

## NETS, SEQUENTIAL COMPONENTS AND CONCURRENCY RELATIONS\*

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**Abstract.** A lattice of unmarked nets is introduced and studied. It is proved that unmarked nets representing the static structure of sequential systems are atoms of that lattice. Marking classes defined by the decomposition of nets into sequential components are introduced and properties (safeness, fireability, etc.) of nets with those marking classes are investigated. The notion of concurrency relation on the system level is defined and discussed. Two different definitions of that relation are given. The first one starts with a given in advance decomposition of a net into sequential components; the second one is constructed on the basis of a given in advance marking class. Both definitions follow from a general concept of the symmetric and irreflexive relation defined by a set covering. Petri's postulate about a common element for every global system state and every sequential component is carried up the system level and its strength is discussed. It turns out that if a net is safe and each transition has a possibility to be fired then that postulate implies that the net is decomposable into finite state machines.

### 1. Introduction

The approach presented in the paper follows from the author's conviction that people think sequentially. Of course, our brain works nonsequentially, and we can understand parallel processes, but our mental perception of reality is sequential.

It turns out that not technology but human imagination is the main obstacle in the use of concurrency in computers. Long before now, people have stated that it is very difficult to comprehend the total effect of actions being performed concurrently and with independent speeds (compare Brinch Hansen [2] and his example of troubles with learning the history of the whole of Europe).

People express their thoughts by means of a language, but every language is sequential in the course of nature. Furthermore, the concept of 'time continuum' also sequentialises our perception of reality.<sup>1</sup> If we agree that the way of thinking

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<sup>1</sup> For instance in the Hopi American Indian Tribe Language, there is no 'time continuum', the world is treated as a collection of events, and the flow-time is a relation among events (see [24]).

and the way of speaking are strictly connected,<sup>2</sup> we obtain that people think in a sequential way.

The sequential way of thinking implies two natural methods of specifying concurrent systems. The first method consists in starting with a functionally equivalent sequential system, determining a set of independent actions and then performing a set of transformations of the sequential system resulting in a concurrent system. Although this method is frequently used in practice, especially to describe technological processes, its theoretical principles are insufficiently examined. A mathematical background of that method was given by Janicki [14].

The second method consists in decomposing the problem into components being sequential in the course of their nature (frequently such a decomposition is ambiguous), solving each sequential part of the problem in a relatively independent way, and then composing all sequential solutions into the whole. There is a large number of techniques for specifying concurrent systems, which are based on that method. The well-known ones are the following: Semaphores [6], Synchronised Parallel Processes [9], Communicating Systems [18], Path Expressions [7], Cosy Formalism [16].

This paper also concerns the second method. We shall deal with general properties of concurrent systems decomposed into, and composed from, sequential components. As a base for further considerations we shall use Petri nets [3, 7, 21]. A net model of concurrency seems to be sufficiently wide, and contrary to the models quoted above, it does not assume in advance the existence of sequential components, although it does not exclude that existence either (see [5, 8]).

The essential intent of our approach is to construct rules for decomposition of nets into indivisible components (atoms) with simple, 'primitive' concurrency. In this way one can describe properties of the whole net by means of properties of components. Special attention is paid to such a class of nets whose components represent sequential systems. We will try to define the marking class on the basis of a net decomposition into sequential components. The notion of concurrency relation is defined and precisely investigated on the system level.

It turns out that in our approach the concurrency relation is one of the most important, very convenient, notions. Petri's postulate that every sequential component and every global state must have one element in common, is carried up the system level, and its strength is analysed.

Although in the paper standard mathematical notation will be used, we recall some basic notations. And so, by  $\emptyset$  we shall denote the empty set or empty relation,  $|X|$  will denote the cardinality of  $X$ ,  $\text{id}$  will denote the identity relation,  $R^*$  will denote the relation defined by the equality  $R^* = \bigcup_{i=0}^{\infty} R^i$ , for every equivalence relation  $R \subseteq X \times X$ , the set of all equivalence classes will be denoted by  $X/R$ , and

<sup>2</sup> For the first time this idea was expressed by Wilhelm von Humboldt (19th century), who said: "We think such as we speak, and at the same time we speak such as we think". In the first half of our century those ideas were investigated by the American linguists Sapir and Whorf, who formulated the so-called 'law of language relativism', also known as 'the Sapir-Whorf hypothesis' (see [23]).

the equivalence class containing the set  $Y$  will be denoted by  $[Y]_R$ . The remaining detailed notations will be given in suitable sections.

Some results of the paper have been announced (see [12, 13, 15]).

## 2. A lattice of s-nets

The approach presented in the paper is based on the notion of so-called s-nets. This concept and the notation connected with it enable us to create a convenient algebraic structure of nets. Both the concept and the notation follow from [10].

For every set  $X$ , let  $\text{left}: X \times X \rightarrow X$ ,  $\text{right}: X \times X \rightarrow X$  be the following projections:

$$\forall (x, y) \in X \times X, \quad \text{left}((x, y)) = x, \quad \text{right}((x, y)) = y.$$

By a *simple net* (abbr. s-net) we mean any pair

$$N = (T, P),$$

where  $T$  is a set (of *transitions*),  $P \subseteq 2^T \times 2^T$  is a relation (interpreted as a set of *places*), and

$$\forall a \in T \quad \exists p, q \in P, \quad a \in \text{left}(p) \cap \text{right}(q).$$

In the paper we restrict our attention to finite s-nets. Instead of writing  $(\{a_1, \dots, a_n\}, \{b_1, \dots, b_m\}) \in P$  we shall write  $[a_1, \dots, a_n; b_1, \dots, b_m] \in P$ .

Every s-net  $N = (T, P)$  can be represented, using the graph shown in Fig. 1, to denote that  $[a_1, \dots, a_n; b_1, \dots, b_m] \in P$ .

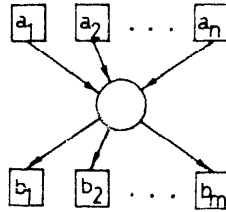


Fig. 1.

**Example 2.1.** Let  $N = (T, P)$ , where

$$T = \{a, b, c, d, e, f, g\},$$

$$P = \{[\emptyset; a], [a, f; b, g], [g; \emptyset], [c; e], [b; e], [b; d], [d; e], [e; f]\}.$$

The pair  $(T, P)$  is an s-net and it can be represented by the graph shown in Fig. 2.

In the literature, nets are usually defined differently, starting with two disjoint sets of transitions and places, and introducing a flow-relation among them (compare [3, 21]). Our approach has an advantage in the sense that it makes it more easy to handle operations among nets.

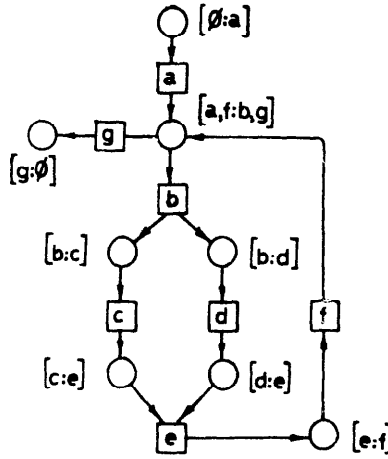


Fig. 2.

Using a standard notation, we define a net as a triple  $N = (T, P, F)$ , where  $T$  is a set of transitions,  $P$  is a set of places,  $T \cap P = \emptyset$ ,  $T \cup P \neq \emptyset$ , and  $F \subseteq T \times P \cup P \times T$  is a flow relation.

To define 'successors' and 'predecessors' under the relation  $F$ , the convenient 'dot' notation is usually used:

$$\forall x \in T \cup P, \quad x' = \{y \mid (y, x) \in F\}, \quad x' = \{y \mid (x, y) \in F\}.$$

The triple  $(T, P, F)$  is an s-net in the above sense iff

- (1)  $\forall p, q \in P, \quad (p' = q' \ \& \ p' = q') \Rightarrow p = q,$
- (2)  $\forall a \in T, \quad a' \neq \emptyset \ \& \ a' \neq \emptyset.$

Then every  $p \in P$  can unambiguously be represented by a pair  $(p, p') \subseteq 2^T \times 2^T$ .

The flow relation, in our approach, can be defined as follows. Let  $N = (T, P)$  be an s-net. Let  $F \subseteq T \times P \cup P \times T$  (or  $F_N$  if  $N$  is not understood) be the following relation:

$$\forall x, y \in T \cup P, \quad (x, y) \in F \Leftrightarrow x \in \text{left}(y) \text{ or } y \in \text{right}(x).$$

In our approach the 'dot' operation can be defined without using the notion of flow relation.

- (1)  $\forall p \in P, \quad p' = \text{right}(p), \quad p = \text{left}(p),$
- (2)  $\forall a \in T, \quad a' = \{p \in P \mid a \in \text{left}(p)\}, \quad a = \{p \in P \mid a \in \text{right}(p)\}.$

Note that the 'dot' operations are correctly defined for every pair  $(T, P)$ , where  $P \subseteq 2^T \times 2^T$ .

**Lemma 2.2.** *A pair  $(T, P)$  where  $P \subseteq 2^T \times 2^T$  is an s-net iff*

$$\forall a \in T, \quad a' \neq \emptyset \ \& \ a' \neq \emptyset.$$

**Proof.** The proof of the lemma follows from the definition of ' $a$ ' and ' $a$ '.  $\square$

For every  $X \subseteq T \cup P$ , let ' $X = \bigcup_{x \in X} x$ ', ' $X' = \bigcup_{x \in X} x'$ '.

Let **SNETS** denote the family of all finite s-nets. Note that the class of **SNETS** is a set.

Let  $\sqsubseteq$  be the following relation on **SNETS**:

$$N_1 = (T_1, P_1) \sqsubseteq N_2 = (T_2, P_2) \Leftrightarrow P_1 \subseteq P_2.$$

Note that  $\sqsubseteq$  is a *partial order* and  $N_1 \sqsubseteq N_2 \Rightarrow T_1 \subseteq T_2$ . Let  $\sup\{N_1, N_2\}$ ,  $\inf\{N_1, N_2\}$  denote respectively the least upper and the greatest lower bound under the relation  $\sqsubseteq$ .

**Theorem 2.3.** For every  $N_1 = (T_1, P_1)$ ,  $N_2 = (T_2, P_2) \in \mathbf{SNETS}$ :

- (1)  $\sup\{N_1, N_2\} = (T_1 \cup T_2, P_1 \cup P_2)$ ,
- (2)  $\inf\{N_1, N_2\} = (P, P)$ , where  $P = \bigcup \{P' \mid P' \subseteq P_1 \cap P_2 \text{ \& } (P') = (P')'\}$ .

**Proof.** (1) Because  $\text{left}(P_1 \cup P_2) = \text{right}(P_1 \cup P_2) = T_1 \cup T_2$ , the pair  $(T_1 \cup T_2, P_1 \cup P_2)$  is an s-net. Denote  $N_{12} = (T_1 \cup T_2, P_1 \cup P_2)$ . Let  $N = \sup\{N_1, N_2\}$ , and let  $N = (T, P)$ . Since obviously  $N_i \sqsubseteq N_{12}$  for  $i = 1, 2$ ,  $N \sqsubseteq N_{12}$ . On the other hand,  $P_i \subseteq P$ ,  $T_i \subseteq T$  for  $i = 1, 2$ , so  $T_1 \cup T_2 \subseteq T$ ,  $P_1 \cup P_2 \subseteq P$ ; but this means that  $N_{12} \sqsubseteq N$ . Thus  $N_{12} = N$ .

(2) Follows directly from the definition of the greatest lower bound.  $\square$

Let us define the well-known lattice operations

$$\begin{aligned} N_1 \cup N_2 &= \sup\{N_1, N_2\}, & N_1 \cap N_2 &= \inf\{N_1, N_2\}, \\ \bigcup_{N \in S} N &= \sup\{N \mid N \in S\}, & \bigcap_{N \in S} N &= \inf\{N \mid N \in S\}. \end{aligned}$$

**Corollary 2.4.** The algebra  $(\mathbf{SNETS}, \cup, \cap)$  is a complete lattice with the greatest lower bound  $(\emptyset, \emptyset)$ .

Since **SNETS** is a lattice, we can introduce the notion of an atom. An s-net  $N$  is said to be an *atom* iff

- (1)  $N \neq (\emptyset, \emptyset)$ ,
- (2)  $(N' \sqsubseteq N) \Rightarrow (N' = N \text{ or } N' = (\emptyset, \emptyset))$ .

In other words, the s-net  $N$  is an atom if it is an atom in the lattice **SNETS**.

For every s-net  $N$ , let  $\text{atoms}(N)$  denote the set of all atoms contained in  $N$ , i.e.,

$$\text{atoms}(N) = \{N' \mid N' \sqsubseteq N \text{ \& } N' \text{ is an atom}\}.$$

An s-net  $N$  is said to be *atomic* iff

$$N = \bigcup_{N' \in \text{atoms}(N)} N'.$$

**Example 2.5.** Let us consider the s-nets shown in Figs. 3 and 4.

We have:  $\text{atoms}(N) = \{N_1\}$  and  $N \neq N_1$ , so  $N$  is not atomic; and  $\text{atoms}(N') = \{N'_1, N'_2\}$ ,  $N' = N'_1 \cup N'_2$ , thus  $N'$  is atomic.

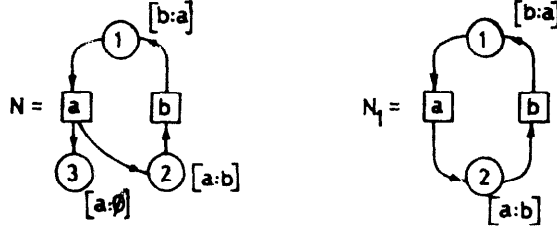


Fig. 3.

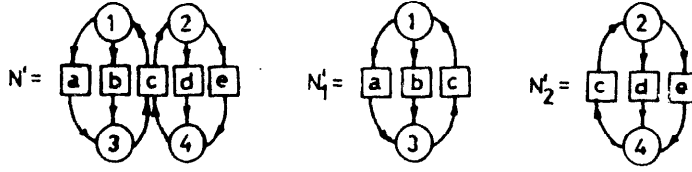


Fig. 4.

An s-net  $N = (T, P)$  is said to be *connected* iff

$$\forall x, y \in T \cup P, \quad (x, y) \in (F_N \cup F_N^{-1})^*.$$

In other words, an s-net is said to be connected if its graph is connected in the usual sense of this word.

Let us put  $C_N = (F_N \cup F_N^{-1})^*$  for every s-net  $N$ . Note that  $C_N$  is an equivalence relation on  $T \cup P$ . Thus we can say that an s-net  $N = (T, P)$  is connected iff  $T \cup P \in (T \cup P)/C_N$ , i.e., if  $T \cup P$  is an equivalence class of  $C_N$ .

**Theorem 2.6.** *Every atom is connected.*

**Proof.** Assume that an s-net  $N = (T, P)$  is disconnected. This means that  $|(T \cup P)/C_N| > 1$ .

Let  $A \in (T \cup P)/C_N$ . Note that  $A \neq T \cup P$ , and  $N_A = (A \cap T, A \cap P)$  is an s-net! But because  $A \neq T \cup P$  then  $N_A \subsetneq N$  &  $N_A \neq N$ . Thus  $N$  is not an atom.  $\square$

### 3 Elementary, quasialementary and proper s-nets

It is a well-known fact that sequential systems can be adequately modelled by finite state machines (see, for example, [3, 7]). In this section we define finite state machines using the notation defined above and show that they are atoms of s-nets.

An s-net  $N = (T, P)$  is said to be *quasialementary* iff

$$\forall a \in T, \quad |a| = |a^\circ| = 1.$$

An s-net  $N = (T, P)$  is said to be *elementary* iff it is elementary and connected.

Elementary nets are equivalent with totally labelled connected finite state machines, and will represent sequential systems or subsystems in our approach. Quasielementary nets will represent disconnected sequential systems.

**Example 3.1.** The net defined in Fig. 5 is elementary.

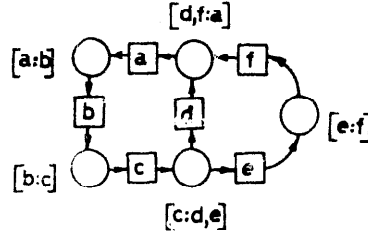


Fig. 5

**Theorem 3.2.** Every elementary s-net is an atom.

**Proof.** Suppose that  $N = (T, P)$  is an elementary net, and that there is an s-net  $N_1 = (T_1, P_1)$  such that

$$(\emptyset, \emptyset) \neq (T_1, P_1) \sqsubset (T, P).$$

We must consider two disjoint cases:

- (1)  $T_1 = T$  &  $P_1 \subsetneq P$ ,
- (2)  $T_1 \subsetneq T$  (of course,  $T_1 \subsetneq T \Rightarrow P_1 \subsetneq P$ ).

*Case 1.* Let  $P_2 = P - P_1$ , and let  $p \in P_2$ .

Since  $N$  is elementary, there is  $a \in T_1 = T$  such that  $\cdot a = \{p\}$  or  $a \cdot = \{p\}$ . But if  $\cdot a = \{p\}$  then  $a \notin \text{right}(P_1)$ , and if  $a \cdot = \{p\}$ , then  $a \notin \text{left}(P_1)$ , in both cases  $(T_1, P_1)$  is not an s-net. Thus, the assumption  $P_2 = P - P_1 \neq \emptyset$  leads to the discrepancy.

*Case 2.* Here  $T_1 \subsetneq T$ ,  $P_1 \subsetneq P$ .

Let  $P_2 = P - P_1$ ,  $T_2 = T - T_1$ . The s-net  $(T, P)$  is connected because it is elementary. But this means that there is  $a \in T_2$  such that  $(\cdot a \cup a \cdot) \cap P_1 \neq \emptyset$ . In other words:  $\exists a \in T_2 \exists p \in P_1, p \in \cdot a \cup a \cdot$ .

Suppose that  $p \in \cdot a$ . Since  $N$  is elementary, it is equivalent to  $\cdot a = \{p\}$ .

But  $\cdot a = \{p\}$  implies  $a \in \text{right}(p) \subseteq \text{right}(P_1)$ . Thus we have  $a \in \text{right}(P_1)$  &  $a \notin T_1$ , so  $T_1 \neq \text{right}(P_1)$ . Similarly, the assumption  $p \in a \cdot$  implies  $T_1 \neq \text{left}(P_1)$ . But this means that the pair  $(T_1, P_1)$  is not an s-net—in spite of the assumption.  $\square$

It turns out that not every atom is an elementary s-net. For example the s-net shown in Fig. 6 is an atom, but is not an elementary s-net.

Quasielementary s-nets are characterised by the following theorem.

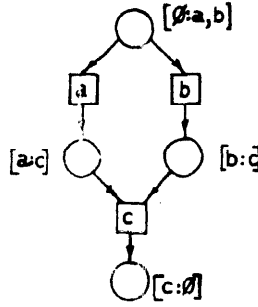


Fig. 6.

**Theorem 3.3.** *Let  $N$  be a quasialementary s-net. Then*

- (1)  $N_1 \subseteq N \Rightarrow N_1$  *quasialementary.*
- (2) *Every connected component of  $N$  is an elementary s-net.*

**Proof.** The proof of the theorem follows directly from the definition.  $\square$

For every s-net  $N$ , let  $\text{elem}(N)$  denote the set of all elementary nets included in  $N$ , i.e.,

$$\text{elem}(N) = \{N' \mid N' \subseteq N \text{ \& } N' \text{ is elementary}\}.$$

Of course,  $\text{elem}(N) \subseteq \text{atoms}(N)$ , and generally this inclusion is a proper one.

The most important class of s-nets is the class of nets decomposable into sequential state machines. These nets represent concurrent systems built by superposition of sequential subsystems, and they are called *proper* in our approach.

An s-net  $N$  is said to be *proper* iff

$$N = \bigcup_{N' \in \text{elem}(N)} N'.$$

Note that every proper net is atomic but not vice versa.

**Example 3.4.** Consider the three s-nets  $N$ ,  $N_1$ ,  $N_2$ , shown in Fig. 7.

Note that  $\text{elem}(N) = \{N_1, N_2\}$ ,  $N = N_1 \cup N_2$ , so  $N$  is proper. Now consider the next three s-nets:  $N'$ ,  $N'_1$ ,  $N'_2$ , shown in Fig. 8.

In this case:  $\text{elem}(N') = \{N'_2\} \subsetneq \text{atoms}(N') = \{N'_1, N'_2\}$ ,  $N' = N'_1 \cup N'_2$ , so  $N'$  is not proper, although it is atomic.

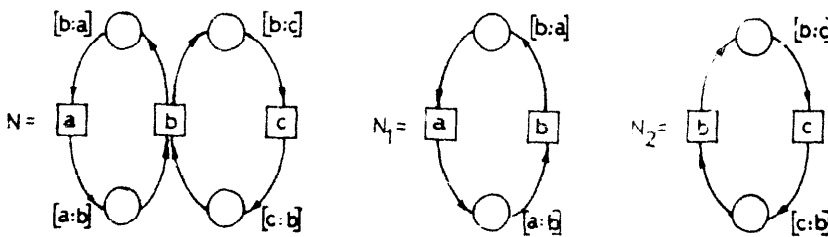


Fig. 7.



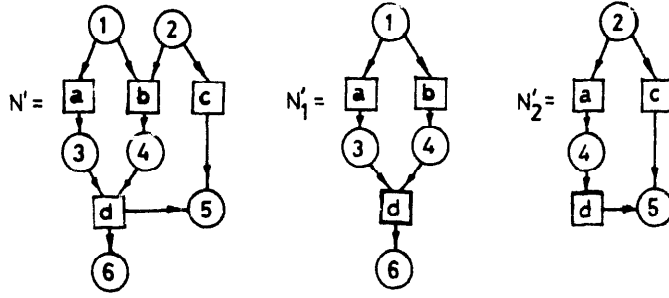


Fig. 8.

**Corollary 3.5.** *An s-net  $N$  is proper iff there is a set  $\{N_1, \dots, N_m\}$  of elementary nets and*

$$N = N_1 \cup \dots \cup N_m.$$

One may prove that the family of all proper nets is not closed under the operation  $\cap$ .

#### 4. Marked s-nets

We are now going to extend the present approach to marked nets. As was mentioned above, unmarked nets represent the static aspects of dynamic systems, while marked nets represent the dynamic aspects of these systems.

Let  $N = (T, P)$  be an s-net.

Let  $R1 \subseteq 2^P \times 2^P$  be the following relation:

$$(M_1, M_2) \in R1 \Leftrightarrow \exists a \in T, M_1 - \cdot a = M_2 - \cdot a \text{ \& \> } \cdot a \subseteq M_1 \text{ \& \> } \cdot a \subseteq M_2.$$

The relation  $R$  is called the *forward reachability in one step*. It can easily be extended to the *forward concurrent reachability in one step* CR1, namely let  $CR1 \subseteq 2^P \times 2^P$  be the relation defined as follows:

$$(M_1, M_2) \in CR1 \Leftrightarrow \exists A \subseteq T, M_1 - \cdot A = M_2 - \cdot A \text{ \& \> } A \subseteq M_1 \text{ \& \> } \cdot A \subseteq M_2.$$

Directly from the definition we have the following lemma.

**Lemma 4.1.**  $(R1 \cup R1^{-1})^* = (CR1 \cup CR1^{-1})^*.$

Let us define  $R = (R1 \cup R1^{-1})^*.$

The relation  $R$  is called the *forward and backward reachability* of  $N$ . If the net  $N$  is not fixed we shall write  $R_N, R1_N$  or  $CR1_N$ , respectively.

In fact we are interested in properties of  $R$ , and the representation of  $R$  in the form  $R = (R1 \cup R1^{-1})^*$  is more convenient for proofs than the representation by CR1. On the other hand, because we admit the possibility of concurrent execution, we have to define the relation CR1.

Note that  $R$  is an *equivalence* relation. For every  $M \in 2^P$ , let  $[M]_R$  denote the equivalence class of  $R$  containing  $M$ .

By a *marked simple net* (abbr. ms-net) we mean any triple

$$MN = (T, P, \text{Mar}),$$

where  $N = (T, P)$  is an s-net,  $\text{Mar} \subseteq 2^P$  is a set of *markings* of  $MN$ , and

$$\text{Mar} = \bigcup_{M \in \text{Mar}} [M]_{R_N}.$$

An ms-net  $MN = (T, P, \text{Mar})$  is called *compact* iff

$$\forall M \in \text{Mar}, \quad \text{Mar} = [M]_{R_N}.$$

In other words, an ms-net is compact if its marking class is the equivalence class of reachability relation. Most authors dealing with nets, restrict their attention to compact nets. Petri [21] has assumed that his Condition-Event-System is compact in the sense defined above. Equivalence classes of  $R_N$  may be interpreted as dynamic realisations of a system. An ms-net is compact if a system has only one dynamic realisation.

A transition  $a \in T$  is called *fireable* iff

$$\exists M_1, M_2 \in \text{Mar}, \quad a \subseteq M_1 \text{ \& \> } a' \subseteq M_2.$$

An ms-net is said to be *locally fireable* if all its transitions are fireable, and it is *fireable* if it is compact and locally fireable.

An ms-net  $MN = (T, P, \text{Mar})$  is said to be *safe* iff  $\forall A \subseteq P \quad \forall a \in T$ ,

$$\begin{aligned} & (a \cap A = \emptyset \text{ \& \> } \exists M \in \text{Mar}, a \cup A \subseteq M) \\ & \Leftrightarrow (a' \cap A = \emptyset \text{ \& \> } \exists M' \in \text{Mar}, a' \cup A \subseteq M'). \end{aligned}$$

Safeness is usually defined differently, starting with the concepts of so-called token capacity of places, and a little different definition of reachability relation. Usually, a marked Petri net is said to be  $k$ -safe if it never has more than  $k$  tokens in a place in any marking reachable from its initial marking (see [3]).

The definition given above follows from [17] and it is equivalent with bilateral 1-safeness.

It turns out that in the case of marked elementary nets, i.e., such ms-nets  $(T, P, \text{Mar})$ , where  $(T, P)$  is an elementary s-net, safeness implies a very regular marking class.

**Lemma 4.2.** *Let  $MN = (T, P, \text{Mar})$  be an ms-net, and let  $N = (T, P)$  be an elementary net. Then:*

$$MN \text{ is safe} \Leftrightarrow \text{Mar} = \{\{p\} \mid p \in P\} \Rightarrow MN \text{ is compact and fireable.}$$

**Proof.** Trivial.  $\square$

In the general case, properties of marked nets are more complicated, and in order to express them we will use the concept of concurrency relation.

**Example 4.3.** Consider the ms-net  $MN = (T, P, \text{Mar})$  (see Fig. 9), where  $N = (T, P)$ ,  $\text{Mar} = \{\{1, 2\}, \{4, 5\}, \{6\}, \{3\}\}$ .

This ms-net is compact, safe, but the transition  $b$  is not fireable, so it is not locally fireable.

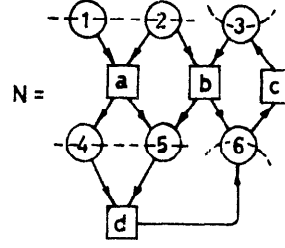


Fig. 9.

**Example 4.4.** Consider the ms-net  $MN = (T, P, \text{Mar})$  (see Fig. 10), where  $N = (T, P)$ ,  $\text{Mar} = \{\{1\}, \{2, 3\}, \{1, 2\}\}$ .

This ms-net is compact, fireable, but it is not safe.

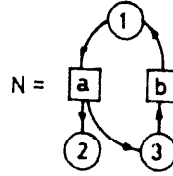


Fig. 10.

**Example 4.5.** Consider the ms-net  $MN = (T, P, \text{Mar})$  (see Fig. 11), where  $N = (T, P)$ ,  $\text{Mar} = \{\{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}\}$ .

This ms-net is safe, locally fireable, but it is not compact because  $(\{1, 3\}, \{1, 4\}) \notin R_N$ . In this case  $\text{Mar}$  consists of two equivalence classes of  $R_N: \{\{1, 3\}, \{2, 4\}\}$  and  $\{\{1, 4\}, \{2, 3\}\}$ .

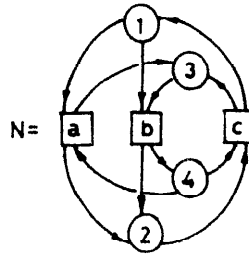


Fig. 11.

**Example 4.6.** Consider the following triple  $(T, P, \text{Mar})$  (see Fig. 12), where  $N = (T, P)$ ,  $\text{Mar} = \{\{1, 2\}, \{3, 4\}\}$ .

This triple is not an ms-net because  $\text{Mar}$  is not a set-theoretic union of equivalence classes of  $R_N$ .

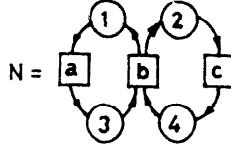


Fig. 12.

**Example 4.7.** Consider the triple  $(T, P, \text{Mar})$ , where  $N = (T, P)$  as in Example 4.6, but  $\text{Mar} = \{\{1, 2\}, \{3, 4\}, \{2, 3\}, \{1, 4\}\}$ .

This triple is a safe, compact and fireable ms-net.

## 5. Concurrency relation and sequential components

The concept of concurrency relation originates from Petri [19], who has shown that a sufficiently comprehensive theory of parallel processes can be established on the basis of that relation. When we deal with concurrent processes, i.e., with partially ordered sets of event occurrences, the concurrency relation can be defined as the complement of the partial order relation. Then many properties of one follows from the well-known theory of partial orders. In particular, the most interesting results of Best [1] and Petri [19, 20] follow from that fact.

A generalisation of the concurrency relation concept was considered in [11]. Unfortunately, [11] contains errors.

In this section we recall and modify for our purposes some results from [11].

Our approach is based on the notion of a symmetric and irreflexive relation defined by a fixed covering of a set. Elements of the covering will represent sequential components of a system.

Let  $X$  be a set, and let  $\text{id} \subseteq X \times X$  be the identity relation on  $X$ .

A relation  $C \subseteq X \times X$  is said to be a *sir-relation* (from symmetric and irreflexive) iff

$$\forall a, b \in X, (a, b) \in C \Leftrightarrow (b, a) \in C \Rightarrow a \neq b.$$

Let  $C$  be a sir-relation, and let  $\text{kens}(C)$ ,  $\overline{\text{kens}}(C)$  be the following families of subsets of  $X$ :

$$\text{kens}(C) = \{A \mid \forall a, b \in A, (a, b) \in C \cup \text{id} \ \& \ \forall c \notin A \ \exists a \in A, (a, c) \notin C\},$$

$$\overline{\text{kens}}(C) = \{A \mid \forall a, b \in A, (a, b) \notin C \ \& \ \forall c \notin A \ \exists a \in A, (a, c) \in C\}.$$

It should be pointed out that  $\text{kens}(C)$ ,  $\overline{\text{kens}}(C)$  are *coverings* of  $X$ . It is obvious when the set  $X$  is finite, in general it follows from the well-known Kuratowski–Zorn Lemma.

From the viewpoint of graph theory, the set  $\text{kens}(C)$  is the set of all cliques of the indirect graph representing  $C$ , while the set  $\overline{\text{kens}}(C)$  is the set of all cliques of the indirect graph representing  $X \times X - C$ .

Note that  $C \cup \text{id}$  is an equivalence relation if and only if  $\text{kens}(C)$  is a partition of  $X$ . Then  $\text{kens}(C) = X / (C \cup \text{id})$ . Similarly,  $X \times X - C$  is an equivalence relation iff  $\overline{\text{kens}}(C)$  is a partition of  $X$ , and then  $\overline{\text{kens}}(C) = X / (X \times X - C)$ . Therefore sir-relations can be treated as a kind of generalisation of equivalence relations. Every equivalence relation describes the partition of a set, while every sir-relation describes the family of set coverings.

We are now going to show how coverings can define sir-relations.

Let  $\text{cov}$  be a covering of  $X$ .

Let  $\text{sir}(\text{cov}) \subseteq X \times X$  be the relation defined as follows:

$$\forall a, b \in X \quad (a, b) \in \text{sir}(\text{cov}) \Leftrightarrow a \neq b \ \& \ \forall A \in \text{cov}, a \notin A \text{ or } b \notin A.$$

**Corollary 5.1.** *If  $\text{cov}$  is a partition of  $X$ , then  $X \times X - \text{sir}(\text{cov})$  is an equivalence relation and  $\text{cov} = \overline{\text{kens}}(\text{sir}(\text{cov}))$ .*

**Corollary 5.2.** *For every covering  $\text{cov}$  of  $X$ :*

$$\forall A \in \text{cov} \exists B \in \overline{\text{kens}}(\text{sir}(\text{cov})), \quad A \subseteq B.$$

In our approach, a covering  $\text{cov}$  will represent an arbitrary set of sequential system components, and the relation  $\text{sir}(\text{cov})$  will represent the concurrency relation defined by this set of components.

Let  $\text{Mar} \subseteq 2^X$  be a family of subsets of  $X$  satisfying the following properties:

- (1)  $\text{Mar} \subseteq \text{kens}(\text{sir}(\text{cov}))$ ,
- (2)  $\text{Mar}$  is a covering of  $X$ .

The family  $\text{Mar}$  will represent the set of global system states (marking class).

Let us put  $\text{DC} = (\text{cov}, \text{Mar})$ . The pair  $\text{DC}$ , called *double covering* of  $X$ , represents the most general information about a system; it describes system sequential components but without information about control flow inside each of the components, and it describes global system states, also without details about communication among them.

Summing up, we have the following interpretations:

- $\text{sir}(\text{cov})$ : the concurrency relation,
- $\text{cov}$ : the set of sequential system components,
- $\text{Mar}$ : the set of global system states,

- $\overline{\text{kens}}(\text{sir}(\text{cov}))$ : the family of all maximal locally dependent sets,  
 where by a *locally dependent set* we mean any set  
 $A \subseteq X$ , such that  $\forall a, b \in A, (a, b) \notin \text{sir}(\text{cov})$ ,
- $\text{kens}(\text{sir}(\text{cov}))$ : the family of all maximal locally concurrent sets,  
 where by a *locally concurrent set* we mean any set  
 $A \subseteq X$ , such that  $\forall a, b \in A, (a, b) \in \text{sir}(\text{cov})$ .

The family  $\overline{\text{kens}}(\text{sir}(\text{cov}))$  is only a set of sequential system components if  $\text{cov} = \overline{\text{kens}}(\text{sir}(\text{cov}))$ , and the family  $\text{kens}(\text{sir}(\text{cov}))$  is only a set of global system states if  $\text{Mar} = \text{kens}(\text{sir}(\text{cov}))$ . That is the main difference between Petri's approach and ours. Petri assumes that  $\overline{\text{kens}}(C)$ , and  $\text{kens}(C)$  represent sequential components and global states respectively. This assumption is only valid on the process level, and it is usually false on the system level.

A sir-relation  $\text{sir}(\text{cov})$  is said to be *consistent* iff

$$\text{cov} = \text{kens}(\text{sir}(\text{cov})).$$

A sir-relation  $\text{sir}(\text{cov})$  is said to be *semiconsistent* iff

$$\text{cov} \subseteq \overline{\text{kens}}(\text{sir}(\text{cov})).$$

The consistency property means that the concurrency relation describes precisely the set of sequential components, while the semi-consistency property means only that every sequential component is defined by the concurrency relation.

In fact, the above properties are rather properties of the covering  $\text{cov}$  than the relation  $\text{sir}(\text{cov})$ , because many coverings can define the same relation.

Nevertheless, in further considerations the covering will usually be fixed, whereas speaking about consistency and semiconsistency as the relation properties enable us more uniform considerations. The same remark concerns notions of KM-, CM- and C-density introduced below.

Considering nets of occurrences, Petri [19] has postulated that for every real process the following condition is fulfilled: every sequential component and every 'case' (global state) have one element in common. This is a generalisation of the well-known postulate of physics that every time sequence and every space must have one common element, or, equivalently: there is no space outside the time. Petri has called this property by *K-density*.

Although *K-density* was formally defined as a property of the concurrency relation (see [19, 20]), in reality, as it was justly noticed by Best [1], it is a property of occurrence net.

*K-density* is formally defined as follows: a sir-relation  $\text{C} \subseteq X \times X$  is said to be *K-dense* iff

$$\forall A \in \text{kens}(C) \forall B \in \overline{\text{kens}}(C), A \cap B \neq \emptyset.$$

In the case of occurrence nets, the notion of *K-density* is consistent with its interpretation [1, 19, 20] but in our approach it has a good interpretation only if

$\text{cov} = \overline{\text{kens}}(\text{sir}(\text{cov}))$  and  $\text{Mar} = \text{kens}(\text{sir}(\text{cov}))$ . Therefore we have to replace it by more adequate notions.

Let  $\text{DC} = (\text{cov}, \text{Mar})$  be an arbitrary double covering of  $X$ .

A sir-relation  $\text{sir}(\text{cov}) \subseteq X \times X$  is said to be *KM-dense* iff

$$\forall A \in \overline{\text{kens}}(\text{sir}(\text{cov})) \forall B \in \text{Mar}, \quad A \cap B \neq \emptyset.$$

A sir-relation  $\text{sir}(\text{cov}) \subseteq X \times X$  is said to be *CM-dense* iff

$$\forall A \in \text{cov} \forall B \in \text{Mar}, \quad A \cap B \neq \emptyset.$$

A sir-relation  $\text{sir}(\text{cov}) \subseteq X \times X$  is said to be *C-dense* iff

$$\forall A \in \text{cov} \forall B \in \text{kens}(\text{sir}(\text{cov})), \quad A \cap B \neq \emptyset.$$

### Corollary 5.3

- (1)  $\text{Mar} = \text{kens}(\text{sir}(\text{cov})) \Rightarrow (\text{KM-density} \Leftrightarrow \text{K-density} \& \text{CM-density} \Leftrightarrow \text{C-density}).$
- (2)  $\text{cov} = \overline{\text{kens}}(\text{sir}(\text{cov})) \Rightarrow (\text{K-density} \Leftrightarrow \text{C-density} \& \text{KM-density} \Leftrightarrow \text{CM-density}).$
- (3)  $\text{cov} \subseteq \overline{\text{kens}}(\text{sir}(\text{cov})) \Rightarrow (\text{K-density} \Rightarrow \text{C-density} \& \text{KM-density} \Rightarrow \text{CM-density}).$

The property of CM-density describes Petri's postulate on a common element for every sequential subsystem and every global system state. KM-density means that the above property concerns not only real sequential system components but all locally dependent sets as well.

The following two theorems characterise the notions considered above.

**Theorem 5.4.** *Let  $X$  be a set, and let  $\text{DC} = (\text{cov}, \text{Mar})$  be a double covering of  $X$ . Then:*

$$\text{sir}(\text{cov}) \text{ is CM-dense} \Rightarrow \text{cov} \subseteq \overline{\text{kens}}(\text{sir}(\text{cov})).$$

**Proof.** Assume that  $\text{cov} - \overline{\text{kens}}(\text{sir}(\text{cov})) \neq \emptyset$ .

Let  $A \in \text{cov}$  and  $A \notin \overline{\text{kens}}(\text{sir}(\text{cov}))$ . From Corollary 5.2 we have  $\exists B \in \overline{\text{kens}}(\text{sir}(\text{cov})), A \subseteq B$ .

Let  $p \in B - A$ . Since  $\text{Mar}$  is a covering of  $X$ , there is a  $D \in \text{Mar}$  such that  $p \in D$ .

Since  $\text{sir}(\text{cov})$  is CM-dense,  $\exists q \in D \cap A$ . But  $D \cap A \subseteq D \cap B$ , so  $q \in D \cap B$ . Thus we have  $\{p, q\} \subseteq D \cap B$ , so by definitions of  $\text{kens}$  and  $\overline{\text{kens}}$ :  $p = q$ . But on the other hand,  $p \in B - A$ ,  $q \in A$ , then  $p \neq q$ .  $\square$

A covering  $\text{cov}$  of  $X$  is called *minimal* iff  $\forall A \in \text{cov}, \text{cov} - \{A\}$  is not a covering of  $X$ .

**Theorem 5.5.** *Let  $C \subseteq X \times X$  be a sir-relation. If  $\overline{\text{kens}}(C)$  is a minimal covering, then  $C$  is  $K$ -dense.*

**Proof.** Assume that  $A \in \overline{\text{kens}}(C)$ ,  $B \in \text{kens}(C)$  and  $A \cap B = \emptyset$ . From the definition of  $\text{kens}(C)$  it follows that  $\forall a \in A \exists b \in B, (a, b) \notin C$ .

This means that

$$\forall a \in A \exists b \in B \exists Q_{ab} \in \overline{\text{kens}}(R), \{a, b\} \subseteq Q_{ab}.$$

Obviously, for every pair  $\{a, b\}$  the set  $Q_{ab}$  differs from  $A$  (because  $A \cap B = \emptyset$  by the assumption).

Since  $\forall a \in A, a \in Q_{ab}$ , then  $A \subseteq \bigcup_{a \in A} Q_{ab}$ . Thus

$$\bigcup_{Q \in \overline{\text{kens}}(C) \setminus \{A\}} Q = \bigcup_{Q \in \overline{\text{kens}}(C)} Q = X,$$

in spite of the assumption that  $\overline{\text{kens}}(C)$  is a minimal covering.  $\square$

Detailed analysis of  $K$ -density and  $C$ -density properties can be found in [22].

## 6. Seminaturally marked s-nets

In this section we shall deal with the relationship between a static net structure (i.e., the pair  $(T, P)$ ) and properties of the marking class (i.e., the set  $\text{Mar}$ ). We restrict our attention to proper s-nets only.

Let  $N = (T, P)$  be an arbitrary proper s-net, and let  $C = \{N_1, \dots, N_m\} \subseteq \text{elem}(N)$  be a set of elementary nets, such that  $N = N_1 \cup \dots \cup N_m$ . Assume that  $N_i = (T_i, P_i)$  for  $i = 1, 2, \dots, m$ .

Every set  $C$  of the above form is said to be an *elementary covering* of  $N$  (abbr. *e-covering*).

Let us define:  $\text{cov}_C = \{P_1, \dots, P_m\}$ . Note that  $\text{cov}_C$  is a covering of  $P$ .

Let  $\text{coex}_C \subseteq P \times P$  be the following relation:

$$\text{coex}_C = \text{sir}(\text{cov}_C).$$

In other words:

$$(p, q) \in \text{coex}_C \Leftrightarrow p \neq q \ \& \ \forall P_i \in \text{cov}_C, p \notin P_i \text{ or } q \notin P_i.$$

The relation  $\text{coex}_C$  is said to be the *coexistence defined by the e-covering  $C$* .

An ms-net  $\text{MN} = (T, P, \text{Mar})$  is said to be *seminaturally marked* with respect to an e-covering  $C$  iff:

- (1)  $C$  is an e-covering of  $N = (T, P)$ .
- (2)  $\text{Mar} \subseteq \text{kens}(\text{coex}_C)$ .
- (3)  $\text{MN}$  is locally fireable.



**Corollary 6.1.** *Let  $MN = (T, P, \text{Mar})$  be seminaturally marked with respect to  $C$ . Then the pair  $(\text{cov}_C, \text{Mar})$  is a double covering of  $P$ .*

Let  $MN = (T, P, \text{Mar})$  be a fixed seminaturally marked s-net with respect to an e-covering  $C$ . Let also  $N = (T, P)$ .

**Lemma 6.2**

$$\forall a \in T \quad \forall P_i \in \text{cov}_C, \quad P_i \cap \cdot a \neq \emptyset \Leftrightarrow P_i \cap a' \neq \emptyset.$$

**Proof.** Assume that  $s \in P_i \cap \cdot a$  and  $P_i \cap a' = \emptyset$ . This means that  $a \in \text{right}(s) \subseteq \text{right}(P_i)$  and  $a \notin \text{left}(P_i)$ —in spite of the assumptions that  $(T_i, P_i)$  is an elementary net.  $\square$

**Theorem 6.3.**  *$MN$  is safe.*

**Proof.** Let  $a \in T$ ,  $A \in 2^P$ ,  $\cdot a \cap A = \emptyset$  &  $\cdot a \cup A \subseteq M \in \text{Mar} \subseteq \text{kens}(\text{coex}_C)$ . First we prove that  $(M - a) \cap a' = \emptyset$ .

Assume that  $s \in M - \cdot a$  &  $s \in a'$ . Assume also that  $s \in P_i$ .

By Lemma 6.2 we obtain  $\exists t \in P_i \cap \cdot a$ . Since  $\cdot a \subseteq M$ , we have  $t \in M$ . But this means that  $\{s, t\} \in M \cap P_i$ .

Since  $M \in \text{kens}(\text{coex}_C)$ , we have  $|M \cap P_i| \leq 1$ , i.e.,  $s = t$ . On the other hand: if  $t \in \cdot a$ ,  $s \in M - \cdot a$ , then  $s \neq t$ . Thus, the assumption that  $(M - a) \cap a' = \emptyset$  leads to a discrepancy.

Hence  $(M - \cdot a) \cap a' = \emptyset$ .

Define  $M' = (M - \cdot a) \cup a'$ . Since  $(M - \cdot a) \cap a' = \emptyset$ , we have  $(M, M') \in R1$  and  $M' \in [M]_R$ . Of course,  $M \in \text{Mar}$ ; then by the definition of  $\text{Mar}$  we obtain  $[M]_R \subseteq \text{Mar}$ , thus  $M' \in \text{Mar}$ .

Since  $\cdot a \cap A = \emptyset$  &  $\cdot a \cup A \subseteq M$ , we have  $A \subseteq M - \cdot a = M' - a'$ . Thus  $A \cap a' = \emptyset$  &  $a' \cup A \subseteq M' \in \text{Mar}$ .

In this way we prove that

$$\begin{aligned} \forall a \in T \quad \forall A \subseteq P, \\ (\cdot a \cap A = \emptyset \quad \& \quad \exists M \in \text{Mar}, \cdot a \cup A \subseteq M) \Rightarrow \\ \Rightarrow (a' \cap A = \emptyset \quad \& \quad \exists M' \in \text{Mar}, a' \cup A \subseteq M') \end{aligned}$$

The implication  $\Leftarrow$  can be proved similarly.  $\square$

Note that notions KM-density and CM-density can be expressed in the terms used in this section only, namely, the relation  $\text{coex}_C$  is said to be KM-dense iff  $\forall A \in \text{Mar} \quad \forall B \in \overline{\text{kens}(\text{coex}_C)}, \quad A \cap B \neq \emptyset$ , and  $\text{coex}_C$  is said to be CM-dense iff  $\forall A \in \text{Mar} \quad \forall B \in \text{cov}_C, \quad A \cap B \neq \emptyset$ .

As a consequence of Theorem 5.4 we have the following.

**Corollary 6.4**

$\text{coex}_C$  is CM-dense  $\Rightarrow \text{coex}_C \subseteq \overline{\text{kens}(\text{coex}_C)}$ .

Thus CM-density of  $\text{coex}_C$  implies its semiconsistency.

We are now going to formulate the main theorem characterising the strength of KM-density.

**Theorem 6.5.** *If  $\text{coex}_C$  is KM-dense, then for every set  $A \in \overline{\text{kens}(\text{coex}_C)}$  the pair  $N_A = (\cdot A, A)$  is a quasialementary s-net.*

**Proof.** Note that  $\cdot A = \text{left}(A)$ . First we prove that  $(\cdot A, A)$  is an s-net. Let  $a \in \text{left}(A)$ . This means that  $\exists p_a \in A, a \in \cdot p_a$ . Because MN is locally fireable  $\exists M \in \text{Mar}, \cdot a \subseteq M$ . Since  $\text{coex}_C$  is KM-dense,  $M \cap A \neq \emptyset$ .

From Theorem 6.3 we have that  $\text{MN} = (T, P, \text{Mar})$  is safe.

Define  $M' = (M - \cdot a) \cup a$ . Since MN is safe,  $(M, M') \in R1$ , and consequently  $M' \in [M] \subseteq \text{Mar}$ . Note that  $M' \cap A = \{p_a\}$ . Thus  $\{p_a\} = a \cap M' \cap A = M' \cap A = a \cap A$ . But this means that  $(M' - \cdot a) \cap A = \emptyset$ .

Because  $(M, M') \in R1$ , we have  $M' - \cdot a = M - \cdot a$ ; thus we can write  $(M - \cdot a) \cap A = \emptyset$ . The relation  $\text{coex}_C$  is KM-dense, so  $M \cap A \neq \emptyset$ . Let  $p \in M \cap A$ . From the facts  $(M - \cdot a) \cap A = \emptyset$  and  $p \in M \cap A$ , it follows that  $p \in \cdot a$ . But  $p \in \cdot a \Leftrightarrow a \in \text{right}(p)$ . Of course,  $\text{right}(p) \subseteq \text{right}(A)$ , so  $a \in \text{right}(A)$ . Hence  $\text{left}(A) \subseteq \text{right}(A)$ .

In a similar way we can prove that  $\text{right}(A) \subseteq \text{left}(A)$ . Thus the pair  $(T_A, A)$ , where  $T_A = \text{left}(A) = \text{right}(A) = \cdot A = A$ , is an s-net.

Now we prove that  $N_A = (T_A, A)$  is quasialementary. Let  $a \in T_A$ . We want to prove that  $|(\cdot a)_{N_A}| = |\cdot a \cap A| = 1$ . Of course  $\cdot a \cap A \subseteq \cdot a$ .

Let  $M_a \in \text{Mar}$  be such marking that  $\cdot a \subseteq M_a$ . We have:  $\cdot a \cap A \subseteq \cdot a \subseteq M_a \in \text{Mar}$ .

Assume that  $\{p, q\} \subseteq \cdot a \cap A$  and  $p \neq q$ . This means that  $\{p, q\} \subseteq M_a \in \text{Mar} \subseteq \text{kens}(\text{coex}_C)$ , so  $(p, q) \in \text{coex}_C$ . But  $(p, q) \in \text{coex}_C \Rightarrow \forall A' \in \overline{\text{kens}(\text{coex}_C)}, p \notin A'$  or  $q \notin A' \Rightarrow p \notin \cdot a \cap A$  or  $q \notin \cdot a \cap A$ —a discrepancy. Thus  $\forall a \in T_A, |\cdot a \cap A| = 1$ .

In a similar way we can prove that  $\forall a \in T_A, |a \cap A| = 1$ .  $\square$

Of course, by the construction we have that every element for  $\text{cov}_C$  describes an elementary s-net, but we do not know anything about elements of  $\overline{\text{kens}(\text{coex}_C)}$ .

Note that, in general, we do not assume the property  $\text{coex}_C \subseteq \overline{\text{kens}(\text{coex}_C)}$ . Theorem 6.5 states that if  $\text{coex}_C$  is KM-dense, then every element of  $\overline{\text{kens}(\text{coex}_C)}$ , i.e., every maximal locally dependent set, creates a *sequential finite state machine* (not necessarily connected).

We consider now two examples illustrating ideas and results formulated above, and proving that the reciprocal of Theorem 6.5 is not true as well as that the word 'quasialementary' cannot be replaced by 'elementary'.

Let  $\overline{\text{coex}_C} = (P \times P - \text{coex}_C) - \text{id}_P$ . Of course  $\overline{\text{kens}(\text{coex}_C)} = \overline{\text{kens}(\overline{\text{coex}_C})}$ .

**Example 6.6.** Let  $N = (T, P)$ ,  $N_i = (T_i, P_i)$  for  $i = 1, \dots, 5$  be the s-nets, given in Fig. 13. Note that  $N = N_1 \cup N_2 \cup \dots \cup N_5$ , and  $\text{elem}(N) = \{N_1, \dots, N_5\}$ . Let  $C = \{N_1, \dots, N_5\}$ . The net family  $C$  is obviously an e-covering of  $N$ . Graphs of  $\text{coex}_C$  and  $\overline{\text{coex}}_C$  are shown in Fig. 14.

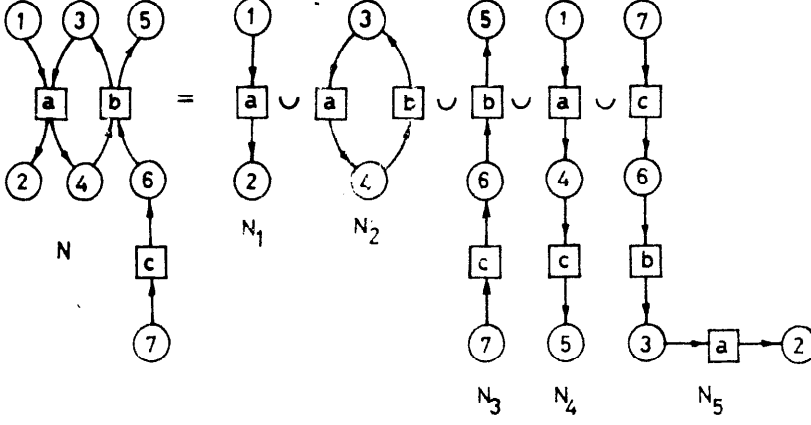


Fig. 13.



Fig. 14.

Here we have

$$\text{kens}(\text{coex}_C) = \{\{1, 3\}, \{1, 6\}, \{1, 7\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{4, 6\}, \{4, 7\}\},$$

$$\overline{\text{kens}}(\text{coex}_C) = \{\{1, 2\}, \{3, 4\}, \{1, 4, 5\}, \{5, 6, 7\}, \{2, 3, 4, 7\}\},$$

$$\text{cov}_C = \overline{\text{kens}}(\text{coex}_C).$$

Let  $\text{Mar} = \{\{1, 3\}, \{2, 4\}, \{3, 5\}, \{4, 6\}, \{4, 7\}\} \subseteq \text{kens}(\text{coex}_C)$ . Note that the triple  $\text{MN} = (T, P, \text{Mar})$  is a seminaturally marked s-net with respect to the set  $C = \{N_1, \dots, N_5\}$ . The ms-net  $\text{MN}$  is safe, locally fireable, but not compact because, for example,  $(\{1, 3\}, \{3, 5\}) \notin R_N$ . Since  $\text{cov}_C = \overline{\text{kens}}(\text{coex}_C)$ , KM-density is equivalent to CM-density. The relation  $\text{coex}_C$  is not CM-dense because, for example,  $\{1, 3\} \cap \{5, 6, 7\} = \emptyset$ . Note that all elements of  $\text{kens}(\text{coex}_C)$  define elementary s-nets, although  $\text{coex}_C$  is not KM-dense, thus *the reciprocal of Theorem 6.5 is false*.

**Example 6.7.** Let  $N = (T, P)$ ,  $N_i = (T_i, P_i)$  for  $i = 1, \dots, 6$  be the s-nets, given in Fig. 15. Note that  $N = N_1 \cup \dots \cup N_6$ .

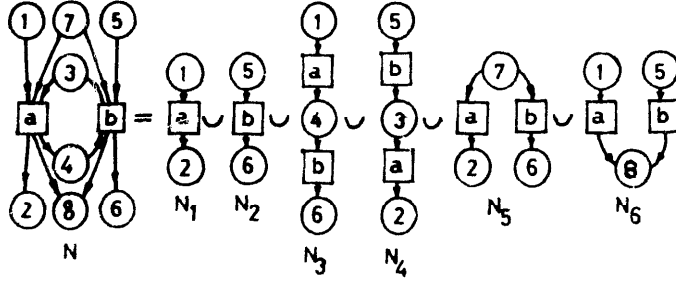


Fig. 15.

Let us put  $C = \{N_1, \dots, N_6\}$ . Of course  $C$  is an e-covering of  $N$ , and  $C \neq \text{elem}(N)$ . In this case  $\text{elem}(N) = C \cup \{N_7, N_8\}$ , where  $N_7 = (\{a, b\}, \{3, 4\})$ ,  $N_8 = (\{a, b\}, \{7, 8\})$ . Graphs of relations  $\text{coex}_C$  and  $\overline{\text{coex}}_C$  are shown in Fig. 16.

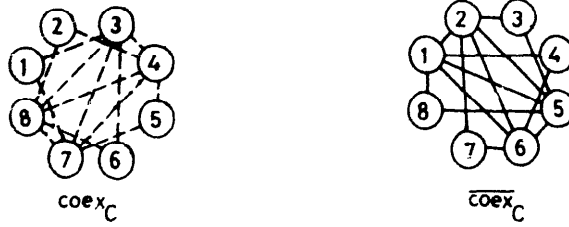


Fig. 16.

We have here

$$\text{kens}(\text{coex}_C) = \{\{1, 3, 7\}, \{2, 4, 8\}, \{3, 6, 8\}, \{4, 5, 7\}, \{3, 4, 7, 8\}\},$$

$$\overline{\text{kens}}(\text{coex}_C) = \{\{1, 2, 5, 6\}, \{2, 3, 5\}, \{1, 5, 8\}, \{2, 7, 6\}, \{1, 4, 6\}\}.$$

$$\text{cov}_C = \{\{1, 2\}, \{5, 6\}, \{2, 3, 5\}, \{1, 5, 8\}, \{2, 7, 6\}, \{1, 4, 6\}\}.$$

Thus, the relation  $\text{coex}_C$  is not semiconsistent.

Let us define:  $\text{Mar} = \{\{1, 3, 7\}, \{2, 4, 8\}, \{3, 6, 8\}, \{4, 5, 7\}\} \subseteq \text{kens}(\text{coex}_C)$ . Note that the triple  $\text{MN} = (T, P, \text{Mar})$  is a seminaturally marked s-net. This net is safe, locally fireable, but not compact, because, for instance,  $(\{1, 3, 7\}, \{3, 6, 7\}) \notin R_N$ .

The relation  $\text{coex}_C$  is not CM-dense, because  $\{1, 2\} \cap \{3, 6, 7\} = \emptyset$ , but it is KM-dense.

The statement  $\forall A \in \overline{\text{kens}}(\text{coex}_C), (A, A)$  is an elementary s-net is not true, because the set  $\{1, 2, 5, 6\}$  defines a disconnected quasialementary s-net. The statement  $\forall A \in \overline{\text{kens}}(\text{coex}_C), (A, A)$  is a quasialementary s-net is obviously true. Thus, in Theorem 6.5, the word 'quasialementary' *cannot be replaced* by 'elementary'.

Compactness is the property, which is frequently required from concurrent systems. Most alternate models of concurrent systems assume the property like compactness defined above. Among others, Petri's condition/event systems and nets equivalent to path expressions [16, 21] are compact.

Compact seminaturally marked s-nets are characterised by the following three theorems.

**Theorem 6.8**

$MN$  is compact  $\Rightarrow$   $\text{coex}_C$  is CM-dense.

**Proof.** For every  $A \in \text{Mar}$ , let  $\alpha_A = \{P_i \mid P_i \in \text{cov}_C \text{ \& } P_i \cap A = \emptyset\}$ . From Lemma 6.2 it follows that

$$(A, B) \in R1 \cup R1^{-1} \Rightarrow \alpha_A = \alpha_B.$$

Since  $MN$  is compact,  $\forall A \in \text{Mar}$ ,  $\text{Mar} = [A]_R$ . Thus  $\forall A, B \in \text{Mar}$ ,  $\alpha_A = \alpha_B$ .

Assume that  $\exists A \in \text{Mar} \exists P_i \in \text{cov}_C$ ,  $A \cap P_i = \emptyset$ . Of course this means that  $P_i \in \alpha_A$ .

But we have shown that  $\forall A, B \in \text{Mar}$ ,  $\alpha_A = \alpha_B$ , then  $\forall B \in \text{Mar}$ ,  $P_i \in \alpha_B$ .

Thus  $\forall B \in \text{Mar}$ ,  $P_i \cap B = \emptyset$ , or equivalently,  $P_i \cap \bigcup_{B \in \text{Mar}} B = \emptyset$ , or  $P_i \cap P = \emptyset$ .

But this is a discrepancy because  $P_i \in \text{cov}_C$ , then  $P_i \subseteq P$ . Thus  $\forall A \in \text{Mar} \forall P_i \in \text{cov}_C$ ,  $A \cap P_i \neq \emptyset$ .  $\square$

Theorem 6.8 states that if a seminaturally marked s-net is compact, then every sequential subsystem and every global system state have one element in common.

From Theorems 5.4 and 6.8 we immediately obtain the following.

**Corollary 6.9**

$MN$  is compact  $\Rightarrow \text{cov}_C \subseteq \overline{\text{kens}(\text{coex}_C)}$ .

This means that in the case of compact seminaturally marked s-nets, every sequential subsystem is described as a clique of the relation  $\text{coex}_C$ . The reciprocals of Theorem 6.8 and Corollary 6.9 are not true. To prove this fact let us consider the ms-net  $MN = (T, P, \text{Mar})$  from Example 4.5. This net is a seminaturally marked s-net with respect to the set  $C = \{N_1, N_2\}$ , where  $N_1 = (\{a, b, c\}, \{1, 2\})$ ,  $N_2 = (\{a, b, c\}, \{3, 4\})$ . In this case the relation  $\text{coex}_C$  is  $K$ -,  $C$ -,  $KM$ -,  $CM$ -dense and  $\text{cov}_C = \overline{\text{kens}(\text{coex}_C)}$ , but  $MN$  is not compact.

If we assume that  $MN$  is compact then the result of Theorem 6.5 can be strengthened.

**Theorem 6.10.** If  $MN$  is compact and  $\text{coex}_C$  is  $KM$ -dense, then for every set  $A \in \overline{\text{kens}(\text{coex}_C)}$  the pair  $N_A = (A, A)$  is an elementary s-net.

**Proof.** By Theorem 6.5 we have that  $N_A$  is quasialementary. Let  $p, q \in A$ . Since  $MN$  is fireable  $\exists M, M' \in \text{Mar}$ ,  $p \in M$ ,  $q \in M'$ . Since  $MN$  is also compact,  $(M, M') \in R_N = (R1_N \cup R1_N^{-1})^*$ . Let  $M_1, \dots, M_n$  be a sequence of markings satisfying the following conditions:

- (1)  $M_1 = M$ ,  $M_n = M'$ ,
- (2)  $(M_i, M_{i+1}) \in R1_N \cup R1_N^{-1}$  for  $i = 1, \dots, n-1$ .

Since  $\text{coex}_C$  is KM-dense,  $\forall i = 1, \dots, n, A \cap M_i \neq \emptyset$ . Let  $p_1, \dots, p_n$  be the following sequence of places:

$$\{p_i\} = A \cap M_i \quad \text{for } i = 1, \dots, n.$$

Note that  $\forall i = 1, \dots, n, (\{p_i\}, \{p_{i+1}\}) \in R1_{N_A} \cup R1_{N_A}^{-1}$ . But this means that  $(\{p\}, \{q\}) \in R_{N_A}$ , and, as a consequence, that  $N_A$  is connected.  $\square$

Of course, if  $MN$  is compact then  $\text{cov}_C \subseteq \overline{\text{kens}}(\text{coex}_C)$  (by Corollary 6.9) and every element of  $\text{cov}_C$  generates, by the definition, an elementary s-net. Theorem 6.10 states that elements of  $\overline{\text{kens}}(\text{coex}_C) - \text{cov}_C$  also generate elementary s-nets. Example 6.6 shows that the reciprocal is false. The ms-net from this example is neither compact nor KM-dense, but every element of  $\overline{\text{kens}}(\text{coex}_C)$  defines an elementary s-net.

It turns out that if  $C = \text{elem}(N)$ , then compactness implies the equivalence of KM-density and consistency.

**Theorem 6.11.** *Let  $C = \text{elem}(N)$ . Then*

$$MN \text{ is compact} \Rightarrow (\text{coex}_C \text{ is KM-dense} \Leftrightarrow \text{cov}_C = \overline{\text{kens}}(\text{coex}_C)).$$

**Proof.** Assume that  $MN$  is compact and  $\text{coex}_C$  is KM-dense. By Corollary 6.9 we have  $\text{cov}_C \subseteq \overline{\text{kens}}(\text{coex}_C)$ . By Theorem 6.10 we have

$$\overline{\text{kens}}(\text{coex}_C) \subseteq \bigcup_{C(A, A) \in \text{elem}(N)} A = \text{cov}_C.$$

Thus,

$$MN \text{ is compact} \& \text{ coex}_C \text{ is KM-dense} \Rightarrow \text{cov}_C = \overline{\text{kens}}(\text{coex}_C).$$

Assume that  $MN$  is compact &  $\text{cov}_C = \overline{\text{kens}}(\text{coex}_C)$ . By Theorem 6.8 and Corollary 5.3 we have that  $\text{coex}_C$  is KM-dense.  $\square$

Seminaturally marked s-nets seem to be a very interesting class of marked s-nets. On the one hand this class is wide (for instance, it contains nets generated by GE\*-path [16]), on the other hand, it has very convenient regular properties. These nets are composed from finite state machines and their marking classes are *strictly connected* with this composition. Furthermore, in Section 9 we show that if any compact ms-net satisfies that mentioned in the previous section, Petri's postulate about a common element, then this net can be treated as seminaturally marked.

Ending this section we consider below two examples characterising the approach presented above.

**Example 6.12.** Let  $N = (T, P)$ ,  $N_i = (T_i, P_i)$  for  $i = 1, \dots, 5$  be the same nets as in the case of Example 6.6.

Let us put  $C = \{N_1, N_2, N_3, N_4\} \subseteq \text{elem}(N) = C \cup \{N_5\}$ . Of course,  $N = N_1 \cup \dots \cup N_4$ , so  $C$  is an e-covering of  $N$ . In this case the relations  $\text{coex}_C$  and  $\overline{\text{coex}}_C$  are of the form shown in Fig. 17. Thus,

$$\text{kens}(\text{coex}_C) = \{\{1, 3, 7\}, \{1, 3, 6\}, \{2, 4, 6\}, \{2, 4, 7\}, \{2, 3, 5\}, \{2, 3, 6\}, \{2, 3, 7\}\},$$

$$\overline{\text{kens}}(\text{coex}_C) = \{\{1, 2\}, \{3, 4\}, \{5, 6, 7\}, \{1, 4, 5\}\},$$

$$\text{cov}_C = \overline{\text{kens}}(\text{coex}_C).$$



Fig. 17.

Let us put

$$\text{Mar} = \{\{1, 3, 7\}, \{1, 3, 6\}, \{2, 4, 6\}, \{2, 4, 7\}, \{2, 3, 5\}\} \subseteq \text{kens}(\text{coex}_C).$$

Note that the triple  $\text{MN} = (T, P, \text{Mar})$  is a seminaturally marked s-net with respect to the set  $C = \{N_1, \dots, N_4\}$ . This ms-net is safe, compact, fireable, the relation  $\text{coex}_C$  is KM-dense and CM-dense.

**Example 6.13.** Let  $N = (T, P)$ ,  $N_i = (T_i, P_i)$  for  $i = 1, 2, 3$  be the s-nets, given in Fig. 18.

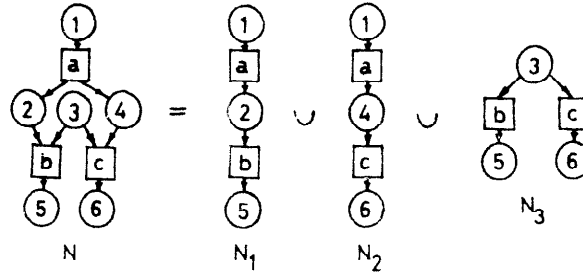


Fig. 18.

Note that  $N = N_1 \cup N_2 \cup N_3$ , and  $\text{elem}(N) = \{N_1, N_2, N_3\}$ . Let us put  $C = \{N_1, N_2, N_3\}$ . Of course  $C$  is the e-covering of  $N$ . Note that in this case there is only one e-covering of  $N$ . Graphs of  $\text{coex}_C$  and  $\overline{\text{coex}}_C$  are shown in Fig. 19. In this case we have

$$\text{kens}(\text{coex}_C) = \{\{1, 3\}, \{2, 3, 4\}, \{2, 6\}, \{4, 5\}\},$$

$$\overline{\text{kens}}(\text{coex}_C) = \{\{1, 2, 5\}, \{1, 4, 6\}, \{3, 5, 6\}, \{1, 5, 6\}\},$$

$$\text{cov}_C = \{\{1, 2, 5\}, \{1, 4, 6\}, \{3, 5, 6\}\},$$



Fig. 19.

thus

$$\text{cov}_C \subseteq \overline{\text{kens}(\text{coex}_C)} = \text{cov}_C \cup \{1, 5, 6\}.$$

Note that  $\text{MN} = (T, P, \text{Mar})$  is a seminaturally marked s-net with respect to the set  $C = \{N_1, N_2, N_3\}$ . The ms-net  $\text{MN}$  is safe, compact and fireable. The relation  $\text{coex}_C$  is CM-dense, but it is not KM-dense, because  $\{1, 5, 6\} \cap \{2, 3, 4\} = \emptyset$ . The statement  $\forall A \in \overline{\text{kens}(\text{coex}_C)}, (A, A)$  is a quasialementary s-net is not true, because the set  $\{1, 5, 6\} \in \overline{\text{kens}(\text{coex}_C)}$  does not define any s-net. Since  $\text{Mar} = \text{kens}(\text{coex}_C)$ , KM-density is equivalent with  $K$ -density and CM-density with  $C$ -density.

## 7. Naturally marked s-nets

Considering seminaturally marked s-nets representing real systems, one can observe that there is frequently the case:  $\text{Mar} = \text{kens}(\text{coex}_C)$ .

Thus one can ask if  $\text{kens}(\text{coex}_C)$  is always a correctly defined marking class. The answer is "Yes" and this kind of net will be called 'naturally marked'.

We shall deal with that kind of net in this section. At first we prove that for every  $C$ -covering  $C$ , the triple  $(T, P, \text{kens}(\text{coex}_C))$  is really an ms-net.

Let  $N = (T, P)$  be a fixed proper net, and let  $C = \{N_1, \dots, N_m\}$ , where  $N_i = (T_i, P_i)$  for  $i = 1, \dots, m$ , be an  $C$ -covering of  $C$ . Let  $\text{cov}_C, \text{coex}_C$  be identically defined as in the previous sections.

### Lemma 7.1

$$\begin{aligned} \forall A \in \text{kens}(\text{coex}_C), (A, B) \in R1 &\Rightarrow B \in \text{kens}(\text{coex}_C), \\ (B, A) \in R1 &\Rightarrow B \in \text{kens}(\text{coex}_C). \end{aligned}$$

**Proof.** Let  $(A, B) \in R1$ , so

$$\exists a \in T, A - a = B - a' \text{ \& \& } a \subseteq A \text{ \& \& } a' \subseteq B.$$

Assume that  $B \notin \text{kens}(\text{coex}_C)$ .

We must consider two cases:

$$(1) \quad \exists p, q \in B \exists P_i \in \text{cov}_C, p, q \in P_i \text{ \& \& } p \neq q.$$



$$(2) \quad \forall p, q \in B, (p, q) \in \text{coex}_C \ \& \ \exists t \notin B \ \forall p \in B, (t, p) \in \text{coex}_C.$$

*Case 1.* Because  $p \neq q$  and  $p, q \in P_i$ , we have  $p, q \notin a'$ , so  $p, q \in B - a'$ . But  $B - a' = A - a'$ , so  $\{p, q\} \subseteq A \cap P_i$ .

But if  $A \in \text{kens}(\text{coex}_C)$  &  $P_i \in \text{cov}_C$ , then  $|A \cap P_i| \leq 1$ —a discrepancy.

*Case 2.* In this case we have

$$\begin{aligned} \forall p, q \in B (\forall P_i \in \text{cov}_C, p \notin P_i \text{ or } q \notin P_i) \ \& \ (\exists t \notin B \ \forall p \in B \\ \forall P_i \in \text{cov}_C, t \notin P_i \text{ or } p \notin P_i). \end{aligned}$$

Since  $a' \subseteq B$ ,  $t \notin a'$ , and by Lemma 6.2 we have  $t \notin a$ . Because  $A - a' = B - a'$ , we have  $\forall p \in A - a'$ ,  $(t, p) \in \text{coex}_C$ . By Lemma 6.2 we also obtain that

$$\forall p \in a', (p, t) \in \text{coex}_C \Rightarrow (\forall p' \in a', (p', t) \in \text{coex}_C).$$

From the above considerations we have

$$\begin{aligned} \forall p \in B, (t, p) \in \text{coex}_C &\Rightarrow \\ \Rightarrow \forall p \in B - a', (t, p) \in \text{coex}_C \ \& \ \forall p \in a', (t, p) \in \text{coex}_C & \\ \Rightarrow \forall p \in A - a', (t, p) \in \text{coex}_C \ \& \ \forall p \in a', (t, p) \in \text{coex}_C. & \end{aligned}$$

Since  $a' \subseteq A$ , we have  $A = (A - a') \cup a'$ .

So we can write  $\forall p \in A$ ,  $(t, p) \in \text{coex}_C$ . On the other hand we have shown that  $t \notin A - a'$  and  $t \notin a'$ . But this means that  $t \notin A$  &  $\forall p \in A$ ,  $(t, p) \in \text{coex}_C$ , thus  $A \notin \text{kens}(\text{coex}_C)$ —in spite of our assumption.

In this way we have proved that  $(A, B) \in R1 \Rightarrow B \in \text{kens}(\text{coex}_N)$ . For  $(B, A) \in R1$  we proceed similarly.  $\square$

**Theorem 7.2.** *The triple  $(T, P, \text{kens}(\text{coex}_C))$  is a safe, locally fireable ms-net.*

**Proof.** From Lemma 7.1 it follows that

$$\forall M \in \text{kens}(\text{coex}_C), [M]_{R_N} \subseteq \text{kens}(\text{coex}_C).$$

Thus,

$$\bigcup_{M \in \text{kens}(\text{coex}_C)} [M]_{R_N} \subseteq \text{kens}(\text{coex}_C).$$

On the other hand:  $\forall M \in \text{kens}(\text{coex}_C)$ ,  $M \in [M]_{R_N}$ , so

$$\text{kens}(\text{coex}_C) \subseteq \bigcup_{M \in \text{kens}(\text{coex}_C)} [M]_{R_N}.$$

Thus the triple  $MN = (T, P, \text{kens}(\text{coex}_C))$  is a marked s-net. From Lemma 7.1 we have that  $MN$  is locally fireable. Thus  $MN$  is seminaturally marked with respect to the covering  $C$ , so by Theorem 6.3 we get that it is safe.  $\square$

Theorem 7.2 makes correct the following definition. If  $C$  is an e-covering of  $(T, P)$ , then the triple  $MN = (T, P, \text{kens}(\text{coex}_C))$  is said to be a *naturally marked s-net*.

Note that every e-covering describes exactly one naturally marked s-net. Of course, every naturally marked s-net is also seminaturally marked. Since in this case  $\text{Mar} = \text{kens}(\text{coex}_C)$ , KM-density is equivalent to  $K$ -density, and CM-density is equivalent to  $C$ -density.

The basic difference between naturally and seminaturally marked s-nets is that in the case of first ones the marking class is fully described by the e-covering  $C$ . Other properties are very similar.

Applying results of Section 6 to the class of naturally marked s-nets we obtain the following theorem.

**Theorem 7.3.** *Let  $MN = (T, P, \text{kens}(\text{coex}_C))$  be a naturally marked s-net with respect to the e-covering  $C$ . Then:*

- (1)  $\text{coex}_C$  is  $K$ -dense  
 $\Rightarrow [\forall A \in \overline{\text{kens}(\text{coex}_C)}, N_A = (\cdot A, A) \text{ is a quasialementary s-net}]$ .
- (2)  $MN$  is compact  $\Rightarrow \text{coex}_C$  is  $C$ -dense  $\Rightarrow \text{cov}_C \subseteq \text{kens}(\text{coex}_C)$ .
- (3)  $MN$  is compact &  $\text{coex}_C$  is  $K$ -dense  $\Rightarrow$   
 $\Rightarrow [\forall A \in \overline{\text{kens}(\text{coex}_C)}, N_A = (\cdot A, A) \text{ is an elementary s-net}]$ .
- (4)  $C = \text{elem}(N)$  &  $MN$  is compact  
 $\Rightarrow (\text{coex}_C \text{ is } K\text{-dense} \Leftrightarrow \text{cov}_C = \overline{\text{kens}(\text{coex}_C)})$ .

## 8. Concurrency relation and global system states

In Section 5 we have started with a given in advance set of sequential component, and next, on the basis of that set, we have constructed the concurrency relation of a system.

We are now going to present the opposite point of view. We shall start with a given in advance set of global system states, and then we shall try to describe the concurrency relation. The set of global system states is also a covering of a set of system local states, so the procedure will be similar to that from Section 5.

Let  $X$  be a set, and let  $\text{cov} \subseteq 2^X$  be a covering of  $X$ . Let  $\overline{\text{sir}}(\text{cov}) \subseteq X \times X$  be the relation defined as follows:

$$\forall a, b \in X, \quad (a, b) \in \overline{\text{sir}}(\text{cov}) \Leftrightarrow a \neq b \ \& \ \exists A \in \text{cov}, a \in A \ \& \ b \in A.$$

Note that  $\overline{\text{sir}}(\text{cov})$  is also a sir-relation. Relationships between  $\overline{\text{sir}}(\text{cov})$  and  $\text{sir}(\text{cov})$  are described by the corollary below.

**Corollary 8.1**

- (1)  $\overline{\text{sir}}(\text{cov}) \cup \text{sir}(\text{cov}) \cup \text{id}_X = X \times X; \quad \overline{\text{sir}}(\text{cov}) \cap \text{sir}(\text{cov}) = \emptyset.$
- (2)  $\text{kens}(\overline{\text{sir}}(\text{cov})) = \overline{\text{kens}}(\text{sir}(\text{cov})); \quad \text{kens}(\text{sir}(\text{cov})) = \overline{\text{kens}}(\overline{\text{sir}}(\text{cov})).$
- (3)  $\text{cov is a partition of } X \Leftrightarrow \overline{\text{sir}}(\text{cov}) \cup \text{id}_X \text{ is an equivalence relation.}$
- (4)  $\overline{\text{sir}}(\text{cov}) \cup \text{id}_X \text{ is an equivalence relation} \Rightarrow \text{cov} = \text{kens}(\overline{\text{sir}}(\text{cov})).$

In the approach presented here, a covering  $\text{cov}$  is interpreted as an arbitrary set of coexisting system states, and the relation  $\overline{\text{sir}}(\text{cov})$  is interpreted as the concurrency relation defined by that set.

As opposed to Section 5, we shall assume that the family  $\overline{\text{kens}}(\overline{\text{sir}}(\text{cov}))$  represent the set of sequential components of a system. The full list of interpretations is the following:

$\text{cov}$ : the set of all global system states,

$\overline{\text{sir}}(\text{cov})$ : the concurrency relation defined by the set of global system states,

$\text{kens}(\overline{\text{sir}}(\text{cov}))$ : the family of all maximal locally concurrent sets,

$\overline{\text{kens}}(\text{sir}(\text{cov}))$ : the set of all sequential components of a system.

Here the family  $\text{kens}(\overline{\text{sir}}(\text{cov}))$  is a set of all global system states only if  $\text{cov} = \text{kens}(\overline{\text{sir}}(\text{cov}))$ . Notions of consistency and semiconsistency are defined, in this case, in a somewhat different way.

A relation  $\overline{\text{sir}}(\text{cov})$  is said to be *consistent* iff  $\text{cov} = \text{kens}(\overline{\text{sir}}(\text{cov}))$ , and it is said to be *semiconsistent* iff  $\text{cov} \subseteq \text{kens}(\overline{\text{sir}}(\text{cov}))$ .

The property of consistency means in this case that the set of global system states and the set of locally concurrent sets are identical. In other words, if  $(a, b)(b, c)(c, a)$  belong to the concurrency relation then the statement ' $(a, b, c)$  are all concurrent' is sensible. This fact means that the concurrency, which by the definition is only a binary relation, can be extensible to more complex structures such as cliques. The property of semiconsistency means in this case that the set of all global system states is defined by the concurrency relation, which is only partially extensible to more complex structures. In fact, in the case of the approach from Section 5 we have also assumed that the set of all system states is included in the set of cliques defined by the concurrency relation.

Note that in order to describe Petri's postulate on a common element, we have to introduce a new kind of density, because densities defined in Section 5 are inadequate.

A relation  $\overline{\text{sir}}(\text{cov})$  is said to be *M-dense* iff

$$\forall A \in \text{cov} \forall B \in \overline{\text{kens}}(\overline{\text{sir}}(\text{cov})), \quad A \cap B \neq \emptyset.$$

Usually,  $X$  is a set of net places and  $\text{cov}$  is a set of net markings, hence the name: M-density (from *marking*).

**Corollary 8.2**

- (1)  $\overline{\text{sir}}(\text{cov})$  is *M-dense*  $\Leftrightarrow \text{sir}(\text{cov})$  is *C-dense*.
- (2)  $\text{cov} = \text{kens}(\overline{\text{sir}}(\text{cov})) \Rightarrow (M\text{-density} \Leftrightarrow K\text{-density})$ .
- (3)  $\text{cov} \subseteq \text{kens}(\overline{\text{sir}}(\text{cov})) \Rightarrow (K\text{-density} \Rightarrow M\text{-density})$ .

From Theorem 5.4 and Corollary 8.2(1) it follows that *M-density* implies semi-consistency, or more formally

**Corollary 8.3**

$$\overline{\text{sir}}(\text{cov}) \text{ is } M\text{-dense} \Rightarrow \text{cov} \subseteq \text{kens}(\overline{\text{sir}}(\text{cov})).$$

**9. Analysis of ms-nets by means of concurrency relation**

In previous sections we dealt with a special kind of net, namely we started with a proper s-net and then we described a marking class on the basis of a given *c-covering*. And so the considerations were restricted to nets decomposable into sequential finite state machines.

In this section we start with an arbitrary ms-net  $MN = (T, P, \text{Mar})$ , and we shall try to design the concurrency relation and its properties on the basis of that triple. We shall use results of Section 8 and prove that the acceptance of Petri's postulate about a common element reduces considerations back to nets decomposable into sequential finite state machines.

Let  $MN = (T, P, \text{Mar})$  be an arbitrary, fixed for the rest of the section, ms-net. Let  $\text{coex}_{MN} \subseteq P \times P$  be the following relation:

$$\text{coex}_{MN} = \overline{\text{sir}}(\text{Mar}).$$

In other words,

$$\forall p, q \in P. \quad (p, q) \in \text{coex}_{MN} \Leftrightarrow p \neq q \ \& \ \exists M \in \text{Mar}, p \in M \ \& \ q \in M.$$

The relation  $\text{coex}_{MN}$  is called the *coexistence defined by markings*.

**Corollary 9.1**

$$\text{coex}_{MN} \text{ is } M\text{-dense} \Rightarrow \text{Mar} \subseteq \text{kens}(\text{coex}_{MN}).$$

Thus, if  $\text{coex}_{MN}$  is *M-dense*, then every marking is a clique defined by the coexistence relation.

We now show that *M-density* connected with safeness and local fireability forms a very strong property of a net.

**Theorem 9.2.** *If an ms-net  $MN = (T, P, \text{Mar})$  is safe, locally fireable and the relation  $\text{coex}_{\text{Mar}}$  is M-dense, then for every set  $A \in \overline{\text{kens}(\text{coex}_{\text{Mar}})}$  the pair  $(\cdot A, A)$  is a quasialementary s-net.*

**Proof.** The way of proving is similar to that of Theorem 6.5. Of course  $\cdot A = \text{left}(A)$ . At first we prove that  $(\cdot A, A)$  is an s-net. To this end it is enough to prove that  $\text{left}(A) = \text{right}(A)$ . Let  $a \in \text{left}(A)$ . This means that  $\exists p_a \in P, a \in p_a$ .

Because  $MN$  is locally fireable,  $\exists M \in \text{Mar}, \cdot a \subseteq M$ . Because  $\text{coex}_{\text{Mar}}$  is M-dense,  $M \cap A \neq \emptyset$ .

Define  $M' = (M - \cdot a) \cup a$ . Since  $MN$  is safe, we have  $(M, M') \in R1_N$ , where  $N = (T, P)$ . Since  $(N, M') \in R1_N$  &  $M \in \text{Mar}$ ,  $M' \in \text{Mar}$ . Note also that  $\{p_a\} = M' \cap A$ .

Let  $p \in M \cap A$ . Since  $a \in \cdot p_a, \cdot a \subseteq M', p_a \in A$ , then  $(M' - a) \cap A = \emptyset$ . Note that  $M' - a = M - \cdot a$ , so  $p \notin M - \cdot a$ . But  $p \in M$ , so  $p \in \cdot a$ . On the other hand:  $p \in \cdot a \Leftrightarrow a \in \text{right}(p) \subseteq \text{right}(A)$ , hence  $\text{left}(A) \subseteq \text{right}(A)$ .

In a similar way we can prove that  $\text{right}(A) \subseteq \text{left}(A)$ . Thus  $N_A = (T_A, A)$ , where  $T_A = \text{left}(A) = \cdot A$ , is an s-net. Let  $a \in T_A$ . We shall prove that  $|\cdot a \cap A| = 1$ . Of course  $\cdot a \cap A \subseteq \cdot a$ . Let  $M_a \in \text{Mar}$  such that  $\cdot a \subseteq M_a$ . Then  $\cdot a \cap A \subseteq M_a \in \text{Mar}$ . Assume that  $\{p, q\} \subseteq \cdot a \cap A$  &  $p \neq q$ . This means that  $\{p, q\} \subseteq M_a \in \text{Mar}$ , so

$$\begin{aligned} (p, q) \in \text{coex}_{\text{Mar}} &\Rightarrow \forall B \in \overline{\text{kens}(\text{coex}_{\text{Mar}})}, p \notin B \text{ or } q \notin B \\ &\Rightarrow p \notin \cdot a \cap A \text{ or } q \notin \cdot a \cap A \end{aligned}$$

—a discrepancy.

Thus  $\forall a \in T, |\cdot(a)_{N_A}| = |\cdot a \cap A| = 1$ .

For  $a$  we proceed similarly. Thus  $N_A$  is a quasialementary s-net.  $\square$

As an immediate consequence of Theorem 9.2 we obtain that if  $MN = (T, P, \text{Mar})$  is safe, locally fireable and  $\text{coex}_{\text{Mar}}$  is M-dense, then  $MN$  is decomposable into sequential finite state machines (but not necessarily connected).

*The reciprocal theorem is false.* To prove that we recall Example 6.6. Let us consider the ms-net  $MN' = (T, P, \text{Mar}')$ , where  $\text{Mar}' = \text{kens}(\text{coex}_C)$ . In this case we have  $\text{coex}_{\text{Mar}'} = \text{coex}_C$ , and every element of  $\overline{\text{kens}(\text{coex}_{\text{Mar}'})}$  is an elementary s-net; but  $\text{coex}_{\text{Mar}'}$  is not M-dense because  $\{1, 3\} \cap \{5, 6, 7\} = \emptyset$ , and  $\{1, 3\} \in \text{Mar}', \{5, 6, 7\} \in \overline{\text{kens}(\text{coex}_{\text{Mar}'})}$ .

Examples 9.5, 9.6 and 9.7 show that the assumptions of Theorem 9.2 cannot be weakened. Example 9.8 shows that the word ‘quasialementary’ cannot be replaced by ‘elementary’.

Local fireability and safeness are rather obvious properties demanded from nets representing correctly defined systems. The first property means that every transition has a possibility to be fired, so there is no useless transition, the second one may be interpreted as: ‘any action cannot disturb other actions’ (see [17]). Adding the property of M-density, we obtain a very regular static (or ‘topological’ in the common

sense of this word) structure of a net. This betokens the fact that  $M$ -density is a strong property of an ms-net.

As was mentioned above, in most approaches it is assumed that systems are compact in our sense. It turns out that in the case of compact ms-nets, the property of  $M$ -density is still stronger.

**Theorem 9.3.** *If  $MN = (T, P, \text{Mar})$  is compact, safe, fireable and  $\text{coex}_{\text{Mar}}$  is  $M$ -dense, then for every set  $A \in \overline{\text{kens}}(\text{coex}_{\text{Mar}})$  the pair  $(\cdot A, A)$  is an elementary s-net.*

**Proof.** Similarly as the proof of Theorem 6.10, using Theorem 9.2 instead of Theorem 6.5.  $\square$

**Theorem 9.4.** *Let  $MN = (T, P, \text{Mar})$  be a compact and fireable ms-net. Then the following are equivalent:*

- (1)  $MN$  is safe and  $M$ -dense,
- (2)  $\text{Mar} \subseteq \text{kens}(\text{coex}_{\text{Mar}})$  and for every  $A \in \overline{\text{kens}}(\text{coex}_{\text{Mar}})$  the pair  $(\cdot A, A)$  is an elementary s-net.

**Proof.** (1) $\Rightarrow$ (2) This is a consequence of Corollary 9.1 and Theorem 9.3.

(2) $\Rightarrow$ (1) Let us put

$$C = \{N_A \mid N_A = (\cdot A, A) \text{ \& } A \in \overline{\text{kens}}(\text{coex}_{\text{Mar}})\}.$$

Note that  $C$  is a correctly defined e-covering of  $N$ , and  $\text{cov}_C = \overline{\text{kens}}(\text{coex}_{\text{Mar}})$ .

But this means that  $\text{sir}(\text{cov}_C) = \text{coex}_{\text{Mar}}$ , and, consequently,  $\text{coex}_{\text{Mar}} = \text{coex}_C$ ,  $\text{kens}(\text{coex}_C) = \text{kens}(\text{coex}_{\text{Mar}})$ .

Since  $MN$  is fireable and  $\text{Mar} \subseteq \text{kens}(\text{coex}_C)$ , the ms-net  $MN$  is seminaturally marked with respect to the e-covering  $C$ .

By Theorem 6.3 we obtain that  $MN$  is safe, and by Theorem 6.8 that  $\text{coex}_C$  is CM-dense. But  $\text{coex}_C = \text{coex}_{\text{Mar}}$  and  $\text{cov}_C = \overline{\text{kens}}(\text{coex}_{\text{Mar}})$ , so CM-density of  $\text{coex}_C$  is equivalent to  $M$ -density of  $\text{coex}_{\text{Mar}}$ .  $\square$

The ms-net  $MN'$  considered after Theorem 9.2 also proves the untruth of the reciprocal of Theorem 9.3. Examples 9.5–9.8 considered below show that the assumptions of Theorem 9.3 cannot be weakened, and Examples 9.6 and 9.8 show that the assumptions of Theorem 9.4 cannot be weakened.

In all examples below we have that  $\overline{\text{coex}}_{\text{Mar}} = (P \times P - \text{coex}_{\text{Mar}}) \dashv \text{id}_P$ .

**Example 9.5.** Let  $MN = (T, P, \text{Mar})$  be the ms-net given in Fig. 20;  $\text{Mar} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4\}\}$ .

Graphs of relations  $\text{coex}_{\text{Mar}}$ ,  $\overline{\text{coex}}_{\text{Mar}}$  are shown in Fig. 21. In this case we have

$$\text{kens}(\text{coex}_{\text{Mar}}) = \{\{1, 2, 3\}, \{4\}\} \not\supseteq \text{Mar},$$

$$\overline{\text{kens}}(\text{coex}_{\text{Mar}}) = \{\{1, 4\}, \{2, 4\}, \{3, 4\}\}.$$

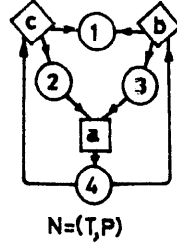


Fig. 20.



Fig. 21.

The relation  $coex_{Mar}$  is  $K$ -dense, but it is not  $M$ -dense, because  $\{2, 3\} \cap \{1, 4\} = \emptyset$ , and it is not semiconsistent. The  $nis$ -net  $MN$  is compact, safe and fireable, but the  $s$ -net  $N = (T, P)$  is not proper, thus the assumption:  $coex_{Mar}$  is  $M$ -dense cannot be removed from Theorems 9.2 and 9.3.

**Example 9.6.** Let  $MN = (T, P, Mar)$  be the net defined in Example 4.3. Graphs of relations  $coex_{Mar}$ ,  $\overline{coex}_{Mar}$  are of the form shown in Fig. 22. Here we have

$$kens(coex_{Mar}) = Mar = \{\{1, 2\}, \{3, 4\}, \{5\}, \{6\}\},$$

$$\overline{kens}(coex_{Mar}) = \{\{1, 3, 5, 6\}, \{1, 4, 5, 6\}, \{2, 3, 5, 6\}, \{2, 4, 5, 6\}\}.$$

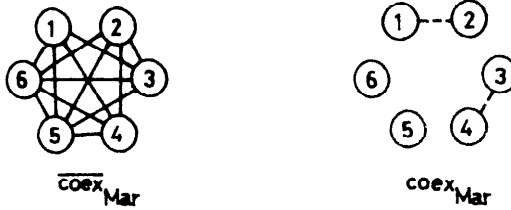


Fig. 22.

The relation  $coex_{Mar}$  is  $K$ -dense,  $M$ -dense and consistent. The  $ms$ -net  $MN$  is safe and compact but it is not locally fireable. The  $s$ -net  $(T, P)$  is not proper. This proves that the assumption of local fireability cannot be removed from Theorems 9.2, 9.3 and 9.4.

**Example 9.7.** Let  $MN = (T, P, Mar)$  be the  $ms$ -net, as shown in Fig. 23;  $Mar = \{\{1, 2\}\}$ .

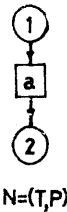


Fig. 23.

Here graphs of  $\text{coex}_{\text{Mar}}$  and  $\overline{\text{coex}}_{\text{Mar}}$  are as shown in Fig. 24.



Fig. 24.

Note that  $\text{kens}(\text{coex}_{\text{Mar}}) = \text{Mar}$ ,  $\overline{\text{kens}}(\text{coex}_{\text{Mar}}) = \{\{1\}, \{2\}\}$ , the relation  $\text{coex}_{\text{Mar}}$  is consistent, M-dense and K-dense, the ms-net MN is fireable and compact, but not safe. The s-net  $N = (T, P)$  is proper indeed, but elements of  $\overline{\text{kens}}(\text{coex}_{\text{Mar}})$ , i.e., sets  $\{1\}, \{2\}$ , do not define any s-net. Thus the assumption of safeness cannot be removed from Theorems 9.2 and 9.3.

**Example 9.8.** Let  $\text{MN} = (T, P, \text{Mar})$  be the ms-net shown in Fig. 25;  $\text{Mar} = \{\{1, 3\}, \{2, 4\}, \{4, 5\}, \{3, 6\}\}$ .

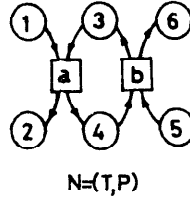


Fig. 25.

, Graphs of  $\text{coex}_{\text{Mar}}$  and  $\overline{\text{coex}}_{\text{Mar}}$  are shown in Fig. 26. In this case,

$$\text{kens}(\text{coex}_{\text{Mar}}) = \text{Mar},$$

$$\overline{\text{kens}}(\text{coex}_{\text{Mar}}) = \{\{1, 2, 5, 6\}, \{2, 3, 5\}, \{1, 4, 6\}, \{3, 4\}\}.$$

The relation  $\text{coex}_{\text{Mar}}$  is consistent, M-dense and K-dense.

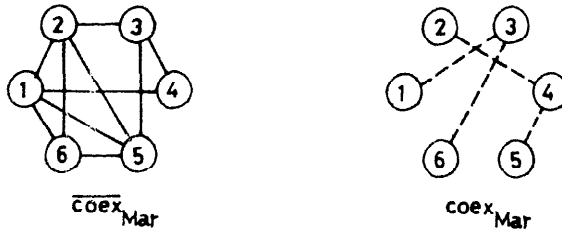


Fig. 26.

The ms-net MN is safe and locally fireable, but not compact because for instance,  $(\{1, 3\}, \{3, 6\}) \notin R_N$ . The family of quasialementary s-nets defined by the set  $\overline{\text{kens}}(\text{coex}_{\text{Mar}})$  consists of the s-nets shown in Fig. 27.

Of course,  $N = (T, P)$  is proper and  $N = N_1 \cup \dots \cup N_4$ . This example proves that in Theorem 9.2 the word 'quasialementary' cannot be replaced by 'elementary', as well as that the assumption of compactness cannot be removed from Theorem 9.4.



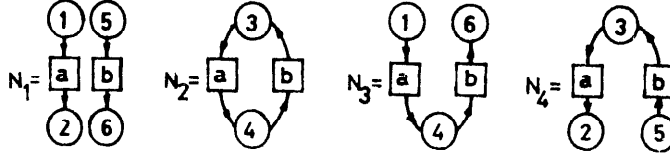


Fig. 27.

Now we consider a very regular case.

**Example 9.9.** Let  $MN = (T, P, \text{Mar})$  be the following ms-net: the pair  $N = (T, P)$  is the same as in Example 9.8, and  $\text{Mar} = \{\{1, 3, 5\}, \{2, 4, 5\}, \{2, 3, 5\}\}$ .

Now graphs of  $\text{coex}_{\text{Mar}}$  and  $\overline{\text{coex}_{\text{Mar}}}$  are shown in Fig. 28. In this case,

$$\text{kens}(\text{coex}_{\text{Mar}}) = \text{Mar} \cup \{\{2, 3, 5\}\},$$

$$\overline{\text{kens}(\text{coex}_{\text{Mar}})} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{1, 4, 6\}\}.$$

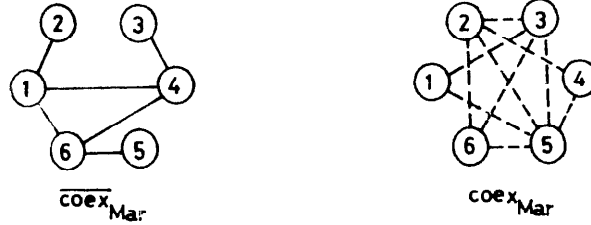


Fig. 28.

The relation  $\text{coex}_{\text{Mar}}$  is  $M$ -dense and semiconsistent, but it is not  $K$ -dense. The ms-net is safe, compact and fireable, the s-net  $(T, P)$  is proper. The family  $\overline{\text{kens}(\text{coex}_{\text{Mar}})}$  describes the elementary nets shown in Fig. 29.

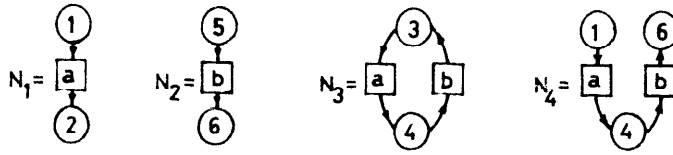


Fig. 29.

Note that  $N = N_1 \cup \dots \cup N_4$  and  $\{N_1, N_2, N_3, N_4\} \subsetneq \text{elem}(N)$ . Note also that  $MN$  is seminaturally marked with respect to the covering  $C = \{N_1, \dots, N_4\}$ .

In this approach the notion of  $M$ -density expresses Petri's postulate that every sequential component and every global system state must have one element in common. From the above considerations it follows that such a postulate implies the reduction of considerations to nets decomposable into sequential finite state machines only.

Besides the  $M$ -density implies semiconsistency, i.e., it makes it possible to talk about maximal sets of independent places.

From the results of this section, it follows that on the system level Petri's postulate is equivalent to the opinion proclaimed by Lauer and others [16] that concurrency is a non-interleaving synchronisation of sequential subsystems.

## 10. Final comment

We wish to point out the importance of Petri's postulate that every sequential component and every global system state has to have one element in common, in the approach presented in this paper. It turns out that nets which do not satisfy this postulate have usually irregular remaining properties. This postulate is not, in the general case, described by the well-known property of  $K$ -density introduced by Petri. According to the need, this postulate is described by CM-density,  $C$ -density, or  $M$ -density. Only in particular cases is it described by  $K$ -density.

In accordance with our intuition, properties: safeness, local fireability and  $M$ -density of  $\text{coex}_{\text{Mar}}$  can be treated as necessary conditions for 'well-defined' concurrent systems. But this means that such a 'well-defined' concurrent system is always a net decomposable into sequential finite state machines.

## Acknowledgment

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