarrow{po} b$, a contradiction since po is an interval order.

Case 2: $a \xleftrightarrow{po} b \xrightarrow{po} c$. Then, $d \xrightarrow{po} b$ and $a \xleftrightarrow{po} d$ for some d . Furthermore, $\neg c \xrightarrow{po} a$. Hence, $d \xrightarrow{po} c$ and $d \xleftrightarrow{po} a \xleftrightarrow{po} c$, yielding $a \succsim c$.

Case 3: $a \xleftrightarrow{po} b \xleftrightarrow{po} c$. Then, $e \xrightarrow{po} b$, $e \xleftrightarrow{po} a$, $f \xrightarrow{po} c$ and $b \xleftrightarrow{po} f$ for some e and f . Hence, since po is an interval order, $e \xrightarrow{po} c$. By $a \xleftrightarrow{po} e$ and $e \xrightarrow{po} c$, $a \neq c$. Also, $\neg c \xrightarrow{po} a$ since $a \xleftrightarrow{po} e \xrightarrow{po} c$, and we assumed $\neg a \xrightarrow{po} c$. Hence $a \xleftrightarrow{po} c$, which together with $a \xleftrightarrow{po} e \xrightarrow{po} c$ yields $a \succsim c$. $\square$

Proof of Theorem 2.14. $\Leftarrow$: Follows from Proposition 2.13.

$\Rightarrow$: Let po be a countable poset and $\Sigma = \text{dom}(po)$. Consider $(\Sigma, \succsim)$ defined as in Lemma A.2. From Lemma A.2 and Szpilrajn-Marczewski extension theorem [45], it follows that there is a total order $t = (\Sigma, \rightarrow_t)$ such that $\succsim \subseteq \rightarrow_t$. From Lemma A.1 it

follows that t has a real injective interval representation $\partial_t = (\Phi_t, \Psi_t)$. Also, we can assume that $\Psi_t(a) < 0$ for all a . (If this does not hold then we can take $\partial_0 = (\Phi_0, \Psi_0)$ defined by $\Phi_0(a) = -2^{-\Phi_t(a)}$ and $\Psi_0(a) = -2^{-\Psi_t(a)}$ for all a , which is another injective interval representation of t .) Hence, $\Psi(a) = \sup\{\Psi_t(c) \mid c = a \vee c \xrightarrow{po} a\}$ is defined for all $a \in \Sigma$. Suppose $a \xrightarrow{po} b$. Then, for all $c \in \Sigma$ we have: $(c = a \vee c \xrightarrow{po} a) \Rightarrow c \not\rightarrow b \Rightarrow c \rightarrow_t b \Rightarrow \Psi_t(c) < \Phi_t(b)$. Hence, $\Psi(a) \leq \Phi_t(b)$. Let $\Phi(a) = \frac{1}{2}(\Phi_t(a) + \Psi_t(a))$ for all $a \in \Sigma$. Clearly, $\Phi_t(a) < \Phi(a) < \Psi_t(a) \leq \Psi(a)$.

We now prove that $\partial = (\Phi, \Psi)$ is an injective interval representation of po . We first observe that if $\Phi(a) = \Phi(b)$ then $\Phi_t(b) < \Phi(b) = \Phi(a) < \Psi_t(a)$. Hence, $\neg a \rightarrow_t b$. Similarly, $\neg b \rightarrow_t a$. Hence, since t is total, $a = b$. To show $a \xrightarrow{po} b \Leftrightarrow \Psi(a) < \Phi(b)$ we observe that

$$a \xrightarrow{po} b \Rightarrow \Psi(a) \leq \Phi_t(b) < \Phi(b) \Rightarrow \Psi(a) < \Phi(b),$$

$$b \xrightarrow{po} a \Rightarrow \Psi(b) < \Phi(a) \Rightarrow \neg \Psi(a) < \Phi(b),$$

$$a = b \Rightarrow \Psi(a) > \Phi(b) \Rightarrow \neg \Psi(a) < \Phi(b),$$

$$a \xleftrightarrow{po} b \Rightarrow \Psi(a) \geq \Psi_t(b) > \Phi(b) \Rightarrow \neg \Psi(a) < \Phi(b). \quad \square$$

Notation. A set of integers J is *gap-free* if $i < j < k$ and $i, k \in J$ implies $j \in J$. If two intervals on real line, $K = [a, b]$ and $L = [c, d]$, satisfy $b < c$ then we will write $K \angle L$.

Proof of Lemma 2.15. From Theorem 2.12 it follows that C_{po} is total. Moreover, by Proposition 2.7, C_{po} is combinatorial. Hence, there is a gap-free set of integers J such that $Cuts_{po} = \{A_j \mid j \in J\}$ and $A_j \xrightarrow{po} A_{j+1}$, for $j, j+1 \in J$.

For every $a \in \text{dom}(po)$, let $K_a = [m_a, M_a]$, where $m_a = \min\{i \mid a \in A_i\}$ and $M_a = \max\{i \mid a \in A_i\}$. Note that m_a and M_a are well defined due to (1). It is not difficult to see that

$$(A.1) \quad \forall a, b \in \text{dom}(po). \quad a \xrightarrow{po} b \Leftrightarrow K_a \angle K_b.$$

Let $K = \{K_a \mid a \in \text{dom}(po)\}$, and let $\delta: \text{dom}(po) \rightarrow \mathbb{R}$ be any injection.

We define $\ll K \times K$ as follows. Let $a, b \in \text{dom}(po)$.

$$\begin{aligned} K_a \ll K_b &:= (m_a < m_b) \vee (m_a = m_b \wedge M_a < M_b) \vee \\ &\quad (m_a = m_b \wedge M_a = M_b \wedge \delta(a) < \delta(b)). \end{aligned}$$

Clearly, $(K, \ll)$ is a total order such that $\angle \subseteq \ll$. Moreover, by (1), $(K, \ll)$ is combinatorial. Hence, there is a gap-free set of integers H such that $\text{dom}(po) = \{a_i \mid i \in H\}$ and $K_{a_i} \ll K_{a_{i+1}}$ for all $i, i+1 \in H$.

For every $i \in H$, let $L_{a_i} = [2i, 2i+1]$, where $l_i = \max\{j \mid K_{a_i} \cap K_{a_j} \neq \emptyset\}$. Clearly, l_i is defined due to (1) and (A.1). We also note that $2l_i + 1 \geq 2i + 1 > 2i$, so each L_{a_i} is

a nondegenerated interval. We also observe that

$$(A.2) \quad \forall i, j \in H. i < j \Rightarrow m_{a_i} \leq m_{a_j}.$$

We now show that

$$(A.3) \quad \forall i, j \in H. K_{a_i} \angle K_{a_j} \Leftrightarrow L_{a_i} \angle L_{a_j}.$$

Suppose $K_{a_i} \angle K_{a_j}$. If there is $p > j$ such that $K_{a_i} \cap K_{a_p} \neq \emptyset$ then, by (A.2), $m_{a_i} \leq m_{a_j} \leq m_{a_p} \leq M_{a_i}$, contradicting $K_{a_i} \cap K_{a_j} = \emptyset$. Hence, $K_{a_i} \angle K_{a_p}$ for all $p \geq j$.

Thus $l_i < j$, which yields $2l_i + 1 < 2j$. Consequently, $L_{a_i} \angle L_{a_j}$. To show the reverse implication, we assume $L_{a_i} \angle L_{a_j}$. Then $2l_i + 1 < 2j$, which yields $l_i < j$. Consequently, $K_{a_i} \angle K_{a_j}$. Hence, (A.3) holds.

Let $a \in \text{dom}(po)$ and $L_a = [x, y]$. Define $\Phi(a) = x$ and $\Psi(a) = y$. From (A.1) and (A.3) it follows that $\hat{\partial} = (\Phi, \Psi)$ is a discrete interval representation of po . Moreover, $\hat{\partial}$ is injective. $\square$

Lemma A.3. Let $\sigma = (\sigma_1, \dots, \sigma_k)$ and $\delta = (\delta_1, \dots, \delta_k)$ be tuples in B . Then, $\neg \Phi_\sigma \simeq \Phi_{\neg \sigma}$, $\Phi_\sigma \vee \Phi_\delta \simeq \Phi_{\sigma \vee \delta}$ and $\Phi_\sigma \wedge \Phi_\delta \simeq \Phi_{\sigma \wedge \delta}$.

Proof. Let $\mu \in RS$, and let $a, b \in \text{dom}(\mu)$ be such that $a \neq b$. We have

$$\begin{aligned} (\neg \Phi_\sigma)(a, b, \mu) &\Leftrightarrow \text{not } \Phi_\sigma(a, b, \mu) \Leftrightarrow \xi_{\text{index}(a, b, \mu)} = \text{false} \\ &\Leftrightarrow \sigma_{\text{index}(a, b, \mu)} = 0 \Leftrightarrow (\neg \sigma)_{\text{index}(a, b, \mu)} = 1 \Leftrightarrow \Phi_{\neg \sigma}(a, b, \mu) \\ (\Phi_\sigma \vee \Phi_\delta)(a, b, \mu) &\Leftrightarrow \Phi_\sigma(a, b, \mu) \text{ or } \Phi_\delta(a, b, \mu) \\ &\Leftrightarrow \sigma_{\text{index}(a, b, \mu)} = 1 \text{ or } \delta_{\text{index}(a, b, \mu)} = 1 \Leftrightarrow (\sigma_{\text{index}(a, b, \mu)} \vee \delta_{\text{index}(a, b, \mu)}) = 1 \\ &\Leftrightarrow \Phi_{\sigma \vee \delta}(a, b, \mu). \end{aligned}$$

Consequently, $\neg \Phi_\sigma \simeq \Phi_{\neg \sigma}$ and $\Phi_\sigma \vee \Phi_\delta \simeq \Phi_{\sigma \vee \delta}$. We also have the following.

$$\begin{aligned} \Phi_\sigma \wedge \Phi_\delta &\equiv \neg(\neg \Phi_\sigma \vee \neg \Phi_\delta) \simeq \neg(\Phi_{\neg \sigma} \vee \Phi_{\neg \delta}) \simeq \neg \Phi_{(\neg \sigma) \vee (\neg \delta)} \\ &\simeq \Phi_{\neg((\neg \sigma) \vee (\neg \delta))} = \Phi_{\sigma \wedge \delta}. \quad \square \end{aligned}$$

Proof of Theorem 3.2. Clearly, $\{\Phi_\sigma \mid \sigma \in B\} \subseteq SRF$. To show $SRF \subseteq \{\Phi_\sigma \mid \sigma \in B\}$ we observe that $\text{true} \simeq \Phi_{1\dots 11}$, $\text{false} \simeq \Phi_{0\dots 00}$, and $\beta r_{i,x} \gamma \simeq \Phi_{(\sigma_1, \dots, \sigma_k)}$, where $\sigma_j = 1 \Leftrightarrow j = i$, for all $i \leq k$. Moreover, by Lemma A.3, $\{\Phi_\sigma \mid \sigma \in B\}$ is closed w.r.t. $\simeq$ under the $\neg$, $\vee$ and $\wedge$ operations. Hence, $SRF \subseteq \{\Phi_\sigma \mid \sigma \in B\}$.

Let $\sigma = (\sigma_1, \dots, \sigma_k)$ and $\delta = (\delta_1, \dots, \delta_k)$. Without loss of generality we assume that $\sigma_1 = 1$ and $\delta_1 = 0$. Since RS is nondegenerated, there is $\mu \in RS$ such that $r_{1,\mu} \neq \emptyset$. Let $(a, b) \in r_{1,\mu}$. We observe that $\Phi_\sigma(a, b, \mu)$ holds, while $\Phi_\delta(a, b, \mu)$ does not hold. Hence, $\Phi_\sigma \simeq \Phi_\delta$ does not hold, which completes the second part of the proof. $\square$

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