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Modeling concurrency with interval traces

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ABSTRACT

Interval order structures are useful tools to model abstract concurrent histories, i.e. sets of equivalent system runs, when system runs are modeled with *interval orders*. This paper shows how interval order structures can be modeled by *partially commutative monoids*, called *interval traces*. The model is then used to provide a semantics of Petri nets with inhibitor arcs, both in terms of interval traces and in terms of interval order structures.

1. Introduction

Most observational[1] semantics of concurrent systems are defined either in terms of sequences (i.e. total orders) or step-sequences (i.e. stratified orders). When concurrent histories[2] are fully described by *causality relations*, i.e. *partial orders*, Mazurkiewicz traces [10,33,34] allow a representation of the entire partial order by a single sequence (plus *independency* relation), which provides a simple and elegant connection between observational and process semantics (i.e. the semantics in terms of concurrent histories) of concurrent systems with static concurrency structure, i.e. if two actions are independent, they are always independent. In such case, all other relevant observations can be derived as just stratified or interval extensions of appropriate partial orders.

It has been observed a long time ago that if priority and concurrency are mixed, it may happen that for two actions a and b, a sequence a followed by b, and a simultaneous execution of a and b are allowed, and they can be considered as equivalent, but the sequence b followed by a is disallowed [16,30,43]. Such situation is often called "not later than" relationship (cf. [17,28]) as a may not follow b (but the opposite order and simultaneity are allowed).

When we want to model both causality and the "not later than" relationship, we have to use *stratified order structures* [14,20,22], when *all* observations are step-sequences, or *interval order structures* [23,20,31], when *all* observations are interval orders.

Comtraces [22] allow a representation of stratified order structures by single step-sequences (with appropriate *simultaneity* and *serializability* relations).

It was argued by Wiener in 1914 [51] (and later more formally in [21]) that any execution that can be observed by a single observer must be an interval order. It implies that the most precise observational semantics is defined in terms of

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[1] 'Observational semantics' is not a generally agreed concept, in this paper this will be just a collection of all system runs (i.e. executions, observations) [5,21,28,49]. A different meaning is used in for example [7].

[2] In this paper a 'concurrent history' is a set of equivalent system runs (executions, observations), represented uniquely by some partial order or order structure, and differs formally (although it is intuitively close) to these of for instance [8,26]. The concept used in this paper was introduced in [21] and is close to that of [34].

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interval orders. However generating interval orders directly is problematic for most models of concurrency. Unfortunately, the only feasible sequence representation of interval order is by using sequences of *beginnings* and *endings* of events involved [11,21].

The goal of this paper is to provide a monoid based model that allows a single sequence of beginnings and endings (enriched with appropriate *simultaneity* and *serializability* relations) to represent the entire *stratified order structures* as well as all equivalent interval order observations. This will be done by introducing and developing the concept of *interval traces*, a mixture of ideas from both Mazurkiewicz traces [10] and representations of interval orders [12], and proving that each interval trace uniquely determines an interval order structure. The interval traces considered in this paper are highly revised, modified and extended version of the concept originally proposed in [25].

We will also show how interval traces can define interval order semantics of elementary nets with inhibitor arcs.

Modeling observational semantics with sequence and concurrent histories with Mazurkiewicz traces (cf. [10]) as well as modeling observational semantics with step sequence and concurrent histories with comtraces (or similar models, cf. [5,19, 22,24,28,32,49]) is well developed and relatively well known. Recently published [19] provides a general model that covers most others as special cases, including these from [5,22,24,49]). For the case where the system runs or observations are represented by intervals or interval orders, the situation is much less impressive [21,25,40,45–49]. Conceptually closest to our approach are models based on the concept of ST-traces (sequences of transition beginnings and ends) and ST-bisimulation [45–49]. We will briefly discuss some relationships of ST-traces to our model in Section 10.5. The beginnings and ends are also used in [40], but the outcome is only step sequence semantics. The paper [21] provides some general abstract results and [25] defines an initial version of interval traces and some preliminary results.

This paper is organized as follows. Section 2 recalls some concepts and results on partial orders, sequences, and their mutual relationship. In Section 3, Mazurkiewicz traces and their basic properties are briefly discussed. Interval traces, the main concept introduced in this paper, are discussed in Section 4. Section 5 is devoted to interval order structures and their partial order representations. The relationship between interval traces and interval order structures, one of the main contributions of this paper, is discussed in Section 6, and the relationship between interval traces and comtraces in Section 7. The concept of concurrent histories and how they relate to interval traces is analyzed in Section 8. In Section 9, it is shown how interval traces can describe various behavioral properties of concurrent systems, and a small example is analyzed in detail. In Section 10 we show how interval traces can be used to provide an adequate semantics of elementary Petri nets with inhibitor arcs. The relationship to the model of [49] is also discussed at the end of Section 10. Section 11 contains some final comments. Technical proofs of two results are given in Appendix A.

2. Partial orders and sequences

In this section, we recall some, often well-known, concepts, notations and results regarding partial orders [12], sequences and representations of partial orders by appropriate sequences [13,18,22,24].

2.1. Partial orders

Partial orders are one of the basic tools used in this paper. They will be used as a full representation of systems runs (or observations) and as a partial representation of concurrent histories.

Definition 1. A relation $<\subseteq X \times X$ is a (*strict*) *partial order* iff it is irreflexive and transitive, i.e. for all $a, c, b \in X$, $a \nless a$ and $a < b < c \implies a < c$. We also define:

$$a \frown_< b \overset{df}{\iff} \neg(a < b) \wedge \neg(b < a) \wedge a \neq b,$$

$$a <^\frown b \overset{df}{\iff} a < b \vee a \frown_< b.$$

Note that $a \frown_< b$ means a and b are *incomparable* (w.r.t. $<$) elements of X. ■

Let $<$ be a partial order on a set X. Then:

1. $<$ is *total* if $\frown_< = \emptyset$. In other words, for all $a, b \in X$, $a < b \vee b < a \vee a = b$. For clarity, we will reserve the symbol $\lhd$ to denote total orders;
2. $<$ is *stratified* if $a \frown_< b \frown_< c \implies a \frown_< c \vee a = c$, i.e., the relation $\frown_< \cup\, id_X$ is an equivalence relation on X;
3. $<$ is *interval* if for all $a, b, c, d \in X$, $a < c \wedge b < d \implies a < d \vee b < c$.

It is clear from these definitions that every total order is stratified and every stratified order is interval. An interval order is *strict* if it is not stratified. In this paper, most partial orders will be represented by Hasse diagrams [12]. The following simple concept will often be used in this paper.

Definition 2. For a relation $R \subseteq X \times X$, any relation $Q \subseteq X \times X$ is an *extension* of R if $R \subseteq Q$. ■

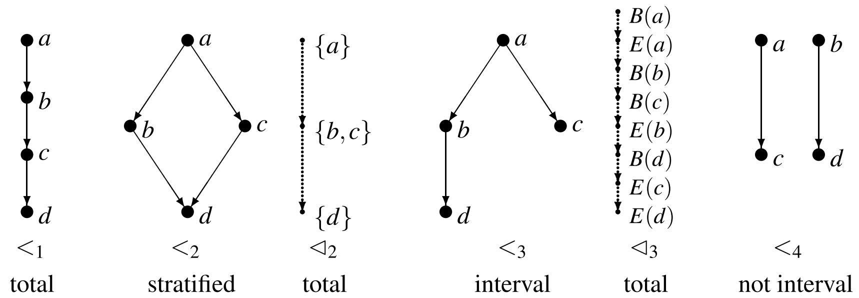


Fig. 1. Various types of partial orders (represented as Hasse diagrams). The partial order $<_1$ is an extension of $<_2$, $<_2$ is an extension of $<_3$, and $<_3$ is and extension of $<_4$. Note that order $<_1$, being total, is uniquely represented by a sequence *abcd*, the stratified order $<_2$ is uniquely represented by a step sequence $\{a\}\{b,c\}\{d\}$, and the interval order $<_3$ is (*not* uniquely) represented by a sequence that represents $\lhd_3$, i.e. $B(a)E(a)B(b)B(c)E(b)B(d)E(c)E(d)$.

For convenience, we define $\mathsf{Total}(<) \overset{df}{=} \{\lhd \subseteq X \times X \mid \lhd \text{ is a total order and } < \subseteq \lhd\}$.
In other words, the set $\mathsf{Total}(<)$ consists of all the *total order extensions* of $<$.
By Szpilrajn's Theorem [44], we know that every partial order $<$ is uniquely represented by the set $\mathsf{Total}(<)$. Szpilrajn's Theorem can be stated as follows:

Theorem 3 *(Szpilrajn [44]). For every partial order $<$,*

$$< \;=\; \bigcap_{\lhd \in \mathsf{Total}(<)} \lhd,$$

i.e. each partial order is the intersection of all its total extensions. ∎

Stratified orders are often defined in an alternative way, namely, a partial order $<$ on X is stratified if and only if there exists a total order $\lhd$ on some T and a mapping $S : X \to T$ such that $\forall x, y \in X.\ x < y \iff S(x) \lhd S(y)$. Usually $S(x)$ is interpreted as a strata. This definition is illustrated in Fig. 1, where $S(a) = \{a\}$, $S(b) = S(c) = \{b, c\}$, $S(d) = \{d\}$. Note that for all $x, y \in \{a, b, c, d\}$ we have $x <_2 y \iff S(x) \lhd_2 S(y)$, where the total order $\lhd_2$ can be concisely represented by a *step sequence* $\{a\}\{b, c\}\{d\}$. As a consequence, stratified orders and step sequences can uniquely represent each other (cf. [13,22]).
For the interval orders, the name and intuition follow from Fishburn's Theorem:

Theorem 4 *(Fishburn [11]). A partial order $<$ on countable[3] set X is interval iff there exists a total order $\lhd$ on some T and two injective mappings with disjoint codomains $B, E : X \to T$ such that for all $x, y \in X$,*

1. $B(x) \lhd E(x)$,
2. $x < y \iff E(x) \lhd B(y)$. ∎

Usually $B(x)$ is interpreted as the beginning and $E(x)$ as the end of an *interval* x. The intuition of Fishburn's theorem is illustrated in Fig. 1 with $<_3$ and $\lhd_3$. For all $x, y \in \{a, b, c, d\}$, we have $B(x) \lhd_3 E(x)$ and $x <_3 y \iff E(x) \lhd_3 B(y)$. For better readability in the future we will write Bx and Ex instead of $B(x)$ and $E(x)$.

2.2. Sequences, enumerated sequences, sequences of beginnings and ends, and their relationship to partial orders

Sequences are the most obvious and popular tool to define an observational semantics of both sequential and concurrent systems, and they can also conveniently represent finite total, stratified, and interval orders (cf. [13,18,22,24]).
Let Σ be a finite set (of events) and $\mathscr{P}(\Sigma)$ its power set. The elements of Σ^* are called *sequences* while the elements of $(\mathscr{P}(\Sigma) \setminus \emptyset)^*$ are called *step sequences*.
While interpreting sequences as partial orders and vice versa is well-known and established fact, a standard notation has not been set up yet. Below we will define the notation that will be used in this paper.
For each sequence $x \in \Sigma^*$ or each step sequence $x \in (\mathscr{P}(\Sigma) \setminus \emptyset)^*$, and each $a \in \Sigma$, let $\#_a(x)$ denote the number or quantity of a in x. For example $\#_a(abbaa) = 3$, $\#_b(abbaa) = 2$ and $\#_c(abbaa) = 0$; $\#_a(\{a, b\}\{b, c\}\{a, b, c\}) = 2$, $\#_b(\{a, b\}\{b, c\}\{a, b, c\}) = 3$, $\#_c(\{a, b\}\{b, c\}\{a, b, c\}) = 2$ and $\#_d(\{a, b\}\{b, c\}\{a, b, c\}) = 0$.
The formal relationship between sequences and total orders, and between step sequences and stratified orders can be defined as follows.

[3] For uncountable X it is additionally required that the equivalence relation $\sim_<$ defined as $a \sim_< b \iff \forall c \in X.(c < a \Leftrightarrow c < b) \wedge (a < c \Leftrightarrow b < c)$ has countably many equivalence classes [11]. But in this paper we need only a simpler version for countable X, cf. [21].

Definition 5.

1. For each set of events Σ, let $\widehat{\Sigma} = \{a^{(i)} \mid a \in \Sigma, i = 1, 2, \ldots\}$. The elements of $\widehat{\Sigma}$ are called *enumerated events*. The operator '$\widehat{\ }$' is idempotent, i.e. $\widehat{\widehat{\Sigma}} = \widehat{\Sigma}$.
2. For each sequence $x \in \Sigma^*$, its *enumerated representation* $\widehat{x} \in \widehat{\Sigma}^*$, is defined as follows:
   - $x = \varepsilon \implies \widehat{x} = \varepsilon$, and $x = a \implies \widehat{x} = a^{(1)}$,
   - $x = ya \implies \widehat{x} = \widehat{y}a^{(i)}$, where $i = \#_a(y) + 1$.

   We also assume $\widehat{\widehat{x}} = \widehat{x}$.
3. For each step sequence $x \in (\mathscr{P}(\Sigma) \setminus \emptyset)^*$, its *enumerated representation* $\widehat{x} \in \widehat{\Sigma}^*$, is defined as follows:
   - $x = \varepsilon \implies \widehat{x} = \varepsilon$, and $x = \{a_1, \ldots, a_k\} \implies \widehat{x} = \{a_1^{(1)}, \ldots, a_k^{(1)}\}$,
   - $x = yA \implies \widehat{x} = \widehat{y}\widehat{A}$, where $\widehat{A} = \{a^{(i)} \mid a \in A \wedge i = \#_a(y) + 1\}$.

   Again we assume $\widehat{\widehat{x}} = \widehat{x}$.
4. For each sequence $x \in \Sigma^*$, or step sequence $x \in (\mathscr{P}(\Sigma) \setminus \emptyset)^*$, Σ_x denotes the set of all elements of Σ that occur in x, and $\widehat{\Sigma}_x$ denotes the set of all enumerated events of $\widehat{x}$.
5. For each sequence $x \in \Sigma^*$, we define the following total order $\lhd_x$ on $\widehat{\Sigma}_x$:
$$a^{(i)} \lhd_x b^{(j)} \iff \widehat{x} = ua^{(i)}vb^{(j)}w,$$
   where $u, v, w \in (\widehat{\Sigma}_x)^*$.
6. For each step sequence $x \in (\mathscr{P}(\Sigma) \setminus \emptyset)^*$, we define the following stratified order $\lhd_x$ on $\widehat{\Sigma}_x$:
$$a^{(i)} \lhd_x b^{(j)} \iff \widehat{x} = uAvBw,$$
   where $a^{(i)} \in A \subseteq \widehat{\Sigma}_x$, $b^{(j)} \in B \subseteq \widehat{\Sigma}_x$ and $u, v, w \in (\widehat{\Sigma}_x)^*$. ■

For example, if $x = abbaa$ then $\widehat{x} = a^{(1)}b^{(1)}b^{(2)}a^{(2)}a^{(3)}$, $\Sigma_x = \{a, b\}$ and $\widehat{\Sigma}_x = \{a^{(1)}, a^{(2)}, a^{(3)}, b^{(1)}, b^{(2)}\}$. If $x = \{a, b\}\{b, c\}\{a, b, c\}$, then $\widehat{x} = \{a^{(1)}, b^{(1)}\}\{b^{(2)}, c^{(1)}\}\{a^{(2)}, b^{(3)}, c^{(2)}\}$, $\Sigma_x = \{a, b, c\}$ and $\widehat{\Sigma}_x = \{a^{(1)}, a^{(2)}, b^{(1)}, b^{(2)}, b^{(3)}, c^{(1)}, c^{(2)}\}$.

The sequence $x = abbaa$ represents a total order:

$$\lhd_x = a^{(1)} \to b^{(1)} \to b^{(2)} \to a^{(2)} \to a^{(3)},$$

while the step sequence $x = \{a, b\}\{b, c\}\{a, b, c\}$ represents the stratified order (represented as total order of equivalence classes):

$$\lhd_x = \{a^{(1)}, b^{(1)}\} \to \{b^{(2)}, c^{(1)}\} \to \{a^{(3)}, b^{(2)}, c^{(2)}\}.$$

If $\widehat{\Sigma}_x \subseteq \{a^{(1)} \mid a \in \Sigma\}$, then we will identify x with $\widehat{x}$. More details can be found for example in [22,18,24].

We will now show how interval orders can be represented by sequences of beginnings and ends. We adapt conventions from Definition 5 and also use Theorem 4.

For a given Σ, let $\mathscr{E}_\Sigma = \{Ba \mid a \in \Sigma\} \cup \{Ea \mid a \in \Sigma\}$, or just $\mathscr{E}$, be the set of all beginnings and ends of events in Σ.

Let $\mathscr{D} \subseteq \mathscr{E}$ and let $s \in \mathscr{E}^*$. We define the projection of s onto $\mathscr{D}$ standardly as:

$$\pi_{\mathscr{D}}(\epsilon) \overset{df}{=} \epsilon, \qquad \pi_{\mathscr{D}}(s\alpha) \overset{df}{=} \begin{cases} \pi_{\mathscr{D}}(s)\alpha & \text{if } \alpha \in \mathscr{D}, \\ \pi_{\mathscr{D}}(s) & \text{if } \alpha \notin \mathscr{D}. \end{cases}$$

For example $\pi_{\{Ba,Ea\}}(BbBaEbBaEaEc) = BaBaEa$, $\pi_{\{Ba,Ea,Bc,Ec\}}(BbBaEbBaEaEc) = BaBaEaEc$.

Definition 6.

1. A string $x \in \mathscr{E}^*$ is an *interval sequence* iff $\forall a \in \Sigma.\ \pi_{\{Ba,Ea\}}(x) \in (BaEa)^*$.
   We use $\mathsf{InSeq}(\mathscr{E}^*)$ to denote the set of all interval sequences of $\mathscr{E}^*$.
2. For every $x \in \mathscr{E}_\Sigma^*$, we define $\widehat{\Sigma}_x^{\mathscr{E}} \subseteq \widehat{\Sigma}^*$ as follows:
$$\widehat{\Sigma}_x^{\mathscr{E}} = \{a^{(i)} \mid Ba^{(i)} \in \widehat{\mathscr{E}}_x\} \cup \{a^{(i)} \mid Ea^{(i)} \in \widehat{\mathscr{E}}_x\},$$
3. Let $x \in \mathsf{InSeq}(\mathscr{E}_\Sigma^*)$, and let $\blacktriangleleft_x$ be a relation on $\widehat{\mathscr{E}}_x^{\Sigma}$, defined by
$$a^{(i)} \blacktriangleleft_x b^{(j)} \iff Ea^{(i)} \lhd_x Bb^{(j)}.$$
   By Theorem 4, the relation $\blacktriangleleft_x$ is an interval order, and it will be called the interval order defined by the sequence x of beginnings and ends. ■

Since the operator '$\widehat{\ }$' is idempotent in all cases, Definitions 5(5,6) and 6(3) imply the following simple but useful result.

Corollary 7. *For every sequence $x \in \Sigma^*$, step sequence $y \in (\mathscr{P}(\Sigma) \setminus \emptyset)^*$, and interval sequence $z \in \mathscr{E}^*$, we have:*

$$\lhd_x = \lhd_{\widehat{x}},\ \lhd_y = \lhd_{\widehat{y}} \text{ and } \blacktriangleleft_z = \blacktriangleleft_{\widehat{z}}\ . \quad \blacksquare$$

Note that if $x \in \mathsf{InSeq}(\mathscr{E}^*)$, then $\widehat{\Sigma}^{\mathscr{E}}_x = \{a^{(i)} \mid Ba^{(i)} \in \widehat{\mathscr{E}_x}\} = \{a^{(i)} \mid Ea^{(i)} \in \widehat{\mathscr{E}_x}\}$. For example a sequence $x = BaBbEbEaBcBaBb EcEbEaBaEa$ is in $\mathsf{InSeq}(\mathscr{E}^*_\Sigma)$ for $\Sigma = \{a, b, c\}$, but sequences $EaBbEbBa$ or $BbEbBaEc$ are not. For this x, we have $\widehat{\Sigma}^{\mathscr{E}}_x = \{a^{(1)}, a^{(2)}, a^{(3)}, b^{(1)}, b^{(2)}, c^{(1)}\}$. For $x = BaEaBbBcEbBdEcEd$, the interval order $\blacktriangleleft_x$ is the same as $<_3$ of Fig. 1 with $a^{(1)}, b^{(1)}, c^{(1)}$, and $d^{(1)}$ represented by a, b, c, and d, and for $y = BaEaBbBaEbBbEaEb$, the interval order $\blacktriangleleft_y$ is also the same as $<_3$ of Fig. 1 with $a^{(1)}$ represented by a, $b^{(1)}$ represented by b, $a^{(2)}$ by c, and $b^{(2)}$ by d.

3. Mazurkiewicz traces

Interval traces, the main contribution of this paper, stemmed from *Mazurkiewicz traces* (cf. [10,33]), a kind of *equational monoids over sequences* [24,38]. The theory of traces has been utilized to tackle problems from quite diverse areas including combinatorics, graph theory, algebra, logic, and especially concurrency theory [10,33]. Applications of traces in concurrency theory are originated from the fact that traces are *sequence representation of partial orders*, which gives traces the ability to model "true concurrency" semantics.

Definition 8 *(Of Mazurkiewicz trace [10,33,34]).*

1. Let Σ be a finite set of events and let the relation $ind \subseteq \Sigma \times \Sigma$ be an irreflexive and symmetric relation (called *independency*). The pair (Σ, ind) is called a *trace alphabet*.
2. Let $\approx \subseteq \Sigma^* \times \Sigma^*$ be a relation defined as follows:
$$x \approx y \iff \exists x_1, x_1 \in \Sigma^*. \exists (a,b) \in ind.\ x = x_1 ab x_2 \wedge y = x_1 ba x_2$$
3. Let $\equiv_{ind}$ be the reflexive and transitive closure of $\approx$, i.e.
$$\equiv_{ind} = \approx^* = \bigcup_{i=0}^{\infty} \approx^i .$$
Since $\approx$ is symmetric, the relation $\equiv_{ind}$ is clearly an *equivalence* relation.
4. For every $x \in \Sigma^*$, the equivalence class $[x]_{\equiv_{ind}}$ is called a ***Mazurkiewicz trace***, or just a ***trace***. $\blacksquare$

We will omit the subscripts ind and $\equiv_{ind}$ from trace description, if it causes no ambiguity, and often write $[x]_{ind}$, or just $[x]$, instead of $[x]_{\equiv_{ind}}$.

One may show that $[x][y] = [x] \circ [y] = [xy]$, where $\circ$ is a concatenation of sets of sequences, a symbol that is usually omitted [10,34].

Formally, an algebra of Mazurkiewicz traces is a *quotient equational monoid over sequences* [10,24,38], however we do not need the full theory of traces as equational monoids in this paper.

Example 9. Consider the trace alphabet (Σ, ind), where $\Sigma = \{a, b, c\}$ and $ind = \{(b, c), (c, b)\}$. Given three sequences $x = abcbca$, $x_1 = abc$ and $x_2 = bca$, we can generate the traces $[x] = \{abcbca, abccba, acbbca, acbcba, abbcca, accbba\}$, $[x_1] = \{abc, acb\}$ and $[x_2] = \{bca, cba\}$. Note that $[x] = [x_1] * [x_2]$ since $[abcbca] = [abc] * [bca] = [abc * bca]$. $\blacksquare$

Note for each trace $[x]$ its *set of all enumerated events* can be defined as $\widehat{\Sigma}_{[x]} = \widehat{\Sigma}_x$. For the trace $[x]$ from Example 9, we have $\widehat{\Sigma}_{[x]} = \left\{a^{(1)}, b^{(1)}, c^{(1)}, b^{(2)}, c^{(2)}, a^{(2)}\right\}$.

Definition 10. For every trace $[x]$, let $\lessdot_{[x]} \subseteq \widehat{\Sigma}_{[x]} \times \widehat{\Sigma}_{[x]}$ be a partial order defined as:

$$\lessdot_{[x]} = \bigcap_{t \in [x]} \lhd_t .$$

The partial order $\lessdot_{[x]}$ is called generated by the trace $[x]$. $\blacksquare$

Theorem 11 *(Follows from [33,34], also Theorem 6.31 in [18]). For every trace* $[x]$, $\mathsf{Total}(\lessdot_{[x]}) = \{\lhd_t \mid t \in [x]\}$. $\blacksquare$

The partial order defined by the trace $[s]$ from Example 9 is presented in Fig. 2. By Theorems 3 and 11 each trace $[x]$ *uniquely determines* the partial order $\lessdot_{[s]}$, that corresponds to an *occurrence graph* from [34], and *vice versa*.

In this model, simultaneity is defined implicitly. Traces are sets of sequences but it is assumed that if a and b are independent, i.e. $(a, b) \in ind$, they can not only commute, but could be executed simultaneously as well [10,34]. In classification from [21], the paradigm π_8, (commutation implies simultaneity and vice versa) is assumed for Mazurkiewicz traces. For

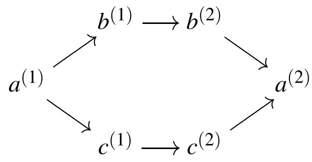


Fig. 2. The partial order $\prec_{[s]}$defined by the trace $[x]$ where $x = abcbca$ and $ind = \{(b, c), (c, b)\}$.

each trace $[x]$, the step sequences that are considered equivalent to sequences from $[x]$ are just step sequence representations of stratified extensions of the partial order $\prec_{[x]}$. They are not part of $[x]$, so the model is simpler, but they can be derived from $[x]$, if needed.

4. Interval traces

Interval traces stem from Mazurkiewicz traces and Fishburn's Theorem (Theorem 4). Traces utilize Szpirlajn's Theorem (Theorem 3) and the fact that finite total orders can be represented by sequences, Fishburn's Theorem allows us to represent interval orders by sequences of beginnings and ends, but not all sequences of beginnings and ends represent interval orders, so some ideas from the previous section must be modified and adapted.

Let Σ be a set of events, $\mathscr{E} = \{Ba \mid a \in \Sigma\} \cup \{Ea \mid a \in \Sigma\}$, and $\mathsf{InSeq}(\mathscr{E}^*)$ be the set of all sequences over $\mathscr{E}$ that define interval orders (see Definition 6(1)).

Definition 12 *(Interval Independency and Interval Trace Alphabet).* Let $ind \subseteq \mathscr{E} \times \mathscr{E}$ be a symmetric and irreflexive relation such that for all $a, b \in \Sigma$

1. $(Ba, Ea) \notin ind$ and $(Ea, Ba) \notin ind$,
2. $(Ba, Bb) \in ind$ and $(Ea, Eb) \in ind$.

Any relation ind that satisfies the properties (1) and (2) above, will be called an ***interval independency***, and the pair $(\mathscr{E}, ind)$ will be called an ***interval trace alphabet***. ■

The condition (1) above follows from the fact that in any representation of any order, the beginning of an event always precedes the end so they cannot commute. The condition (2) follows from the generalization of the observation that the interval sequences $BaBbEaEb$, $BbBaEaEb$, $BaBbEbEa$, and $BbBaEbEa$ represent the same fact, namely that a and b are simultaneous.

Definition 12 differs substantially from the original definition of interval trace independency proposed in [25]. It is simpler and more general, so the definition from [25] can be considered as a special case of Definition 12. The motivations behind these two definitions were also different. Since interval orders can be regarded as a generalization of stratified orders (incomparability may not be an equivalence relation), the definition from [25] treated interval traces as partially stemming from comtraces of [22] (an extension of Mazurkiewicz traces for stratified orders), so it used the relations *sim* and *ser* that are part of comtrace model. In this paper we do not use comtraces as an inspiration, we just show later that our model is equivalent to the comtrace model, if restricted to stratified orders.

Note that $(\mathscr{E}, ind)$ is also a standard trace alphabet, so *we can apply the standard theory of Mazurkiewicz traces*. One of the problems is that not all sequences from $\mathscr{E}^*$ can be interpreted as trace elements. They have to represent interval orders, so only sequences from $\mathsf{InSeq}(\mathscr{E}^*)$ can be used.

Lemma 13. *Let $(\mathscr{E}, ind)$ be an interval trace alphabet.*

1. *For each $x, y \in \mathscr{E}^*$, if $x \in \mathsf{InSeq}(\mathscr{E}^*)$ and $y \in \mathsf{InSeq}(\mathscr{E}^*)$ then $xy \in \mathsf{InSeq}(\mathscr{E}^*)$.*
2. *For each $s \in \mathscr{E}^*$, we have: $s \in \mathsf{InSeq}(\mathscr{E}^*) \iff \forall x \in [s]_{ind}.\ x \in \mathsf{InSeq}(\mathscr{E}^*)$.*
3. *For each $x, y \in \mathscr{E}^*$,*
   if $[x]_{ind} \subseteq \mathsf{InSeq}(\mathscr{E}^*)$ and $[y]_{ind} \subseteq \mathsf{InSeq}(\mathscr{E}^*)$, then $[x]_{ind} * [y]_{ind} = [xy]_{ind} \subseteq \mathsf{InSeq}(\mathscr{E}^*)$.

Proof. (1) Since for each $a \in \Sigma$, $(BaEa)^*(BaEa)^* = (BaEa)^*$.

(2) ($\Leftarrow$) Obvious as $s \in [s]_{ind}$.

($\Rightarrow$) From Definition 12(1) we know that Ba and Ea cannot commute for any $a \in \Sigma$. Hence, if $\pi_{\{Ba,Ea\}}(s) \in (BaEa)^*$ then also $\pi_{\{Ba,Ea\}}(x) \in (BaEa)^*$ for each $x \in [s]_{ind}$.

(3) A consequence of (1) and (2). □

The interval sequence representation of interval orders is not unique, but completeness of the relation $\equiv_{ind}$ requires that all such representations are equivalent, which is given by the following result.

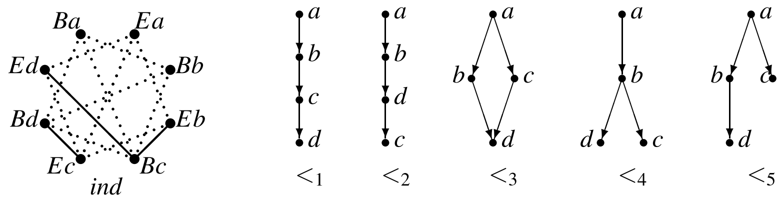


Fig. 3. An interval independency relation *ind* and interval orders generated by an interval trace $[BaEaBbEbBcEcBdEd]_{ind}$. The default part of the relation *ind* given by Definition 12(2) is represented by dotted lines.

Proposition 14. *Let $(\mathcal{E}, ind)$ be an interval trace alphabet, and $x \in \mathsf{InSeq}(\mathcal{E}^*)$, then for each $y \in \mathsf{InSeq}(\mathcal{E}^*)$*

$$\blacktriangleleft_x = \blacktriangleleft_y \implies x \equiv_{ind} y. \quad \blacksquare$$

The proof of the above proposition is in Appendix A as it requires plenty specific results about interval orders, that are not much relevant to the theory of interval traces.

The opposite implication $x \equiv_{ind} y \implies \blacktriangleleft_x = \blacktriangleleft_y$ usually *does not* hold. It holds when the relation *ind* is entirely defined by point (2) of Definition 12, i.e. when

$(\alpha, \beta) \in ind \iff \exists a, b \in \Sigma.(\alpha = Ba \wedge \beta = Bb) \vee (\alpha = Ea \wedge \beta = Eb)$. However if there are $a, b \in \Sigma$ such that $(Ea, Bb) \in ind$, then $x \equiv_{ind} y \implies \blacktriangleleft_x = \blacktriangleleft_y$ may not hold. Consider $x = BaEaBbEb$, $y = BaBbEaEb$ and $(Ea, Bb) \in ind$. Then clearly $x \equiv_{ind} y$, but $\blacktriangleleft_x =$ $a \bullet \!\!\longrightarrow\!\! \bullet b$ differs from $\blacktriangleleft_y =$ $a \bullet \quad \bullet b$.

We can now define *interval trace* as follows:

Definition 15 *(Interval Trace).* A trace $[x]_{ind}$ over the interval trace alphabet $(\mathcal{E}, ind)$ is called an ***interval trace*** if $[x]_{ind} \subseteq \mathsf{InSeq}(\mathcal{E}^*)$. $\blacksquare$

Example 16. Let $\Sigma = \{a, b, c, d\}$ and *ind* be the relation from Fig. 3. Then the set of interval sequences

$$\mathbf{x} = \left\{ \begin{array}{l} \underline{BaEaBbEbBcEcBdEd}, BaEaBbEbBdEdBcEc, BaEaBbBcEbEcBdEd, \\ BaEaBcBbEbEcBdEd, BaEaBcBbEcEbBdEd, BaEaBbBcEcEbBdEd, \\ BaEaBbEbBcBdEcEd, BaEaBbEbBdBcEcEd, BaEaBbEbBdBcEdEc, \\ BaEaBbEbBcBdEdEc, BaEaBbBcEbBdEcEd, BaEaBbBcEbBdEdEc, \\ BaEaBcBbEbBdEdEc, BaEaBcBbEbBdEcEd \end{array} \right\},$$

is an interval trace, $\mathbf{x} = [x]_{ind}$ for any $x \in \mathbf{x}$, for example $x = BaEaBbEbBcEcBdEd$ (underlined above), so $\mathbf{x} = [BaEaBb\,EbBcEcBdEd]_{ind}$. $\blacksquare$

Since every element of every interval trace is an interval sequence, by Theorem 4, every element of the trace defines a unique interval order. However interval orders that are not total are represented by more than one sequence from the trace.

Definition 17. For every interval trace $\mathbf{x} = [x]_{ind}$, let

$$Interv(\mathbf{x}) = \{\blacktriangleleft_t \mid t \in \mathbf{x}\}$$

denote the set of all interval orders defined by the elements of $\mathbf{x}$ (see Definition 6(3) for $\blacktriangleleft_t$). $\blacksquare$

For the interval trace $\mathbf{x}$ from Example 16, $Interv(\mathbf{x}) = \{<_1, <_2, <_3, <_4, <_5\}$, where $<_1, <_2, <_3, <_4$, and $<_5$ are partial orders from Fig. 3, with $a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)}$ represented just by a, b, c, d. In this case

- $BaEaBbEbBcEcBdEd$ represents a total order $<_1$,
- $BaEaBbEbBdEdBcEc$ represents a total order $<_2$,
- each of the sequences $BaEaBbBcEbEcBdEd$, $BaEaBcBbEbEcBdEd$, $BaEaBcBbEcEbBdEd$, and $BaEaBbBcEcEbBdEd$, represents a stratified order $<_3$,
- each of the sequences $BaEaBbEbBcBdEcEd$, $BaEaBbEbBdBcEcEd$, $BaEaBbEbBdBcEdEc$, and $BaEaBbEbBcBdEdEc$ represents a stratified order $<_4$,
- and each of the sequences $BaEaBbBcEbBdEcEd$, $BaEaBbBcEbBdEdEc$, $BaEaBcBbEbBdEdEc$, and $BaEaBcBbEbBdEcEd$ represents the strict interval order $<_5$.

Mazurkiewicz traces are sets of sequences, i.e. sets of total orders, representing equivalent sequential observations. Theorem 11 allows interpreting each trace $[x]$ as a partial order $<_{[x]}$, cf. Fig. 2, that could be interpreted as a concurrent history when a 'true concurrency' model is used [10,34].

Interval traces are sets of interval sequences, i.e. sets of interval orders, that also represent equivalent observations, but are modeled with interval orders instead. We will show that, in this case, each interval trace uniquely defines an *interval order structure*, i.e. a pair of relations, that also can be interpreted as a representation of some concurrent history [18,23].

5. Interval order structures and their partial order representations

While partial orders can adequately model 'earlier–later' relationship, to model 'not later than' relationship we need more sophisticated tools, especially when system runs/observations are represented by interval orders.

Interval order structures provide a more general formalism for analysis of concurrent systems than partial orders and stratified order structures, as discussed in [17,23].

Definition 18 *([20,23,31]).* An ***interval order structure*** is a relational structure $S = (X, \prec, \sqsubset)$, such that for all $a, b, c, d \in X$:

I1: $a \not\sqsubset a$	I4: $a \prec b \sqsubset c \lor a \sqsubset b \prec c \implies a \sqsubset c$
I2: $a \prec b \implies a \sqsubset b$	I5: $a \prec b \sqsubset c \prec d \implies a \prec d$
I3: $a \prec b \prec c \implies a \prec c$	I6: $a \sqsubset b \prec c \sqsubset d \implies a \sqsubset d \lor a = d.$

The relation $\prec$ is called *causality* and the relation $\sqsubset$ is called *weak causality*. ■

The above definition comes from [23] and was derived from its earlier versions of [31] and [20]. Many properties of interval order structures have been presented in [23], yet their theory is not as well-developed and much less often applied than for instance simpler stratified order structures (cf. [17,22,27,32]), not to mention just plain partial orders.

In this model the *causality* relation $\prec$ represents the "earlier than' relationship, and the *weak causality* relation $\sqsubset$ represents the "not later than". We also assume that the system runs are interval orders.The relation $\prec$ is always a partial order, while the relation $\sqsubset$ may not be.

From Definition 18 we can get immediately that, if $<$ is an interval order on X, then $(X, <, <^{\frown})$ is an interval order structure, i.e. interval orders can be interpreted as simple instances of interval order structures.

Definition 19 *([23]).*

1. An interval order $<$ on X is an *interval extension* of an interval order structure $S = (X, \prec, \sqsubset)$ if $\prec \subseteq <$ and $\sqsubset \subseteq <^{\frown}$, i.e. if $<$ is an extension of $\prec$ and $<^{\frown}$ is an extension of $\sqsubset$.
2. The set of all interval extensions of S will be denoted by $\mathsf{Interv}(S)$. ■

Theorem 3 (Szpilrajn Theorem) states that each partial order is uniquely represented by its set of total extensions. We have a similar relationship between interval order structures and interval orders.

Theorem 20 *([23]). For each interval order structure $S = (X, \prec, \sqsubset)$, we have*

$$S = \Big(X, \bigcap_{<\in \mathsf{Interv}(S)} <, \bigcap_{<\in \mathsf{Interv}(S)} <^{\frown}\Big),$$

i.e. S is entirely defined by the set of all its extensions. ■

The above theorem is a generalization of Szpilrajn's Theorem to interval order structures. It shows that if the system's operational semantics is fully described in terms of interval orders, then the interval order structures uniquely represent sets of equivalent system runs (see [17,23] for details).

An example of a simple interval order structure which illustrates the main ideas behind this concept is shown in Fig. 4. The orders $<_1$ and $<_2$ are total, $<_3$ and $<_4$ are stratified, and $<_5$ is interval but not stratified. In the present case $\prec$ equals $<_5$, as there are not so many partial orders over the four elements set, but the interpretations of $<_5$ and $\prec$ are different. The incomparability in $<_5$ is interpreted as *simultaneity* while in $\prec$ as *having no causal relationship*.

It turns out that every interval order structure can be represented by an appropriate partial order of beginnings and ends. We will later use this result to construct a relationship between interval traces and interval order structures.

Theorem 21 *([1]). A triple $S = (X, \prec, \sqsubset)$, with countable X, is an interval order structure if and only if there exists a partial order $<$ on some Y and two mappings $B, E : X \to Y$ such that $B(X) \cap E(X) = \emptyset$ and for each $x, y \in X$:*

1. $B(x) < E(x)$,
2. $x \prec y \iff E(x) < B(y)$,
3. $x \sqsubset y \iff B(x) < E(y) \land x \neq y$. ■

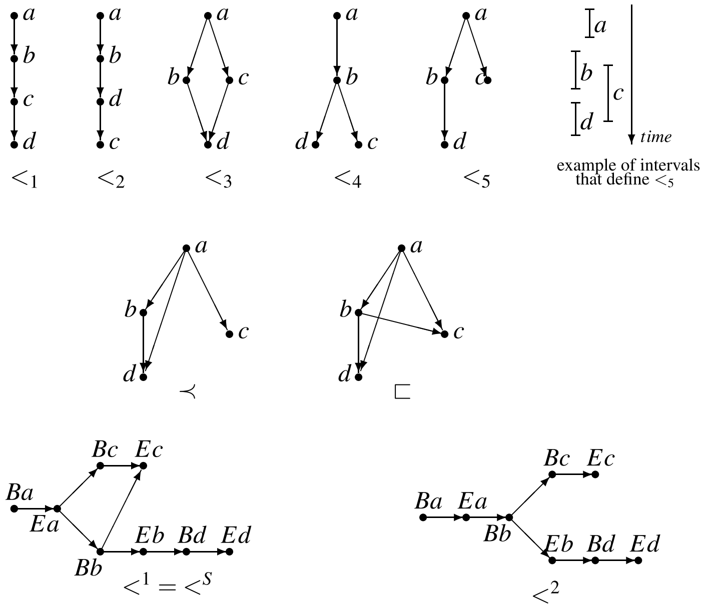


Fig. 4. An example of a simple interval order structure $S = (X, \prec, \sqsubset)$, with $X = \{a, b, c, d\}$. Its set of all interval extensions $\mathsf{Interv}(S)$ equals to $\{<_1, <_2, <_3, <_4, <_5\}$. Partial orders $<^1$ and $<^2$ represent the interval order structure S via Theorem 21. The partial order $<^1$ is also the minimal partial order for S that satisfies Theorem 21.

Theorem 21 can be seen as a generalization of Theorem 4 (Fisburn's Theorem) from interval orders to interval order structures. It will play a crucial role in the next section.

The partial order from Theorem 21 is not unique (see Fig. 4), but the least partial order that satisfies Theorem 21 clearly does exist. It is just an intersection of all partial orders satisfying Theorem 21. Moreover one can show that the original construction from [1] is such least partial order.

6. Interval traces and interval order structures

We will now show the exact relationship between interval traces and interval order structures. We expect this relationship to be similar to the relationships between Mazurkiewicz traces and partial orders, and, as explained in Section 7, between comtraces and stratified order structures of [22]. *This section is the main result of this paper.*

First we recall how one can construct a partial order of beginnings and ends from an interval trace. Assume that a finite set of events Σ and an interval trace alphabet $(\mathscr{E}, ind)$ are given. Recall that for each sequence $x \in \mathscr{E}^*$, $\widehat{\mathscr{E}}_x$ is the set of all elements of $\hat{x}$, the enumerated version of x, $\lhd_x \subseteq \widehat{\mathscr{E}}_x \times \widehat{\mathscr{E}}_x$ is the *total* order that is equivalent to the sequence x (see Definition 5(5)), and $\lessdot_{[x]} \subseteq \widehat{\mathscr{E}}_x \times \widehat{\mathscr{E}}_x$ is the partial order that is equivalent to the trace $[x]$ (see Definition 10).

We are now ready to define an interval order structure induced by a single sequence $x \in \mathscr{E}^*$.

Definition 22. For each $x \in \mathscr{E}^*$, let $S^x = (\widehat{\Sigma}^{\mathscr{E}}_x, \prec_x, \sqsubset_x)$, where $\prec_x$ and $\sqsubset_x$ are relations on $\widehat{\Sigma}^{\mathscr{E}}_x$ (see Definition 6(2)) defined as follows, for all $a, b \in \Sigma$:

1. $a^{(i)} \prec_x b^{(j)} \stackrel{df}{\iff} Ea^{(i)} \lessdot_{[x]} Bb^{(j)}$.
2. $a^{(i)} \sqsubset_x b^{(j)} \stackrel{df}{\iff} Ba^{(i)} \lessdot_{[x]} Eb^{(j)}$. ■

The resemblance of Definition 22 to the points (2) and (3) of Theorem 21 is not a coincidence. The triple $S^x = (\widehat{\Sigma}^{\mathscr{E}}_x, \prec_x, \sqsubset_x)$, is indeed an interval order structure.

Proposition 23. *If $x \in \mathsf{InSeq}(\mathscr{E}^*)$, then $S^x = (\widehat{\Sigma}^{\mathscr{E}}_x, \prec_x, \sqsubset_x)$ is an interval order structure.*

Proof. Since $x \in \mathsf{InSeq}(\mathscr{E}^*)$, then the property (1) of Theorem 21 is satisfied. Definition 22 implies satisfying (2) and (3) of Theorem 21. Hence, by Theorem 21, S^x is an interval order structure. □

We will call $S^x = (\widehat{\Sigma}^{\mathscr{E}}_x, \prec_x, \sqsubset_x)$ the *interval order structure S^x induced by an interval sequence x*. We will show that S^x plays the same role in our model as a partial order derived from a single sequence plays in standard trace theory [34]. To do this we need to show that $x \equiv y \iff S^x = S^y$, and that the set of interval orders $\mathsf{Interv}(S^x)$ is uniquely defined by the elements of $[x]$.

We need the following two lemmas to prove one of our main results. The first lemma is an easy technical one, but it will be used often. It characterizes the relationships $Ba^{(i)} \lessdot_{[x]} Bb^{(j)}$ and $Ea^{(i)} \lessdot_{[x]} Eb^{(j)}$. Because (Ba, Bb) and (Ea, Eb) are in *ind* (Definition 12(2)), these relationships are not arbitrary.

Lemma 24. *For every $x \in \mathsf{InSeq}(\mathscr{E}^*)$, and for all $a^{(i)}, b^{(j)} \in \widehat{\Sigma}^{\mathscr{E}}_x$, we have:*

1. $Ba^{(i)} \lessdot_{[x]} Bb^{(j)} \iff \exists c^{(k)} \in \widehat{\Sigma}^{\mathscr{E}}_x.\ Ba^{(i)} \lessdot_{[x]} Ec^{(k)} \lessdot_{[x]} Bb^{(j)}$,
2. $Ea^{(i)} \lessdot_{[x]} Eb^{(j)} \iff \exists c^{(k)} \in \widehat{\Sigma}^{\mathscr{E}}_x.\ Ea^{(i)} \lessdot_{[x]} Bc^{(k)} \lessdot_{[x]} Eb^{(j)}$.

Proof. 1.($\Leftarrow$) Obvious. ($\Rightarrow$) Since for all $c, d \in \Sigma$, $(Bc, Bd) \in ind$, so all beginnings can commute, there must be some $Ec^{(k)}$ between Ba^i and Bb^j, otherwise $\neg(Ba^{(i)} \lessdot_{[x]} Bb^{(j)})$.

2. Dually, by exchanging B with E. □

The second lemma shows that the relationship between $\lessdot_{[x]}$ and S^x is a one-to-one correspondence.

Lemma 25. *For all $x, y \in \mathsf{InSeq}(\mathscr{E}^*)$, $\lessdot_{[x]} = \lessdot_{[y]}$ if and only if $S^x = S^y$.*

Proof. ($\Rightarrow$) From Definition 22, we clearly have $S^x = S^y$.

($\Leftarrow$) Suppose that $\lessdot_{[x]} \neq \lessdot_{[y]}$. We may assume $\widehat{\mathscr{E}}_x = \widehat{\mathscr{E}}_y$, otherwise $S^x \neq S^y$. This means we have $\alpha \lessdot_{[x]} \beta$ and $\neg(\alpha \lessdot_{[y]} \beta)$, for some $\alpha, \beta \in \widehat{\mathscr{E}}_x$. If $\alpha = Ea^{(i)}$ and $\beta = Bb^{(j)}$, then by Definition 22(1), $\prec_x \neq \prec_y$, if $\alpha = Ba^{(i)}$ and $\beta = Eb^{(j)}$, then by Definition 22(2), $\sqsubset_x \neq \sqsubset_y$; so $S^x \neq S^y$ in both cases. If $\alpha = Ba^{(i)}$ and $\beta = Bb^{(j)}$, then by Lemma 24(1) and Definition 22, there is $c^{(k)}$ such that $a^{(i)} \sqsubset_x c^{(k)} \prec_x b^{(j)}$ but $\neg(a^{(i)} \sqsubset_y c^{(k)} \prec_y b^{(j)})$, so $S^x \neq S^y$ again. Similarly for $\alpha = Ea^{(i)}$ and $\beta = Eb^{(j)}$, but here we use Lemma 24(2). Hence $\lessdot_{[x]} \neq \lessdot_{[y]} \implies S^x \neq S^y$, i.e. $S^x = S^y \implies \lessdot_{[x]} = \lessdot_{[y]}$. □

We are now able to prove one of our main results, namely that every interval trace uniquely determines an interval order structure.

Theorem 26. *For all $x, y \in \mathsf{InSeq}(\mathscr{E}^*)$, $x \equiv y$ if and only if $S^x = S^y$.*

Proof. ($\Rightarrow$) If $x \equiv y$ then $[x] = [y]$, so $\lessdot_{[x]} = \lessdot_{[y]}$. Then by Lemma 25, $S^x = S^y$.

($\Leftarrow$) If $S^x = S^y$ then, by Lemma 25, we have $\lessdot_{[x]} = \lessdot_{[y]}$, and now by Theorem 11, $\{\lhd_t \mid t \in [x]\} = \{\lhd_t \mid t \in [y]\}$. From Definition 5(5) it follows that $t = u \iff \lhd_t = \lhd_u$, so $[x] = [y]$, i.e. $x \equiv y$. □

The above theorem makes possible the following definition.

Definition 27. For each interval trace $[x]$, the interval order structure $S^{[x]}$ induced by $[x]$, in defined as $S^{[x]} = (\widehat{\Sigma}^{\mathscr{E}}_x, \prec_{[x]}, \sqsubset_{[x]}) = S^t = (\widehat{\Sigma}^{\mathscr{E}}_t, \prec_t, \sqsubset_t)$, where $t \in [x]$. ■

Theorem 26 alone is not enough to claim that interval traces can represent all the properties of interval order structures. We also have to show that for any $x \in \mathsf{InSeq}(\mathscr{E}^*)$, $\mathsf{Interv}(S^x)$, the set of all interval order extensions of S^x (see Definition 19) is equal to the set of all interval orders generated via Fishburn's Theorem (Theorem 4) from all $\hat{t}$ (enumerated version of t) such that $t \in [x]$. Interval orders generated by appropriate sequences from $\mathscr{E}^*$, and denoted by $\blacktriangleleft_x$ for $x \in \mathscr{E}^*$, are described by Definition 6(3).

Our second main result is the following.

Theorem 28. *For every $x \in \mathsf{InSeq}(\mathscr{E}^*)$,*

$$\mathsf{Interv}(S^x) = Interv([x]) = \{\blacktriangleleft_t \mid t \in [x]\}.$$

Proof. By definition $Interv([x]) = \{\blacktriangleleft_t \mid t \in [x]\}$.

($\supseteq$) Let $t \in [x]$ and $a^{(i)}, b^{(j)} \in \widehat{\Sigma}^{\mathscr{E}}_x$. By Theorem 26, $S^x = S^t$, so we only have to consider $t = x$. Consider the relation $\prec_x$. We have $a^{(i)} \prec_x b^{(j)} \overset{\text{Definition 22}}{\iff} Ea^{(i)} \lessdot_{[x]} Bb^{(j)} \overset{\text{Definition 10}}{\implies} Ea^{(i)} \lhd_t Bb^{(j)} \overset{\text{Definition 6(3)}}{\iff} a^{(i)} \blacktriangleleft_x b^{(j)}$. Hence, by Definition 2, the relation $\blacktriangleleft_x$ is an extension of $\prec_x$. Let us now consider the relation $\sqsubset_x$. Here we have $a^{(i)} \sqsubset_x b^{(j)} \overset{\text{Definition 22}}{\iff} Ba^{(i)} \lessdot_{[x]} Eb^{(j)} \overset{\text{Definition 10}}{\implies} Ba^{(i)} \lhd_t Eb^{(j)}$. Because $\lhd_t$ is a total order, $Ba^{(i)} \lhd_t Eb^{(j)} \iff \neg(Eb^{(j)} \lhd_t Ba^{(i)})$. But $\neg(Eb^{(j)} \lhd_t Ba^{(i)}) \overset{\text{Definition 6(3)}}{\iff} \neg(b^{(j)} \blacktriangleleft_x a^{(i)}) \iff a^{(i)} \blacktriangleleft^{\frown}_x b^{(j)}$. Hence $a^{(i)} \sqsubset_x b^{(j)} \implies a^{(i)} \blacktriangleleft^{\frown}_x b^{(j)}$, so, by Definition 2, $\blacktriangleleft_x$ an extension of $\sqsubset_x$ as well, which means, now by Definition 19, $\blacktriangleleft_x \in \mathsf{Interv}(S^x)$.

($\subseteq$) We need to show that for each $<\in \mathsf{Interv}(S^x)$ there exists $t_< \in [x]$ such that $< = \blacktriangleleft_{t_<}$. We start with constructing some $t_<$ that satisfies $< = \blacktriangleleft_{t_<}$, and then we will show that our $t_< \in [x]$.

Let $<\in \mathsf{Interv}(S^x)$ and let $\lhd_< \subseteq \widehat{\Sigma}^{\mathcal{E}}_x \times \widehat{\Sigma}^{\mathcal{E}}_x$ be a total order representation of $<$ via Fishburn Theorem (Theorem 4), i.e. $a^{(i)} < b^{(j)} \iff Ea^{(i)} \lhd_< Bb^{(j)}$. Furthermore let $t_< \in \mathcal{E}^*$ be the sequence representation of the total order $\lhd_<$, i.e. $\lhd_< = \lhd_{t_<}$, where $\lhd_{t_<}$ is the total order generated by $t_<$ as in Definition 5(5). Note that, by Definition 6(3), the interval order $<$ equals the interval order $\blacktriangleleft_{t_<}$.

We will now show that $t_< \in [x]$.

Since $<\in \mathsf{Interv}(S^x)$ then $<$ is an extension of $\prec_x$ and $\sqsubset_x$, i.e., by Definition 19, $\prec_x \subseteq <$ and $\sqsubset_x \subseteq <^\frown$. We will show that $\lhd_<$ is a total extension of $\lessdot_{[x]}$, i.e. $\lhd_< \in \mathsf{Total}(\lessdot_{[x]})$. To prove this we will just show that for all $\alpha, \beta \in \{Ba^{(i)}, Ea^{(i)}, Bb^{(j)}, Eb^{(j)}\}$ we have $\alpha \lessdot_{[x]} \beta \implies \alpha \lhd_< \beta$.

First note that from Theorem 21(1) and Theorem 4(1) we have $Ba^{(i)} \lessdot_{[x]} Ea^{(i)}$, $Bb^{(j)} \lessdot_{[x]} Eb^{(j)}$, and $Ba^{(i)} \lhd_< Ea^{(i)}$, $Bb^{(j)} \lhd_< Eb^{(j)}$. Now we have to consider the remaining four cases.

(*Case 1*). Consider $Ea^{(i)}$ and $Bb^{(j)}$. By Definitions 22, 19 and Theorem 4(2), we have: $Ea^{(i)} \lessdot_{[x]} Bb^{(j)} \overset{\text{Definition 22}}{\iff} a^{(i)} \prec_x b^{(j)} \overset{\text{Definition 19}}{\implies} a^{(i)} < b^{(j)} \overset{\text{Th.4(2)}}{\iff} Ea^{(i)} \lhd_< Bb^{(j)}$.

(*Case 2*). Consider $Ba^{(i)}$ and $Eb^{(j)}$. Again by Definitions 22, 19 and Theorem 4(2), we have: $Ba^{(i)} \lessdot_{[x]} Eb^{(j)} \overset{\text{Definition 22}}{\iff} a^{(i)} \sqsubset_x b^{(j)} \overset{\text{Definition 19}}{\implies} a^{(i)} <^\frown b^{(j)} \iff \neg(b^{(j)} < a^{(i)}) \overset{\text{Th.4(2)}}{\iff} \neg(Eb^{(j)} \lhd_< Ba^{(i)}) \iff Ba^{(i)} \lhd_< Eb^{(j)}$.

(*Case 3*). Consider $Ba^{(i)}$ and $Bb^{(j)}$. From Lemma 24(1) it follows: $Ba^{(i)} \lessdot_{[x]} Bb^{(j)} \iff (Ea^{(i)} \lessdot_{[x]} Bb^{(j)}) \vee (\exists c^{(k)} \in \widehat{\Sigma}^{\mathcal{E}}_x.\ Ba^{(i)} \lessdot_{[x]} Ec^{(k)} \lessdot_{[x]} Bb^{(j)})$. From Case 1 we obtain $Ea^{(i)} \lessdot_{[x]} Bb^{(j)} \implies Ea^{(i)} \lhd_< Bb^{(j)}$, i.e., by Theorem 4(2), $Ba^{(i)} \lhd_< Ea^{(i)} \lhd_< Bb^{(j)}$, so $Ba^{(i)} \lhd_< Bb^{(j)}$. Similarly from Case 2 and Case 1 we obtain $Ba^{(i)} \lessdot_{[x]} Ec^{(k)} \lessdot_{[x]} Bb^{(j)} \implies Ba^{(i)} \lhd_< Ec^{(k)} \lhd_< Bb^{(j)} \implies Ba^{(i)} \lhd_< Bb^{(j)}$.

(*Case 4*). Consider $Ea^{(i)}$ and $Eb^{(j)}$. Similarly as Case 3 but using Lemma 24(2) instead.

This means that indeed $\lhd_< \in \mathsf{Total}(\lessdot_{[x]})$. By Theorem 11, $\lhd_< \in \{\lhd_t \mid t \in [x]\}$. But $t_<$ is by the definition a sequence representation of $\lhd_<$, i.e. $\lhd_< = \lhd_{t_<}$, so $t_< \in [x]$, which end the proof of ($\subseteq$). $\square$

Theorems 26 and 28 show that interval traces, i.e. sets of legal sequences of beginnings and ends, correspond to interval order structures in the same way as Mazurkiewicz traces correspond to partial orders (dependency graphs of [34]).

From the definition of S^x (Definition 22), it follows that $\lessdot_{[x]}$ satisfies Theorem 21 for S^x. We will show that in fact $\lessdot_{[x]}$ is the smallest order that does this.

Proposition 29. *For every $x \in \mathsf{InSeq}(\mathcal{E}^*)$, $\lessdot_{[x]}$ is the least partial order that satisfies Theorem 21 for the interval order structure S^x.*

Proof. We will show that for each $<$ that satisfies Theorem 21, and every $\alpha, \beta \in \widehat{\mathcal{E}}_x$, we have $\alpha \lessdot_{[x]} \beta \implies \alpha < \beta$. Since α and β are of the form $Ba^{(i)}$ or $Ea^{(i)}$ where $a \in \Sigma$, we have to consider four cases.

(*Case 1*). $\alpha = Ba^{(i)}, \beta = Eb^{(j)}$. In this case we have

$$Ba^{(i)} \lessdot_{[x]} Eb^{(j)} \overset{\text{Definition 22}}{\iff} a^{(i)} \sqsubset_x b^{(j)} \overset{\text{Th.21}}{\iff} Ba^{(i)} < Eb^{(j)}.$$

(*Case 2*). $\alpha = Ea^{(i)}, \beta = Bb^{(j)}$. Now we have

$$Ea^{(i)} \lessdot_{[x]} Bb^{(j)} \overset{\text{Definition 22}}{\iff} a^{(i)} \prec_x b^{(j)} \overset{\text{Th.21}}{\iff} Ea^{(i)} < Bb^{(j)}.$$

(*Case 3*). $\alpha = Ba^{(i)}, \beta = Bb^{(j)}$. By Lemma 24 we have

$$Ba^{(i)} \lessdot_{[x]} Bb^{(j)} \iff (Ea^{(i)} \lessdot_{[x]} Bb^{(j)}) \vee (\exists c^{(k)} \in \widehat{\Sigma}^{\mathcal{E}}_x.\ Ba^{(i)} \lessdot_{[x]} Ec^{(k)} \lessdot_{[x]} Bb^{(j)}).$$

If $Ea^{(i)} \lessdot_{[x]} Bb^{(j)}$ the case is reduced to Case 2, so assume $(\exists c^{(k)} \in \widehat{\Sigma}^{\mathcal{E}}_x.\ Ba^{(i)} \lessdot_{[x]} Ec^{(k)} \lessdot_{[x]} Bb^{(j)})$. Thus $Ba^{(i)} \lessdot_{[x]} Ec^{(k)} \lessdot_{[x]} Bb^{(j)} \overset{\text{Definition 22}}{\iff} a^{(i)} \sqsubset_x c^{(k)} \prec_x b^{(j)} \overset{\text{Th.21}}{\iff} Ba^{(i)} < Ec^{(k)} < Bb^{(j)} \implies Ba^{(i)} < Bb^{(j)}$, so $Ba^{(i)} \lessdot_{[x]} Bb^{(j)} \implies Ba^{(i)} < Bb^{(j)}$.

(*Case 4*). $\alpha = Ba^{(i)}, \beta = Bb^{(j)}$. Dually to Case 3, by exchanging B with E. $\square$

Let us now analyze a simple example.

Example 30. Let $\Sigma = \{a, b, c, d\}$. Then we have $\mathcal{E} = \{Ba, Ea, Bb, Eb, Bc, Ec, Bd, Ed\}$. Let $ind \subseteq \mathcal{E} \times \mathcal{E}$ be the interval independency from Fig. 3.

Take $x = BaEaBbEbBcEcBdEd \in \mathcal{E}^*$. Since $x \in \mathsf{InSeq}(\mathcal{E}^*)$ then the interval trace $[x]_{ind}$ is defined, and $[x]_{ind} = \mathbf{x}$, where $\mathbf{x}$ is that from Example 16 (it contains fourteen interval sequences).

The interval order structure $S^{[x]} = S^x = (\widehat{\Sigma}^{\mathcal{E}}_x, \prec, \sqsubset)$, where $\widehat{\mathcal{E}}^{\Sigma}_x = \{a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)}\}$, and the relations $\prec$ and $\sqsubset$ are these from Fig. 4, after replacing a with $a^{(1)}$, b with $b^{(1)}$, etc. The set $\widehat{\mathcal{E}}_x = \{Ba^{(1)}, Ea^{(1)}, Bb^{(1)}, Eb^{(1)}, Bc^{(1)}, Ec^{(1)}, Bd^{(1)}, Ed^{(1)}\}$ and the relation $\lessdot_{[x]} \subseteq \widehat{\mathcal{E}}_x \times \widehat{\mathcal{E}}_x$ equals $<^1$ also from Fig. 4, after replacing Ba with $Ba^{(1)}$, Ea with $Ea^{(1)}$, etc.

The set $\mathsf{Interv}(S^{[x]}) = \{<_1, <_2, <_3, <_4, <_5\}$, where $<_1$, $<_2$, $<_3$, $<_4$,x and $<_5$ are interval orders from Fig. 4, again after replacing a with $a^{(1)}$, b with $b^{(1)}$, etc., and clearly $\mathsf{Interv}(S^{[x]}) = Interv([x]_{ind})$, as expected.

Moreover $<_1 = \blacktriangleleft_{BaEaBbEbBcEcBdEd}$, $<_2 = \blacktriangleleft_{BaEaBbEbBdEdBcEc}$, $<_3 = \blacktriangleleft_{BaEaBbBcEbEcBdEd} = \blacktriangleleft_{BaEaBcBbEbEcEdEd} = \blacktriangleleft_{BaEaBcBbEcEbBdEd} = \blacktriangleleft_{BaEaBbBcEcEbBdEd}$, $<_4 = \blacktriangleleft_{BaEaBbEbBcBdEcEd} = \blacktriangleleft_{BaEaBbEbBdBcEcEd} = \blacktriangleleft_{BaEaBbEbBdBcEdEc} = \blacktriangleleft_{BaEaBbEbBcBdEdEc}$, $<_5 = \blacktriangleleft_{BaEaBbBcEbBdEcEd} = \blacktriangleleft_{BaEaBbBcEbBdEdEc} = \blacktriangleleft_{BaEaBcBbEbBdEdEc} = \blacktriangleleft_{BaEaBcBbEbBdEcEd}$.

Finally note that the results would be the same if x would be replaced by any interval sequence $t \in [x]_{ind}$. ■

7. Interval traces vs comtraces

While every stratified order is an interval order, every stratified order structure is an interval order structure [23], and every Mazurkiewicz trace can be interpreted as a simplified comtrace [24], the similar relationship is far more complex for comtraces and interval traces. We start with recalling basic ideas and results of stratified order structures, followed by the same for comtraces.

7.1. Stratified order structures

When all system runs/observations are represented by stratified orders or step sequences, the interval order structures can be replaced by simpler stratified order structures.

Definition 31 *([14,20]).* A *stratified order structure* is a relational structure $S = (X, \prec, \sqsubset)$, such that for all $a, b, c \in X$:

S1: $a \not\sqsubset a$ S3: $a \sqsubset b \sqsubset c \wedge a \neq c \implies a \sqsubset c$

S2: $a \prec b \implies a \sqsubset b$ S4: $a \sqsubset b \prec c \vee a \prec b \sqsubset c \implies a \prec c$.

The relation $\prec$ is called *causality* while $\sqsubset$ is called *weak causality*. ■

Stratified order structures were independently introduced in [14] and [20]. Their comprehensive theory has been presented in [17,23,24,28]. The interpretation of the relations $\prec$ and $\sqsubset$ is the same as for interval order structures, i.e. the *causality* relation $\prec$ represents the "earlier than" relationship, and the *weak causality* relation $\sqsubset$ represents the "not later than" relationship, however in this case we assume that *the system runs are stratified orders*. Similarly as for interval order structures, the relation $\prec$ is always a partial order, while the relation $\sqsubset$ may not be. Moreover, if $<$ is a stratified order on X, then $(X, <, <^{\frown})$ is a stratified order structure, i.e. stratified orders can be interpreted as simple instances of stratified order structures.

Definition 32 *([23]).*

1. A stratified order $<$ on X is an *stratified extension* of an stratified order structure $S = (X, \prec, \sqsubset)$ if $\prec \subseteq <$ and $\sqsubset \subseteq <^{\frown}$, i.e. if $<$ is stratified, it is an extension of $\prec$, and $<^{\frown}$ is an extension of $\sqsubset$.
2. The set of all *stratified extensions* of S will be denoted by $\mathsf{Strat}(S)$. ■

As expected, every stratified order structure is also an interval order structure.

Proposition 33 *([23]).*

1. *Every stratified order structure S is also an interval order structure.*
2. *For every stratified order structure S, $\mathsf{Strat}(S) \subseteq \mathsf{Interv}(S)$.* ■

Theorem 20 states that each interval order structure order is uniquely represented by its set of interval extension extensions. We have the similar relationship between stratified order structures and stratified orders.

Theorem 34 *([23]). For each stratified order structure $S = (X, \prec, \sqsubset)$, we have*

$$S = \Big(X, \bigcap_{<\in \mathsf{Strat}(S)} <, \bigcap_{<\in \mathsf{Strat}(S)} <^{\frown}\Big),$$

i.e. S is entirely defined by the set of all its extensions. ■

The above theorem is a generalization of Szpilrajn's Theorem to stratified order structures (cf. [17,28]).

The relational structure S from Fig. 4 is a also simple example of a stratified order structure with $\mathsf{Strat}(S) = \{<_1, <_2, <_3, <_4\}$ (but $<_5 \notin \mathsf{Strat}(S)$). However not every interval order structure is a stratified order structure. The relational structure $S_0 = (\{a, b, c\}, \prec_0, \sqsubset_0)$, where the relations $\prec_0$ and $\sqsubset_0$ are described below

is an interval order structure, but not a stratified order structure. We have here $a \prec_0 b \sqsubset_0 c$ but $\neg(a \prec_0 c)$, so the axiom $S5$ of Definition 31 is not satisfied.

7.2. Comtraces

Comtraces are extensions of Mazurkiewicz traces that handle both 'not later than' relationship and system runs/observations modeled by step sequences, and can be uniquely represented by stratified order structures [22,28].

Definition 35 *([22]).*

1. Let Σ be a finite set, $ser \subseteq sim \subset \Sigma \times \Sigma$ be two relations called *serialisability* and *simultaneity* respectively. The triple (Σ, sim, ser) is called the *comtrace alphabet*. We assume that sim is irreflexive and symmetric.
2. We define $\mathbb{S}$, the set of all (potential) *steps*, as the set of all cliques of the graph (Σ, sim), i.e. $\mathbb{S} = \{A \mid A \neq \emptyset \wedge (\forall a, b \in A.\, a = b \vee (a, b) \in sim)\}$.
3. Let $\approx \subseteq \mathbb{S}^* \times \mathbb{S}^*$ be the relation comprising all pairs (x, y) of step sequences such that
$$x = x_1 A x_2 \text{ and } y = x_1 B C x_2.$$
where $x_1, x_2 \in \mathbb{S}^*$ and A, B, C are steps in $\mathbb{S}$ satisfying $A = B \cup C$, $B \cup C = \emptyset$ and $B \times C \subseteq ser$.
4. Let $\equiv_{(sim,ser)}$ be the reflexive and transitive closure of $\approx \cup \approx^{-1}$, i.e.
$$\equiv_{(sim,ser)} = (\approx \cup \approx^{-1})^*.$$
Clearly the relation $\equiv_{(sim,ser)}$ is an *equivalence* relation.
5. For every $x \in \mathbb{S}^*$, the equivalence class $[x]_{\equiv_{(sim,ser)}}$ is called a ***comtrace***. ■

We will often write $[x]_{(sim,ser)}$, instead of $[x]_{\equiv_{(sim,ser)}}$. One may show that $[x][y] = [x] \circ [y] = [xy]$, where $\circ$ is a concatenation of sets of step sequences, a symbol that is usually omitted. Formally, an algebra of comtraces is a *quotient equational monoid over step sequences* [22,19].

The comtraces were invented to handle explicitly 'transitive simultaneity' and 'not later than' relationships. The relation sim, called *simultaneity*, is symmetric and irreflexive, the relation ser, called *serializability* is a subset of sim. If $(a, b) \in sim$ then a and b can be executed simultaneously, while $(a, b) \in ser$ means a and b can either be executed simultaneously, or a precedes b. When operational semantics is expressed in terms of stratified orders or step sequences, $(a, b) \in sim$ means the step $\{a, b\}$ is allowed, and $(a, b) \in ser$ means the both the step $\{a, b\}$ and the sequence $\{a\}\{b\}$ are allowed.

If $sim = ser$ then a comtrace can fully be represented by an appropriate Mazurkiewicz trace with $ind = sim$ [22,19].

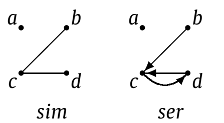
For every comtrace $\mathbf{x} = [x]_{(sim,ser)}$ over (Σ, sim, ser), the set $Strat(\mathbf{x}) = \{\lhd_t \mid t \in \mathbf{x}\}$, is the set of all stratified orders defined by the elements of $\mathbf{x}$, and let $S^{\mathbf{x}} = (\widehat{\Sigma}_{\mathbf{x}}, \prec_{\mathbf{x}}, \sqsubset_{\mathbf{x}})$, be the relational structure given by

$$\prec_{\mathbf{x}} = \bigcap_{<\in Strat(\mathbf{x})} <, \qquad \sqsubset_{\mathbf{x}} = \bigcap_{<\in Strat(\mathbf{x})} <^{\frown}.$$

Theorem 36 *([22]). For every comtrace $\mathbf{x} = [x]_{(sim,ser)}$ over (Σ, sim, ser), the relational structure $S^{\mathbf{x}} = (\widehat{\Sigma}_{\mathbf{x}}, \prec_{\mathbf{x}}, \sqsubset_{\mathbf{x}})$ is a* stratified order structure *and* $\mathrm{Strat}(S^{\mathbf{x}}) = Strat(\mathbf{x})$. ■

The relational structure $S^{\mathbf{x}} = (\widehat{\Sigma}_{\mathbf{x}}, \prec_{\mathbf{x}}, \sqsubset_{\mathbf{x}})$ is a *stratified order structure generated by the comtrace* $\mathbf{x}$. Since $\mathrm{Strat}(S^{\mathbf{x}}) = Strat(\mathbf{x})$, both $\mathbf{x}$ and $S^{\mathbf{x}}$ represent the same behavior. More details about comtrace theory and applications can be found in [18,22,24,28].

For example if $\Sigma = \{a, b, c, d\}$, sim and ser are relations as the ones below:



and $x = \{a\}\{b, c\}\{d\}$ is a step sequence over a comtrace alphabet (Σ, sim, ser), then the set of step sequences

$$[x]_{(sim,ser)} = \{\ \{a\}\{b\}\{c\}\{d\}, \{a\}\{b\}\{d\}\{c\}, \{a\}\{b, c\}\{d\}, \{a\}\{b\}\{c, d\}\ \}$$

is the comtrace generated by the step sequence x. Note that in this case, step sequences $[x]_{(sim,ser)}$, when interpreted as stratified orders, i.e. $Strat([x]_{(sim,ser)})$, satisfy

$$Strat([x]_{(sim,ser)}) = \{<_1, <_2, <_3, <_4\} = \mathrm{Strat}(S),$$

where S is exactly the stratified order structure from Fig. 4.

Moreover, $S = S^{[x]_{(sim,ser)}}$ (cf. [22]).

7.3. Representing comtraces by interval traces

Mazurkiewicz traces can be represented by comtraces with $sim = ser = ind$, and stratified order structures can be regarded as special cases of interval order structures. Do we have a similar relationship between comtraces and interval traces? It turns out this case is more complicated.

Let (Σ, sim, com) be a comtrace alphabet, x a step sequence, and $\mathbf{x} = [x]_{(sim,ser)}$ be a comtrace defined by x.

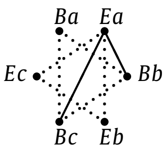
It is usually ***false*** *that there is an interval trace alphabet $(\mathscr{E}, ind)$ and an interval sequence y such that the interval trace $\mathbf{y} = [y]_{ind}$ satisfies $Strat(\mathbf{x}) = Interv(\mathbf{y})$.*

Consider $\Sigma = \{a, b, c\}$, sim and ser as below

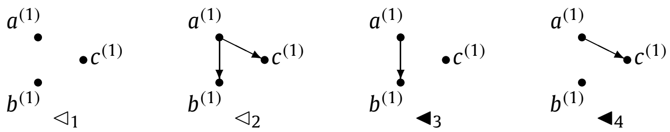
and $\mathbf{x} = [\{a, b, c\}]_{(sim,ser)} = \{\{a, b, c\}, \{a\}\{b, c\}\}$.

Suppose there is a relation ind on $\mathscr{E} = \{Ba, Bb, Bc, Ea, Eb, Ec\}$ and an interval sequence $y \in \mathscr{E}^*$ such that the interval trace $\mathbf{y} = [y]_{ind}$ satisfies $Strat(\mathbf{x}) = Interv(\mathbf{y})$. The stratified order $\lhd_{\{a,b,c\}}$ can be represented by the interval sequence $y_1 = BaBbBcEaEbEc$, so $\mathbf{y} = [y_1]_{ind}$, and the stratified order $\lhd_{\{a\}\{b,c\}}$ can be represented by the interval sequence $y_2 = BaEaBbBcEbEc$, so $y_1, y_2 \in \mathbf{y}$, i.e. we must have $y_1 \equiv_{ind} y_2$. To obtain y_1 from y_2 we must move Ea from after Bc to before Bb, hence $(Ea, Bb) \in ind$ and $(Ea, Bc) \in ind$. But this means that $y_3 = BaBbEaBcEbEc \in \mathbf{y}$ but the order $\blacktriangleleft_{y_3}$ is *not* stratified!

It can be shown by inspection that if $\mathbf{y}$ must contain all interval sequence representations of $\lhd_{\{a,b,c\}}$ and $\lhd_{\{a\}\{b,c\}}$, and the only stratified orders included in $Interv(\mathbf{y})$ are $\lhd_{\{a,b,c\}}$ and $\lhd_{\{a\}\{b,c\}}$, then the relation ind must be as the one below:



The interval trace $\mathbf{y} = [y_1]_{ind}$ generates the set of interval orders $Interv(\mathbf{y}) = \{\lhd_1, \lhd_2, \blacktriangleleft_3, \blacktriangleleft_4\}$, where the orders $\lhd_1 = \lhd_{\{a,b,c\}}$, $\lhd_2 = \lhd_{\{a\}\{b,c\}}$, and $\blacktriangleleft_3$, $\blacktriangleleft_4$ are given below:



The orders $\lhd_1$ and $\lhd_2$ are stratified while $\blacktriangleleft_3$ and $\blacktriangleleft_4$ are not.

However, we have (see Definition 1 for the meaning of $<^\frown$)

$\prec_{\mathbf{x}} = \lhd_1 \cap \lhd_2 = \lhd_1 \cap \lhd_2 \cap \blacktriangleleft_3 \cap \blacktriangleleft_4 = \prec_{\mathbf{y}}$, and

$\sqsubset_{\mathbf{x}} = \lhd_1^\frown \cap \lhd_2^\frown = \lhd_1^\frown \cap \lhd_2^\frown \cap \blacktriangleleft_3^\frown \cap \blacktriangleleft_4^\frown = \sqsubset_{\mathbf{y}}$,

which implies that the stratified order structure $S^{\mathbf{x}}$ and $S^{\mathbf{y}}$ are identical, i.e. $S^{\mathbf{x}} = S^{\mathbf{y}}$, as $S^{\mathbf{x}} = (\{a^{(1)}, b^{(1)}, c^{(1)}\}, \prec_{\mathbf{x}}, \sqsubset_{\mathbf{x}})$ and $S^{\mathbf{y}} = (\{a^{(1)}, b^{(1)}, c^{(1)}\}, \prec_{\mathbf{y}}, \sqsubset_{\mathbf{y}})$.

We show that this pattern holds in general.

For every comtrace alphabet (Σ, sim, ser), let $(\mathscr{E}, ind_{(sim,ser)})$ be an interval trace alphabet such that the relation $ind_{(sim,ser)}$ satisfies:

$$(Bb, Ea) \in ind_{(sim,ser)} \iff (a, b) \in ser.$$

Theorem 37. *Let (Σ, sim, ser) be a comtrace alphabet, x be a step sequence, y be any interval sequence such that $\lhd_x = \blacktriangleleft_y$, $\mathbf{x} = [x]_{(sim,ser)}$, $\mathbf{y} = [y]_{ind}$. Furthermore let $S^{\mathbf{x}} = (\widehat{\Sigma}_x, \prec_{\mathbf{x}}, \sqsubset_{\mathbf{x}})$ be the stratified order structure generated by the comtrace $\mathbf{x}$, and $S^{\mathbf{y}} = (\widehat{\Sigma}_y^{\mathscr{E}}, \prec_{\mathbf{y}}, \sqsubset_{\mathbf{y}})$ be the interval order structure generated by the interval order $\mathbf{y}$.*

Then we have $S^{\mathbf{x}} = S^{\mathbf{y}}$.

Proof. Clearly $\widehat{\Sigma}_x = \widehat{\Sigma}_y^{\mathscr{E}}$. We will show that $\prec_{\mathbf{x}} = \prec_{\mathbf{y}}$ and $\sqsubset_{\mathbf{x}} = \sqsubset_{\mathbf{y}}$.

Let $w = u_1 A u_2 \in \mathbf{x}$, $v = u_1 BC u_2 \in \mathbf{x}$, $A = B \cup C$, $B \cap C = \emptyset$ and $B \times C \subseteq ser$, i.e. $w \approx_{(sim,ser)} v$. Note that $\lhd_w \subseteq \lhd_v$ and $\lhd_{\widehat{v}} \subseteq \lhd_{\widehat{w}}$. Assume $B = \{b_1, \ldots, b_k\}$, $C = \{c_1, \ldots, c_m\}$, so $A = \{b_1, \ldots, b_k, c_1, \ldots, c_m\}$. For every set $X \subseteq \Sigma$, let $\mathsf{B}(X) = \{Ba \mid a \in X\}$ and $\mathsf{E}(X) = \{Ea \mid a \in X\}$. Let w^{itv}, v^{itv}, u_1^{itv} and u_2^{itv} be *some* interval sequence representations of stratified orders $\lhd_w$, $\lhd_u$, $\lhd_{u_1}$ and $\lhd_{u_2}$ respectively, i.e. $\blacktriangleleft_{w^{itv}} = \lhd_w$, $\blacktriangleleft_{u^{itv}} = \lhd_u$, $\blacktriangleleft_{u_1^{itv}} = \lhd_{u_1}$ and $\blacktriangleleft_{u_2^{itv}} = \lhd_{u_2}$. We may assume that $w^{itv} = u_1^{itv} z_A^{\mathsf{B}} z_A^{\mathsf{E}} u_2^{itv}$, where $z_A^{\mathsf{B}} \in perm(\mathsf{B}(A))$ and $z_A^{\mathsf{E}} \in perm(\mathsf{E}(A))$, and $u^{itv} = u_1^{itv} z_B^{\mathsf{B}} z_B^{\mathsf{E}} z_C^{\mathsf{B}} z_C^{\mathsf{E}} u_2^{itv}$, where $z_B^{\mathsf{B}} \in perm(\mathsf{B}(B))$, $z_B^{\mathsf{E}} \in perm(\mathsf{E}(B))$, $z_C^{\mathsf{B}} \in perm(\mathsf{B}(C))$ and $z_C^{\mathsf{E}} \in perm(\mathsf{E}(C))$.

Because $(Bb, Ea) \in ind_{(sim,ser)} \Leftrightarrow (a,b) \in ser$, and $ind_{(sim,ser)}$ satisfies property (2) of Definition 12, we have $w^{itv} \equiv_{ind_{(sim,ser)}} v^{itv}$. Assume $w^{itv} \approx_{ind_{(sim,ser)}} s_1 \approx_{ind_{(sim,ser)}} \ldots \approx_{ind_{(sim,ser)}} s_n \approx_{ind_{(sim,ser)}} u^{itv}$.

Consider $r = r_1 BaEbr_2$ and $t = r_1 EbBar_2$ where $(Ba, Eb) \in ind$. Note that $\blacktriangleleft_r \subseteq \blacktriangleleft_t$ and $\blacktriangleleft_{\widehat{t}} \subseteq \blacktriangleleft_{\widehat{r}}$.

But this means the $\blacktriangleleft_{w^{itv}} \subseteq \blacktriangleleft_{s_i} \subseteq \blacktriangleleft_{u^{itv}}$, and $\blacktriangleleft_{\widehat{u^{itv}}} \subseteq \blacktriangleleft_{\widehat{s_i}} \subseteq \blacktriangleleft_{w^{itv}}$, for all $i = 1, \ldots, n$. Hence $\lhd_w \cap \lhd_u = \blacktriangleleft_{w^{itv}} \cap \blacktriangleleft_{s_1} \cap \ldots \cap \blacktriangleleft_{s_n} \cap \blacktriangleleft_{u^{itv}}$ and $\lhd_{\widehat{w}} \cap \lhd_{\widehat{u}} = \blacktriangleleft_{\widehat{w^{itv}}} \cap \blacktriangleleft_{\widehat{s_1}} \cap \ldots \cap \blacktriangleleft_{\widehat{s_n}} \cap \blacktriangleleft_{\widehat{u^{itv}}}$.

Let $x, x', x_1, \ldots, x_l$ be step sequences such that $x \equiv_{(sim,ser)} x'$ and $x \approx_{(sim,ser)} x_1 \approx_{(sim,ser)} \ldots \approx_{(sim,ser)} x_l \approx_{(sim,ser)} x'$, and let y, y' be interval sequences such that $\lhd_x = \blacktriangleleft_y$ and $\lhd_{x'} = \blacktriangleleft_{y'}$. By the property of $ind_{(sim,ser)}$ we have $y \equiv_{ind_{(sim,ser)}} y'$, so let $y_1, \ldots, y_j$ be interval sequences such that $y \approx_{ind_{(sim,ser)}} y_1 \approx_{ind_{(sim,ser)}} \ldots \approx_{ind_{(sim,ser)}} y_j \approx_{ind_{(sim,ser)}} y'$. From what we have proved above, we may conclude that

$\lhd_x \cap \bigcap_{i=1}^{l} \lhd_{x_i} \cap \lhd_{x'} = \blacktriangleleft_y \cap \bigcap_{i=1}^{j} \blacktriangleleft_{y_i} \cap \blacktriangleleft_{y'}$, and

$\lhd_{\widehat{x}} \cap \bigcap_{i=1}^{l} \lhd_{\widehat{x_i}} \cap \lhd_{\widehat{x'}} = \blacktriangleleft_{\widehat{y}} \cap \bigcap_{i=1}^{j} \blacktriangleleft_{\widehat{y_i}} \cap \blacktriangleleft_{\widehat{y'}}$.

Define $\prec_{xx'} = \lhd_x \cap \bigcap_{i=1}^{l} \lhd_{x_i} \cap \lhd_{x'}$, $\sqsubset_{xx''} = \lhd_{\widehat{x}} \cap \bigcap_{i=1}^{l} \lhd_{\widehat{x_i}} \cap \lhd_{\widehat{x'}}$, and

$\prec_{yy'} = \blacktriangleleft_y \cap \bigcap_{i=1}^{j} \blacktriangleleft_{y_i} \cap \blacktriangleleft_{y'}$, $\sqsubset_{yy'} = \blacktriangleleft_{\widehat{y}} \cap \bigcap_{i=1}^{j} \blacktriangleleft_{\widehat{y_i}} \cap \blacktriangleleft_{\widehat{y'}}$.

Note that $\prec_{\mathbf{x}} = \bigcap_{t \in [x]_{(sim,ser)}} \lhd_t = \bigcap_{x' \in [x]_{(sim,ser)}} \prec_{xx'}$,

$\prec_{\mathbf{y}} = \bigcap_{r \in [y]_{ind_{(sim,ser)}}} \blacktriangleleft_r = \bigcap_{y' \in [x]_{ind_{(sim,ser)}}} \prec_{yy'}$, so $\prec_{\mathbf{x}} = \prec_{\mathbf{y}}$.

Similarly, $\sqsubset_{\mathbf{x}} = \bigcap_{t \in [x]_{(sim,ser)}} \lhd_{\widehat{t}} = \bigcap_{x' \in [x]_{(sim,ser)}} \sqsubset_{xx'}$,

$\sqsubset_{\mathbf{y}} = \bigcap_{r \in [y]_{ind_{(sim,ser)}}} \blacktriangleleft_{\widehat{r}} = \bigcap_{y' \in [x]_{ind_{(sim,ser)}}} \sqsubset_{yy'}$, so $\sqsubset_{\mathbf{x}} = \sqsubset_{\mathbf{y}}$. Hence $S^{\mathbf{x}} = S^{\mathbf{y}}$. $\square$

Theorem 37 states that *while comtraces cannot literally be simulated by interval traces, the stratified order structures they represent, can.*

8. Interval traces as concurrent histories

We claimed that interval traces and the interval order structures induced by them, describe concurrent histories, i.e. sets of equivalent observations. Can we provide any evidence of that?

In general, concurrent behaviors can be investigated at the level of individual observations as well as at the level of some structures, such as causal partial orders, stratified order structures, or interval order structures. These structures capture the essential invariant dependencies between events and represent *complete sets of equivalent observations*. A key link between these two levels comes from the notion of a *concurrent history* [21] which is an *invariant closed* set Δ of observations (system runs). The latter means that Δ can be derived in full from a structure built from simple invariant relationships on events Σ occurring in Δ, such as causality ($a \prec_\Delta b$ if a precedes b in *all* observations in Δ) and weak causality ($a \sqsubset_\Delta b$ if a precedes or is simultaneous with b in *all* observations in Δ).

Formally, a *concurrent history* [21] is defined as follows. Let X be a set and let $\mathfrak{I}(X)$ and $\mathfrak{S}(X)$ denote, respectively, sets of all *interval orders* and *stratified orders* on X. Let $\Delta \subseteq \mathfrak{I}(X)$ with its elements interpreted as observations (system runs).

An *invariant* of Δ, $R \in inv(\Delta)$, is any relation $R \subseteq X \times X$ defined by:

$$(x, y) \in R \iff \forall < \in \Delta.\ \Phi_R(x, y, <),$$

where $\Phi_R(x, y, <)$ is any propositional formula built from atoms $x < y$, $y < x$, $x \frown_< y$ and $True$. For example, a formula $\Phi_R(x, y, <) = x < y \lor x \frown_< y$ generates an invariant $R = \bigcap_{<\in\Delta} (< \lor \frown_<) = \{(x, y) \mid \forall < \in \Delta.\, x < y \lor x \frown_< y\} \in inv(\Delta)$.

In principle, an invariant is a precedence property that is shared by all elements of the set Δ.

Despite a seemingly general definition, there are only at most eight different relational invariants, and at most two of them are independent, i.e. they cannot be obtained from each other by using the standard set theory operators union, intersection and complement [21].

We say that a set $\Delta_{interval}^{inv}$ is the *interval invariant closure* of $\Delta \subseteq \mathfrak{I}(X)$ if and only if:

$$< \in \Delta_{interval}^{inv} \iff [< \in \mathfrak{I}(X)] \land [\forall R \in inv(\Delta). \forall x, y \in X.\ (x, y) \in R \implies \Phi_R(x, y, <)].$$

If $\Delta \subseteq \mathfrak{S}(X)$, we define the *stratified invariant closure* Δ_{strat}^{inv} in the same way, replacing $\mathfrak{I}(X)$ with $\mathfrak{S}(X)$.

Definition 38 *([21])*.

1. A set $\Delta \subseteq \mathfrak{I}(X)$ is an ***interval concurrent history*** iff $\Delta = \Delta^{inv}_{interval}$.
2. A set $\Delta \subseteq \mathfrak{S}(X)$ is an ***stratified concurrent history*** iff $\Delta = \Delta^{inv}_{interval}$. $\square$

For every set of partial orders Δ over X, we define: $\prec_\Delta = \bigcap_{<\in\Delta} <$, $\sqsubset_\Delta = \bigcap_{<^\frown\in\Delta} <^\frown$, and $S_\Delta = (X, \prec_\Delta, \sqsubset_\Delta)$.

Proposition 39 *([23]). Let $\Delta \in \mathfrak{I}(X)$, and let π_Δ be the following predicate:*

$$\pi_\Delta = (\exists < \in \Delta.\, x < y) \wedge (\exists < \in \Delta.\, y < x) \implies (\exists < \in \Delta.\, x \frown_< y).$$

1. *The following two conditions are equivalent:*
   (a) *$\Delta = \Delta^{inv}_{interval}$ and π_Δ is true, i.e. Δ is an interval concurrent history satisfying π_Δ.*
   (b) *S_Δ is an interval order structure and $\Delta = \mathsf{Interv}(S_\Delta)$.*
2. *The following two conditions are equivalent:*
   (a) *$\Delta = \Delta^{inv}_{strat}$ and π_Δ is true, i.e. Δ is a stratified concurrent history satisfying π_Δ.*
   (b) *S_Δ is a stratified order structure and $\Delta = \mathsf{Strat}(S_\Delta)$.* $\square$

The predicate π_Δ used in Proposition 39 is called a *concurrency paradigm* in [17,19,21,23]. In principle, a paradigm describes how simultaneity (represented by $\frown_<$ for $< \in \Delta$) is handled in concurrent histories. The most popular concurrency paradigm, used both in true concurrency and interleaving models, often called a 'diagonal rule' is the following [6,10,17,41]:

$$(\exists < \in \Delta.\, x < y) \wedge (\exists < \in \Delta.\, y < x) \iff (\exists < \in \Delta.\, x \frown_< y).$$

Due to the $\iff$ between the left and right side, the above paradigm may make, in some circumstances, the concept of simultaneous executions redundant, as they might be fully represented by interleavings from the left hand side. This is exploited by popular interleaving models (cf. [6]) and Mazurkiewicz traces [10]. When the 'diagonal rule' paradigm is assumed for concurrent histories, neither interval nor stratified order structures are needed as in this case they are just partial orders in disguise, namely in both cases: $\sqsubset = \prec^\frown$ [17,21,23]. Moreover, if the 'diagonal rule' paradigm holds, both comtraces of [22] and interval traces of this paper can be represented by standard Mazurkiewicz traces (a simple consequence of Theorems 11, 36, 28, about equivalence of trace based model and relation based models).

However, if we assume 'diagonal rule' paradigm, we cannot express 'not later than' phenomenon as simultaneity always implies both interleavings. The paradigm π_Δ allows expressing conveniently the 'not later than' phenomenon. Nevertheless a question that has to be asked is 'do we need the assumption π_Δ at all?'. While Theorems 20 and 34 do not have any assumption like π_Δ, if π_Δ does not hold for $\mathsf{Interv}(S)$ or $\mathsf{Strat}(S)$ respectively, the sets $\mathsf{Interv}(S)$ and $\mathsf{Strat}(S)$ may not be appropriate concurrent histories (cf. [17,21]). With the general case, one without π_Δ or any other assumption about treatment of simultaneity, different more general relational structures and more sophisticated traces must be used. The case when all observations/system runs are assumed to be represented by step sequences, i.e. stratified orders, has been analyzed in detail in [19,24,32], and a complete solution in terms of so called 'invariant structures' and 'step traces' was proposed very recently in [19]. Stratified order structures are special cases of invariant structures and both comtraces and Mazurkiewicz traces are special cases of step traces. The case when all observations/system runs are represented by interval orders is an open research problem, with some very preliminary results presented in [17].

In the rest of this paper we will assume that interval order structures and stratified order structures represent concurrent histories, i.e. the paradigm π_Δ holds.

For example $\Delta_1 = \{<_1, <_2, <_3, <_4\}$ and $\Delta_2 = \{<_1, <_2, <_3, <_4, <_5\}$, where $<_1$, $<_2$, $<_3$, $<_4$, and $<_5$ are these from Fig. 4, are concurrent histories and $S_{\Delta_1} = S_{\Delta_2} = S$ as $\mathsf{Strat}(S_{\Delta_1}) = \Delta_1$ and $\mathsf{Interv}(S_{\Delta_2}) = \Delta_2$. However $\Delta_4 = \{<_2, <_3, <_4\}$ for example is *not* a concurrent history as $S_{\Delta_4} = S$ and $\mathsf{Strat}(S_{\Delta_4}) = \Delta_1 \neq \Delta_4$. For more details the reader is referred to [17,21,23,18].

9. Analysis of a toy example

Consider the following simple program written using Dijkstra's *cobegin*'s and *coend*'s.

```
Q: cobegin
   a : begin worka; lock(r) end;
   b : begin unlock(r); workb end;
   []
   c : workc
   coend
```

Assume that the subroutines a, b, and c are atomic, and *worka*, *workb*, and *workc* require the resource r, which can be used simultaneously by any finite number of subroutines. The resource r is initially unlocked and available to use. Clearly none of the subroutines a, b, and c is instantaneous, they all need some time intervals to execute.

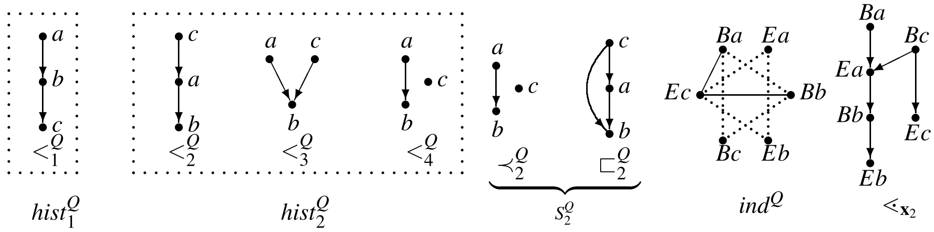


Fig. 5. Behavioural properties of the program Q: its two concurrent histories, $hist_1^Q$ and $hist_2^Q$, the interval order structure S_2^Q that represents the history $hist_2^Q$, interval independency relation ind^Q and the partial order $\lessdot_{\mathbf{x}_2}$ generated by the interval trace $\mathbf{x}_2 = [BaBcEaBbEbEc]_{ind^Q}$. The partial order $\lessdot_{\mathbf{x}_2}$ also defines S_2^Q via Theorem 21.

The program Q illustrates the difficulties of modeling 'simultaneity' and 'not later than' relationships when no restrictions on the shape of system runs is assumed. For time intervals 'simultaneity' is standardly modeled by intervals overlapping [3], an approach used in our concept of interval traces. It can be shown by inspection that, if some specific assumptions about time and event duration are made, the program Q could have two potential *sequential* runs (observations), represented by sequences abc and cab, and two non-sequential runs (observations), one where a and c are executed simultaneously first and then are followed by b, and the other where c is simultaneously executed with a sequence a followed by b. In Fig. 5, these runs are represented by total orders $<_1^Q$ and $<_2^Q$, by a stratified order $<_3^Q$ and by an interval order $<_4^Q$, respectively. We may say that the program Q, all possible observations (system runs) that involve all three events a, b, c are represented by the set of partial orders $Obs(Q) = \{<_1^Q, <_2^Q, <_3^Q, <_4^Q\}$.

One can check by inspection that sets $hist_1^Q = \{<_1^Q\}$ and $hist_2^Q = \{<_2^Q, <_3^Q, <_4^Q\}$ are concurrent histories as $hist_1^Q = (hist_1^Q)^{inv}_{interval}$ and $hist_2^Q = (hist_2^Q)^{inv}_{interval}$, and additionally $Obs(Q) = hist_1^Q \cup hist_2^Q$. Moreover $hist_1^Q$ trivially satisfies both 'diagonal rule' and $\pi_\triangle$, while $hist_2^Q$ does not satisfy the 'diagonal rule' as for example $b(<_4^Q)^\frown c$ and for all $< \in hist_2^Q$ we have $\neg(b < c)$. However $hist_2^Q$ satisfies $\pi_\triangle$ as for $\triangle = hist_2^Q$, the predicate $(\exists < \in \triangle.\, x < y) \wedge (\exists < \in \triangle.\, y < x)$ is always false.

It can also be shown by inspection that $hist_1^Q = \mathsf{Interv}(S_{hist_1^Q})$, $hist_2^Q = \mathsf{Interv}(S_{hist_2^Q})$, and $S_{hist_2^Q} = S_2^Q$, where $S_2^Q = (\{a, b, c\}, \prec_2^Q, \sqsubset_2^Q)$ is shown in Fig. 5. The history $hist_1^Q$ represents system runs (observations) where a occurs before c (or c is later than a), while the history $hist_2^Q$ represents observations where c is not later than a. The history $hist_1^Q$ comprises only one observation, a total order $<_1^Q$, while $hist_2^Q$ contains three observations, a total order $<_2^Q$, a stratified order $<_3^Q$ and an interval (but not stratified) order $<_4^Q$.

Analyzing the program Q for possible commuting (independent) beginnings and ends, we observe that, if mandatory relationships enforced by Definition 12 are not counted, only Bb and Ec may commute, which results in the interval independency relation ind^Q from Fig. 5. As $hist_1^Q = \{<_1^Q\}$, its representation in any model is trivial. For example the interval trace $\mathbf{x}_1 = [BaEbBbEbBcEc]_{ind^Q}$, or the interval order structure $S_1^Q = (\{a, b, c\}, <_1^Q, <_1^{Q\frown})$ both describe $hist_1^Q$, as $Interv(\mathbf{x}_1) = \mathsf{Interv}(S_1^Q) = \{<_1^Q\}$.

The concurrent history $hist_2^Q = \{<_2^Q, <_3^Q, <_4^Q\}$ is uniquely represented by the *interval order structure* $S_2^Q = (\{a, b, c\}, \prec_2^Q, \sqsubset_2^Q)$ from Fig. 5. One can verify by inspection that the set of all interval order extensions of S_2^Q satisfies $\mathsf{Interv}(S_2^Q) = hist_2^Q = \{<_2^Q, <_3^Q, <_4^Q\}$. Moreover the set of interval sequences $\mathbf{x}_2$ defined as

$$\mathbf{x}_2 = \left\{ \begin{array}{l} BcEcBaEaBbEb,\ BaBcEcEaBbEb,\ BaBcEaEcBbEb,\ BcBaEcEaBbEb, \\ BcBaEaEcBbEb,\ \underline{BaBcEaBbEbEc},\ BaBcEaBbEcEb,\ BcBaEaBbEbEc, \\ BcBaEaBbEcEb \end{array} \right\}.$$

is an interval trace over the interval trace alphabet $(\{Ba, Ea, Bb, Eb, Bc, Ec\}, ind^Q)$, where ind^Q is the interval independency relation from Fig. 5. Hence $\mathbf{x}_2 = [x]_{ind^Q}$ for any $x \in \mathbf{x}$, for example $x = BaBcEaBbEbEc$ (underlined), so $\mathbf{x}_2 = [BaBcEaBbEbEc]_{ind^Q}$. Since $\mathsf{Interv}(\mathbf{x}_2) = hist_2^Q$, the trace $\mathbf{x}_2$ represents the history $hist_2^Q$. The partial order $\lessdot_{\mathbf{x}_2}$ that represents both the interval order structure $S^{\mathbf{x}_2}$ and the interval order $\mathbf{x}_2$, is also shown in Fig. 5.

Interval sequences from $S^{\mathbf{x}_2}$ represent the elements of $hist_2^Q$ as follows:

- $BcEcBaEaBbEb$ represents the total order $<_2^Q$,
- each of the sequences $BaBcEcEaBbEb$, $BaBcEaEcBbEb$, $BcBaEcEaBbEb$ and $BcBaEaEcBbEb$ represents the stratified order $<_3^Q$, and
- each of the sequences $BaBcEaBbEbEc$, $BaBcEaBbEcEb$, $BcBaEaBbEbEc$ and $BcBaEaBbEcEb$ represents the strict interval order $<_4^Q$.

Analyzing systems or programs is usually easier if they can be represented by some formal models with well developed semantics, as various types of automata, Petri nets, process algebras, etc. In the next section we show that interval traces can be used as a convenient semantics of some version of Petri nets with inhibitor arcs.

10. Inhibitor Peri nets and their interval traces semantics

Inhibitor arcs allow a transition to check for an *absence* of a token. They have been introduced in [2] to solve a synchronization problem not expressible in classical Petri nets. In principle they allow 'test for zero', an operator the standard Petri nets do not have (cf. [36]).

Elementary nets with inhibitor arcs [22] are very simple. They are just classical *elementary nets* of [37,42], i.e. one-safe place-transition nets without self-loops (cf. [9]), extended with inhibitor arcs. Nevertheless they can easily express complex behaviors involving 'not later than' cases [5,22,27,28], priorities, various versions of simultaneities, etc. [17,23,49]. However the expressiveness of elementary nets with inhibitor arcs is often misunderstood and misinterpreted. While for most known models each elementary net with inhibitor arcs can always equivalently be represented by an appropriate elementary net with *activator*[4] arcs [22,28], the activator arcs *can not* always be simulated by self-loops. If only firing sequences, i.e. languages, generated by nets are concerned, then both inhibitor and activator elementary nets can be represented by equivalent one-safe nets with self-loops. However this is absolutely not true if simultaneous executions, for instance steps, are allowed (cf. [5,22,49]). Moreover, we will show later in Section 10.5 that the relationship between inhibitor nets and activator nets is more complex than it was assumed in the existing models.

In general, the theory of Petri nets does not make any specific assumptions about the nature of transitions. They can represent both instantaneous and non-instantaneous (i.e. lasting some time) entities (cf. [39,9,41]). If we are interested only in firing sequence semantics, as for example in popular [36], then the distinction between instantaneous or non-instantaneous transitions is often negligible. However for step sequences [5,22,28] or interval orders [23] this distinction is important. In this paper we assume that the transitions of standard elementary nets with inhibitor arcs, like the net N_Q in Fig. 6, are not instantaneous, their execution takes some time. In principle this is the same model that was used in [22,28,49] for example.

In this section we will show how some rich semantics, that includes non-transitive simultaneity, of elementary inhibitor nets can be expressed with interval traces.

10.1. Inhibitor Petri nets and their standard operational semantics

An inhibitor net is a tuple $N = (P, T, F, I, m_0)$, where P is a set of *places*, T is a set of *transitions*, P and T are disjoint, $F \subseteq (P \times T) \cup (T \times P)$ is a *flow relation*, $I \subseteq P \times T$ is a set of *inhibitor arcs*, and $m_0 \subseteq P$ is the *initial marking*. An inhibitor arc $(p, e) \in I$ means that e can be enabled only if p is not marked. In diagrams (p, e) is indicated by an edge with a small circle at the end. Any set of places $m \subseteq P$ is called a *marking*.

For every $x \in P \cup T$, the set ${}^\bullet x = \{y \mid (y, x) \in F\}$ denotes the *input* nodes of x and the set $x^\bullet = \{y \mid (x, y) \in F\}$ denotes the *output* nodes of x. The set $x^\circ = \{y \mid (x, y) \in I \cup I^{-1}\}$ is the set of nodes connected by an inhibitor arc to x. The dot-notation extends to sets in the natural way, e.g. the set $X^\bullet$ comprises all outputs of the nodes in X. We assume that for every $t \in T$, both ${}^\bullet t$ and $t^\bullet$ are non-empty and disjoint. These requirements do not always appear in the literature, but following [37,42] we use them for two reasons. Firstly because they are quite natural, and secondly because they allow us to avoid many unnecessary technicalities (cf. [42]). Additionally, both of ${}^\bullet t$ and $t^\bullet$ must have an empty intersection with t°.

Example 40. The tuple $N_P = (P, T, F, I, m_0)$, with $P = \{s_1, s_2, s_3, s_4, s_5\}, T = \{a, b, c\}$, $F = \{(s_1, a), (a, s_3), (s_2, c), (c, s_4), (s_3, b), (b, s_5)\}$, $I = \{(s_3, c)\}$ and $m_0 = \{s_1\}$ is an inhibitor net. This is the net N_Q from Fig. 6. We have here ${}^\bullet a = \{s_1\}$, $a^\bullet = \{s_3\}$, ${}^\bullet b = \{s_3\}$, $b^\bullet = \{s_5\}$, ${}^\bullet c = \{s_2\}$, $c^\bullet = \{s_4\}$, ${}^\bullet s_1 = \emptyset$, $s_1^\bullet = \{a\}$, ${}^\bullet s_2 = \emptyset$, $s_2^\bullet = \{c\}$, ${}^\bullet s_3 = \{a\}$, $s_3^\bullet = \{c\}$, ${}^\bullet s_4 = \{c\}$, $s_4^\bullet = \emptyset$, ${}^\bullet s_5 = \{b\}$, $s_5^\bullet = \emptyset$, and $s_3^\circ = \{c\}$, $c^\circ = \{s_3\}$. ■

We will show in this section that the net N_Q from Fig. 6 can be considered as a model of the program Q analyzed in Section 9 as it has the same behavioral properties as the properties of the program Q (illustrated in Fig. 5). The orders $<_1^Q$, $<_2^Q$, and $<_3^Q$, represented by step sequences $\{a\}\{b\}\{c\}$, $\{c\}\{a\}\{b\}$, and $\{a, c\}\{b\}$ can easily be derived by practically any step sequence semantics proposed for inhibitor nets (cf. [5,22,49]), and the model proposed later in this section will allow us to derive the rest of the properties depicted in Fig. 5.

Since in our model transitions are not instantaneous, one may imagine 'holding a token' when firing particular transitions. For the net N_Q, holding a token in c may overlap with holdings a token in a and next in b, and then a possible outcome could be for example an interval sequence $BaBcEaBbEbEc$, or the interval order $<_4^Q$. The idea of 'holding a token' is not new, it is used in all semantics of timed Petri nets where time is assigned to transitions (cf. [50,52]). We will formally embed the idea of 'holding a token' in our interval trace semantics of elementary nets with inhibitor arcs.

[4] *Activator arcs* (also called 'read', or 'contextual' arcs [5,35,49]), formally introduced in [22,35], are conceptually orthogonal to the inhibitor arcs, they allow a transition to check for a *presence* of a token.

We will now briefly recall the firing sequences and the firing step sequences semantics. We will use the notation from [22,28], which in principle is the notation used for elementary nets [37,42] extended with rules for handling inhibitor arcs. The rules are simple but use the assumption that both ${}^\bullet t$ and $t^\bullet$ are non-empty and disjoint.

10.1.1. Firing sequence and firing step sequence semantics

The *firing sequences semantics*, the simplest operational semantics, is defined in almost the same way as any other kind of Petri nets. The only difference is that for the inhibitor nets, a transition can be enabled only if no place with which it is joined by an inhibitor arc is marked.

Formally, a transition t is *enabled* at marking m if ${}^\bullet t \subseteq m$ and $(t^\bullet \cup t^\circ) \cap m = \emptyset$. For each marking m, the set of all enabled transitions at m is denoted by $enabled_N(m)$.

An enabled t can *occur* leading to a new marking $m' = (m \setminus^\bullet t) \cup t^\bullet$, which is denoted by $m[t\rangle m'$.

A *firing sequence* from the marking m_1 to m_{k+1} is any sequence of transitions $t_1...t_k$ for which there are markings $m_2, ..., m_k$ satisfying: $m_1[t_1\rangle m_1[t_2\rangle m_2...m_k[t_k\rangle m_{k+1}$.

In such case we write: $m_1[t_1...t_k\rangle m_{k+1}$.

The set of all firing sequences from the marking m to the marking m' is defined as

$$\mathsf{FS}_N(m \to m') = \{x \in T^* \mid m[x\rangle m'\}.$$

The *firing step sequence semantics* is defined in a similar fashion. The only difference is that sets of mutually independent transitions can be fired simultaneously.

Let $A \subseteq T$ be a non-empty set such that for all distinct $t, r \in A$, we have $(t^\bullet \cup^\bullet t) \cap (r^\bullet \cup^\bullet r) = \emptyset$. Every such set A is called a *step*, and a step A is a *step enabled* at marking m if ${}^\bullet A \subseteq m$ and $(A^\bullet \cup A^\circ) \cap m = \emptyset$. For each marking m, the set of all step enabled sets of transitions at m is denoted by $senabled_N(m)$.

We also denote $m[A\rangle m'$, where $m' = (m \setminus^\bullet A) \cup A^\bullet$.

A *firing step sequence* from the marking m_1 to m_{k+1} is any sequence of non-empty sets of transitions $A_1...A_k$ for which there are markings $m_2, ..., m_k$ satisfying:

$m_1[A_1\rangle m_1[A_2\rangle m_2...m_k[A_k\rangle m_{k+1}$.

In such case we may write: $m_1[A_1...A_k\rangle m_{k+1}$.

The set of all firing step sequences from the marking m to the marking m' is defined as follows:

$$\mathsf{FSS}_N(m \to m') = \{x \in (\mathscr{P}(T) \setminus \emptyset)^* \mid m[x\rangle m'\}.$$

It can easily be shown [22] that $\{\{a_1\} \ldots \{a_k\} \mid a_1 \ldots a_k \in \mathsf{FS}_N(m \to m') \subseteq \mathsf{FSS}_N(m \to m')$, which, by a small abuse of notation can be stated as $\mathsf{FS}_N(m \to m') \subseteq \mathsf{FSS}_N(m \to m')$.

For example for the net N_Q from Fig. 6 where the initial marking is $\{s_1, s_2\}$ and the final marking $\{s_4, s_5\}$, we have

$$\mathsf{FS}_{N_Q}(\{s_1, s_2\} \to \{s_4, s_5\}) = \{abc, cab\},$$

$$\mathsf{FSS}_{N_Q}(\{s_1, s_2\} \to \{s_4, s_5\}) = \{\ \{a\}\{b\}\{c\}, \{c\}\{a\}\{b\}, \{a, c\}\{b\}\ \}.$$

When sequences and step sequences are replaced by the partial orders they describe, we get $\{<_1^Q, <_2^Q\}$ for FS_{N_Q} and $\{<_1^Q, <_2^Q, <_3^Q\}$ for FFS_{N_Q} (see Fig. 5).

We will now show how the idea of 'holding a token' can be implemented by using the concept of an interval sequence.

10.2. Firing interval sequences semantics

As it was aptly stated in [49]: "*If transitions have a beginning and an end, a system state cannot adequately be described by a marking alone; instead, it consists of a marking together with some transitions that have started, but have not finished yet*". One way of describing such system state is the concept of ST-marking, proposed in [47] and explored in [45,49]. For a given net (with or without inhibitor arcs), an *ST-marking* is a pair (m_{ST}, c_{ST}), where $m_{ST} \subseteq P$ is a marking of N and $c_{ST} \subseteq T$ is the set of *currently firing* transitions. The ST-markings can be defined for general place/transitions nets [41], that are not considered in this paper.

In this paper we use only *elementary* inhibitor nets, so we opted for a simpler model that is briefly presented in Fig. 6. If inhibitor arcs are not involved, to represent transitions by their beginnings and ends we might just replace each transition $\boxed{t}$ by the net $\boxed{Bt} \to (t) \to \boxed{Et}$ as proposed for example in [52] for Timed Petri nets.

However, the inhibitor arcs cause some problems. Consider our example net N_Q. The 'obvious' transformation of the net N_Q into the net N_Q^2 (in N_Q a token in s_3 prevents c from being enabled, so a token in s_3 but in N_Q^2 prevents *starting* c, i.e. Bc is not enabled), *does not work for at least two reasons*. No matter how we implement 'holding a token', we want our model to be consistent with most well established step sequence semantics of elementary inhibitor nets. *None* of the well established step sequence semantics of elementary inhibitor nets consider the step sequence $\{a\}\{b, c\}$ (or stratified order $<_5^{\neg Q}$ from Fig. 5) to be a legal step sequence generated by the net N_Q [5,22,19,49]. But the interval sequence $BaEaBbBcEbEc$ is a firing sequence of N_Q^2 and represents the stratified order $<_5^{\neg Q}$. Moreover the interval sequence $BaEaBcBbEbEc$ is *not* a firing sequence of N_Q^2, but it *also represents* the stratified order $<_5^{\neg Q}$.

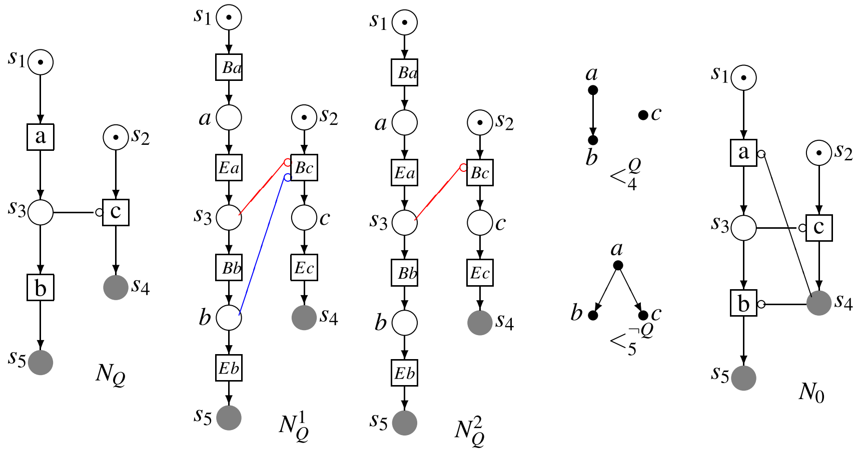


Fig. 6. Elementary inhibitor nets N_Q and N_0, and two potential models of 'holding a token' idea for the net N_Q. Both N_Q^1 and N_Q^2 fire all interval sequences that represent the strict interval order $<_4^Q$, N_Q^1 also fires two (out of four) interval sequences generating the stratified order $<_5^Q$. All firing sequences (starting from $\{s_1, s_2\}$ and ending at $\{s_4, s_5\}$) of the net N_0 represent the strict interval order $<_4^Q$. For the net N_0 the sets of total and interval orders it generates, are empty.

In other words, 'holding a token' is more restrictive when inhibitor arcs are involved. When ${}^\bullet t_1 \cap t_2^\circ \neq \emptyset$ and t_1 holds a token but t_2 does not, then t_2 cannot start until t_1 ends. Otherwise 'holding a token' is inconsistent with probably all reasonable step sequence semantics. Transformation from N_Q to N_Q^1 preserves this property. One may verify that $\mathsf{FS}_{N_Q^1}(\{s_1, s_2\} \to \{s_4, s_5\}) = \{x_1\} \cup \mathbf{x}_2$, where $x_1 = BaEaBbEbBcEc$, and $\mathbf{x}_2$ is the interval trace that represents the concurrent history $hist_2^Q$ and is discussed in Section 9. In other words, $\mathsf{FS}_{N_Q^1}(\{s_1, s_2\} \to \{s_4, s_5\})$ is the interval sequence representation of $Obs(Q) = hist_1^Q \cup hist_2^Q$, and this is what we expect. For N_Q^2 we have $\mathsf{FS}_{N_Q^2}(\{s_1, s_2\} \to \{s_4, s_5\}) = \{x_1\} \cup \mathbf{x}_2 \cup \{y_1, y_2\}$, where $y_1 = BaEaBbBcEcEb$, $y_2 = BaEaBbBcEbEc$, and for each $x \in \{y_1, y_2\}$, $\blacktriangleleft_x = <^{\neg Q_5}$. The remaining two interval sequences that also define $<^{\neg Q_5}$, namely $BaEaBcBbEcEb$ and $BaEaBcBbEbEc$ are not firing sequences of N_Q^2.

While defining the meaning of one entity by transforming it into another is good for providing intuition and motivation, it is not necessarily a good way to do it in a general case. Hence we will formally define firing interval sequences, $\mathsf{FIS}_N(m \to m')$, in terms of the net N alone, without explicitly using the transformation illustrated in Fig. 6 (from N_Q into N_Q^1). The key idea is to allow tokens not only in places but in transitions as well. A token in a transition t could be interpreted as 't is active', and removing all tokens from ${}^\bullet t$ and placing one token in t can be interpreted as an execution of Bt, while removing the token from t and placing tokens in $t^\bullet$ can be interpreted as executing Et. The whole definition is given below. It creates a basic structure that we will use to formally define 'holding a token' semantics (but without explicit notion of time) in terms of interval sequences (for runs/observations) and interval traces (for concurrent histories).

Definition 41. Let $N = (P, T, F, I, m_0)$ be a given elementary nets with inhibitor arcs.

1. For each $t \in T$ we define Bt – the beginning of t and Et – the end of t, and the set $\mathscr{T} = \{Bt \mid t \in T\} \cup \{Et \mid t \in T\}$. The elements of $\mathscr{T}$ are called ***BE-transitions***.
2. For each $t \in T$ we define:
   (a) ${}^\bullet Bt = {}^\bullet t$,
   (b) $Bt^\bullet = \{t\}$,
   (c) ${}^\bullet Et = \{t\}$,
   (d) $Et^\bullet = t^\bullet$,
   (e) $Bt^\circ = t^\circ \cup (t^\circ)^\bullet$, and
   (f) $Et^\circ = \emptyset$.
3. We say that a set $m \subseteq P \cup T$ is an ***extended marking*** if $m \cap ({}^\bullet m \cup m^\bullet) = \emptyset$.
4. A BE-transition $\tau \in \mathscr{T}$ is ***enabled*** at ***extended marking*** $m \subseteq P \cup T$ if ${}^\bullet \tau \subseteq m$ and $(\tau^\bullet \cup \tau^\circ) \cap m = \emptyset$. For each extended marking m, the set of all enabled elements of $\mathscr{T}$ at m is denoted by $enabled_N^{ext}(m)$.
5. An enabled τ can ***occur*** leading to a ***new extended marking*** $m' = (m \setminus {}^\bullet \tau) \cup \tau^\bullet$, which is denoted by: $m[\![\tau\rangle\!\rangle m'$.
6. An ***extended firing sequence*** from the extended marking m_1 to the extended marking m_{k+1} is any sequence of BE-transitions $\tau_1 \ldots \tau_k$ for which there are extended markings $m_2, \ldots, m_k$ satisfying: $m_1[\![\tau_1\rangle\!\rangle m_2 \ldots m_k[\![\tau_k\rangle\!\rangle m_{k+1}$.
   In such case we write: $m_1[\![\tau_1 \ldots \tau_k\rangle\!\rangle m_{k+1}$. ■

The above definition is pretty much self explained as it mimics the standard firing sequence semantics approach, with the exception of condition 2(e). In the standard model, a token in a place $p \in t^\circ$ means t cannot be fired until this token is removed. In the new model it means that Bt cannot be fired. But if this token is removed for instance by firing Bt_1 where $t_1 \in p^\bullet$, then Bt could be enabled, and potentially fired *before* firing Et_1, which can be interpreted as simultaneous execution of t and t_1, contrary to the fact that $p \in t^\circ \cap {}^\bullet t_1$ was supposed to prevent it. This is the case for Bc, Bb and Eb in the net N_Q^2 in Fig. 6. To prevent this we need to extend Bt° (see the rule $(\tau^\bullet \cup \tau^\circ) \cap m = \emptyset$ in condition 4 of Definition 41) by $(t^\circ)^\bullet$, which leads to $Bt^\circ = t^\circ \cup (t^\circ)^\bullet$.

In particular each marking is an extended marking. For the net N_Q from Fig. 6, for example $\{s_2, s_3\}, \{s_1, c\}$, and $\{a, c\}$ are extended markings, but $\{s_3, b\}$ is not as $s_3^\bullet = \{b\}$ and ${}^\bullet b = \{s_3\}$. Furthermore, if $m \subseteq P$, i.e. m is a marking, then for each $a \in T$, $a \in enabled_N(m) \iff Ba \in enabled_N^{ext}(m)$, so $enabled_N$ and $enabled_N^{ext}$ are consistent.

We will now show that the concepts introduced in Definition 41 allow us to do the following for any given elementary net N with inhibitor arcs:

1. Derive all system runs/observations, including these represented by strict interval orders, encoded as interval sequences, i.e. $<_1^Q$, $<_2^Q$, $<_3^Q$, and $<_4^Q$ for the net N_Q.
2. Derive a (Mazurkiewicz type) independency relation ind_N, such that for every interval sequence that represents a system run x, the interval trace $[x]_{ind_N}$ represents an appropriate concurrent history. For the net N_Q, $[BaEaBbEbBcEc]_{ind_N}$ should represent $hist_1^Q$ and $[BcEcBaEaBbEb]_{ind_N}$ should represent $hist_2^Q$ (see Fig. 5).
3. Transform obtained interval traces into appropriate interval order structures by applying the results of Section 6 of this paper.

We will start with a formal definition of firing interval sequences.

Definition 42. The set of all ***firing interval sequences*** from the marking m to the marking m' is defined as

$$\mathsf{FIS}_N(m \to m') = \{x \in \mathscr{T}^* \mid m[\![x\rangle\!\rangle m'\}.$$

Note that we assume $m, m' \subseteq P$. ■

For the net N_Q, we have $\mathsf{FIS}_{N_Q}(\{s_1, s_2\} \to \{s_4, s_5\}) = \mathsf{FIS}_{N_Q^1}(\{s_1, s_2\} \to \{s_4, s_5\}) = \{BaEaBbEbBcEc\} \cup \mathbf{x}_2$, where $\mathbf{x}_2$ represents the concurrent history $hist_2^Q$ and is discussed in Section 9. Hence N_Q generates the same set of interval sequences as we derived from the program Q in the previous section.

It is not immediately obvious that Definition 42 is sound and complete. This would require the set $\mathsf{FIS}_N(m \to m')$ to satisfy the following two properties

- every element of $\mathsf{FIS}_N(m \to m')$ must be an interval sequence, and
- since all total order representations of a given interval order are considered equivalent and none is preferred, if $x \in \mathsf{FIS}_N(m \to m')$, then $\blacktriangleleft_x = \blacktriangleleft_y$ should imply $y \in \mathsf{FIS}_N(m \to m')$.

Note that if we replace $Bt^\circ = t^\circ \cup (t^\circ)^\bullet$ with $Bt^\circ = t^\circ$ in Definition 41.2(e) (which for the nets in Fig. 6, corresponds using the net N_Q^2 to represent the net N_Q), the second property does not hold!

The following result guarantees the first property.

Proposition 43. *For all markings $m, m' \subseteq P$, we have $\mathsf{FIS}_N(m \to m') \subseteq \mathsf{InSeq}(\mathscr{T}^*)$.*

Proof. It suffices to show that if $m[\![x\rangle\!\rangle m'$, then $x \in \mathsf{InSeq}(\mathscr{T}^*)$, or (see Definition 6(1)) to show that for each $a \in T$, $\pi_{\{Ba,Ea\}}(x) \in (BaEa)^*$.

Let $x = y\ Ba\ z$ and $m[\![y\ Ba\rangle\!\rangle m''$. Since $Ba^\bullet = \{a\}$, $a \in m''$. We also have: for any $m_a \subseteq P \cup T$, if $a \in m_a$, then Ba is not enabled in m_a, and the only way to remove a from m_a is to fire Ea (as ${}^\bullet Ea = \{a\}$). Hence we must have $x = y\ Ba\ w\ Ea\ v$, where $\pi_{\{Ba,Ea\}}(w) = \varepsilon$. □

The second property requires a proposition like the one below.

Proposition 44. *For every $x \in \mathsf{FIS}_N(m \to m')$ and every $y \in \mathscr{T}^*$, if $\blacktriangleleft_x = \blacktriangleleft_y$ then $y \in \mathsf{FIS}_N(m \to m')$.* ■

The proof of the above proposition is in Appendix A as it requires plenty specific results about interval orders, that are not much relevant to this section.

The next two results show that the interval sequence semantics is consistent with both sequence semantics and step sequence semantics. First we show consistency of standard firing sequences and extended firing sequences.

Lemma 45. *For every two $m, m' \subseteq P$, then for each $t \in T$,*

$$m[t\rangle m' \iff m[\![BtEt\rangle\!\rangle m'.$$

Proof. Since ${}^\bullet Bt = {}^\bullet t$, $Bt^\circ = t^\circ \cup (t^\circ)^\bullet$, and $m \cap (t^\circ)^\bullet = \emptyset$, t is enabled at m if and only if Bt is enabled at m.

($\Rightarrow$) If $m[t\rangle m'$ then $m' = (m \setminus {}^\bullet t) \cup t^\bullet$. Let $m[\![Bt\rangle\!\rangle m_B$, i.e. $m_B = (m \setminus {}^\bullet Bt) \cup Bt^\bullet = (m \setminus {}^\bullet t) \cup \{t\}$. Hence Et is enabled at m_B. Let $m_B[\![Et\rangle\!\rangle m_E$. And $m_E = (m_B \setminus {}^\bullet Et) \cup Et^\bullet = (((m \setminus {}^\bullet t) \cup \{t\}) \setminus \{t\}) \cup t^\bullet = (m \setminus {}^\bullet t) \cup t^\bullet = m'$. Hence $m[\![Bt\rangle\!\rangle m_B[\![Et\rangle\!\rangle m'$, i.e. $m[\![BtEt\rangle\!\rangle m'$.

($\Leftarrow$) If $m[\![BtEt\rangle\!\rangle m'$ then reasoning as in the proof of ($\Rightarrow$) we can show that $m' = (m \setminus {}^\bullet t) \cup t^\bullet$. Hence $m[t\rangle m'$. □

We have a similar relationship between firing step sequences and extended firing sequences. For every *step* $A = \{t_1, ..., t_k\} \subseteq T$, let $A^{BE} \subseteq \mathscr{T}^*$ be defined as follows

$$A^{BE} = \{Bt_{i_1}...Bt_{i_k}Et_{j_1}...Et_{j_k} \mid i_1, ..., i_k \text{ and } j_1, ..., j_k \text{ are permutations of } 1, 2, ..., k\}.$$

For example $\{a, b\}^{BE} = \{BaBbEaEb, BaBbEbEa, BbBaEaEb, BbBaEbEa\}$. In [49], the elements of A^{BE} are called 'linearizations' of the step A.

Lemma 46. *For every two markings $m, m' \subseteq P$ and every step $A \subseteq T$,*

$$m\,[A\rangle\, m' \iff (\forall x \in A^{BE}.\, m\,[\![x\rangle\!\rangle\, m').$$

Proof. Let $A = \{t_1, ..., t_k\}$. Since A is a step then $(t_i^\bullet \cup {}^\bullet t_i) \cap (t_j^\bullet \cup {}^\bullet t_j) = \emptyset$ for $i \neq j$, and, by Definition 41(2), $(Et_i^\bullet \cup {}^\bullet Bt_i) \cap (Et_j^\bullet \cup {}^\bullet Bt_j) = \emptyset$ for $i \neq j$.

($\Rightarrow$) $m\,[A\rangle\, m'$ implies ${}^\bullet A \subseteq m$, $(A^\bullet \cup A^\circ) \cap m = \emptyset$, and $m' = (m \setminus {}^\bullet A) \cup A^\bullet$. Let $y = Bt_{i_1}...Bt_{i_k}$ and $z = Et_{j_1}...Et_{j_k}$, where $i_1, ..., i_k$ and $j_1, ..., j_k$ are permutations of $1, 2, ..., k$. Since for all $t_i \in A$, ${}^\bullet t_i = {}^\bullet Bt_i$, $Bt_i^\bullet = \{t_i\}$, $(t_i^\circ)^\bullet \subseteq T$ and $t_i^\circ \cup (t_i^\circ)^\bullet = Bt_i^\circ$, then ${}^\bullet Bt_{i_1} \subseteq m$ and $(Bt_{i_1}^\bullet \cup Bt_i^\circ) \cap m = \emptyset$. Hence Bt_{i_1} is enabled at m, and $m[Bt_{i_1}\rangle m_{t_1}$ where $m_{t_{i_1}} = (m \setminus {}^\bullet Bt_{i_1}) \cup \{t_{i_1}\}$. Moreover, since for all $t_i \in A$, ${}^\bullet t_i = {}^\bullet Bt_i$, then $(Bt_i^\bullet \cup {}^\bullet Bt_i) \cap (Bt_j^\bullet \cup {}^\bullet Bt_j) = \emptyset$ if $i \neq j$. Hence ${}^\bullet Bt_{i_2} \subseteq m_{t_1}$ and $(Bt_{i_2}^\bullet \cup Bt_{i_2}^\circ) \cap m_{t_{i_1}} = \emptyset$, which means Bt_{i_2} is enabled at $m_{t_{i_1}}$, and $m[Bt_{i_2}\rangle m_{t_{i_2}}$ where $m_{t_{i_2}} = (m_{t_{i_1}} \setminus {}^\bullet Bt_{i_2}) \cup \{t_{i_2}\}$. The arguments for t_2 also hold for $t_{i_3} \ldots t_{i_k}$, which means that $m\,[\![y\rangle\!\rangle\, m_B$, where $m_B = (m \setminus ({}^\bullet Bt_{i_1} \cup ... \cup {}^\bullet Bt_{i_k})) \cup (Bt_{i_1}^\bullet \cup ... \cup Bt_{i_k}^\bullet) = (m \setminus {}^\bullet A) \cup (t_{i_1}^\bullet \cup ... \cup t_{i_k}^\bullet) = (m \setminus {}^\bullet A) \cup A$.

For all ET_i we just have ${}^\bullet Et_i = \{t_i\}$, $Et_i^\bullet = t_i^\bullet$ for all i. Hence $m_B\,[\![z\rangle\!\rangle\, m_E$, where $m_E = (m_B \setminus ({}^\bullet Et_{j_1} \cup ... \cup {}^\bullet Et_{j_k})) \cup (Et_{j_1}^\bullet \cup ... \cup Et_{j_k}^\bullet) = (((m \setminus {}^\bullet A) \cup A) \setminus A) \cup A^\bullet = (m \setminus {}^\bullet A) \cup A^\bullet = m'$, which means $m\,[A\rangle\, m' \implies \forall x \in A^{BE}.\, m\,[\![x\rangle\!\rangle\, m'$.

($\Leftarrow$) Let $A = \{t_1, ..., t_k\}$ and assume $\forall x \in A^{BE}.\, m\,[\![x\rangle\!\rangle\, m'$. This means each Bt_i is enabled at m, i.e. for each i, ${}^\bullet Bt_i \subseteq m$ and $Bt_i^\circ \cap m = \emptyset$. The latter implies $t_i^\circ \cap m = \emptyset$. Hence ${}^\bullet A \subseteq m$ and $A^\circ \cap m = \emptyset$.

Assume $x = yz$ where y and z are these from the proof of ($\Rightarrow$). Clearly $m\,[\![y\,\rangle\!\rangle m_y\,[\![z\rangle\!\rangle\, m'$, for some $m_y \in P \cup T$. By Definition 41(5), $m_y = (m \setminus ({}^\bullet Bt_{i_1} \cup ... \cup {}^\bullet Bt_{i_k})) \cup (Bt_{i_1}^\bullet \cup ... \cup Bt_{i_k}^\bullet) = (m \setminus {}^\bullet A) \cup (t_{i_1}^\bullet \cup ... \cup t_{i_k}^\bullet) = (m \setminus {}^\bullet A) \cup A$, and $m' = (m_y \setminus ({}^\bullet Et_{j_1} \cup ... \cup {}^\bullet Et_{j_k})) \cup (Et_{j_1}^\bullet \cup ... \cup Et_{j_k}^\bullet) = (m_y \setminus A) \cup A^\bullet$. Hence $m' = (m_y \setminus ({}^\bullet Et_{j_1} \cup ... \cup {}^\bullet Et_{j_k})) \cup (Et_{j_1}^\bullet \cup ... \cup Et_{j_k}^\bullet) = (m \setminus {}^\bullet A) \cup A^\bullet$.

What remain is to show that $A^\bullet \cap m = \emptyset$. Suppose $A^\bullet \cap m \neq \emptyset$, i.e. $Et_i^\bullet \cap m \neq \emptyset$ for some i. We may assume $j_1 = i$. Now we should have $m[\![y\rangle\!\rangle m_y$, where $m_y = (m \setminus {}^\bullet A) \cup A$ and Et_i is enabled at m_y. Since ${}^\bullet A \cap A^\bullet = \emptyset$ (as A is a step), $Et_i^\bullet \cap m \neq \emptyset$ implies $Et_i^\bullet \cap m_y \neq \emptyset$, but this means Et_i is *not* enabled at m_y, a contradiction.

In this way we have proved ${}^\bullet A \subseteq m$, $(A^\bullet \cup A^\circ) \cap m = \emptyset$, and $m' = (m \setminus {}^\bullet A) \cup A^\bullet$, so $m\,[A\rangle\, m'$. □

We will say that *an interval sequence x and a step sequence y are equivalent* iff $\blacktriangleleft_x = \lhd_y$, i.e. when they both represent the same partial order. By using the sets A^{BE} we can define this relationship in terms of sequences only. Namely,

- an interval sequence x and a step sequence $A_1 \ldots A_k$ are equivalent iff

  $$x \in A_1^{BE} \circ \ldots \circ A_k^{BE},$$

  where '$\circ$' denotes concatenation of languages, i.e. $A \circ B = \{xy \mid x \in A \wedge y \in B\}$.

We may now define an *interval representation of a step sequence* $A_1 \ldots A_k$ as

$$irs(A_1 \ldots A_k) = A_1^{BE} \circ \ldots \circ A_k^{BE},$$

and formulate the following result characterizing the relationship between firing step sequences and firing interval sequences.

Proposition 47. *For every inhibitor net N we have:*

$$irs(\mathsf{FSS}_N(m \to m')) \subseteq \mathsf{FIS}_N(m \to m').$$

Proof. A straightforward conclusion from Lemma 46. □

The sets of total, stratified, and interval orders represented by firing sequences, firing step sequences, and firing interval sequences respectively,can be defined as follows:

$$\mathsf{TO}_N(m\to m') = \{\lhd_x \mid x \in \mathsf{FS}_N(m\to m')\},$$
$$\mathsf{SO}_N(m\to m') = \{\lhd_x \mid x \in \mathsf{FSS}_N(m\to m')\}$$
$$\mathsf{IO}_N(m\to m') = \{\blacktriangleleft_x \mid x \in \mathsf{FIS}_N(m\to m')\}.$$

From Proposition 47 and the fact that each sequence $a_1 \dots a_k$ is equivalent to the step sequence $\{a_1\}\dots\{a_k\}$, we have:

Corollary 48. *For every inhibitor net we have:*

$$\mathsf{TO}_N(m\to m') \subseteq \mathsf{SO}_N(m\to m') \subseteq \mathsf{IO}_N(m\to m'). \quad \blacksquare$$

For the net N_Q from Fig. 6, with $m_0 = \{s_1, s_2\}$ and $m_f = \{s_4, s_5\}$, we have: $\mathsf{TO}_{N_Q}(m_0\to m_f) = \{<_1^Q, <_2^Q\}$, $\mathsf{SO}_{N_Q}(m_0\to m_f) = \{<_1^Q, <_2^Q, <_3^Q\}$, and $\mathsf{IO}_{N_Q}(m_0\to m_f) = \{<_1^Q, <_2^Q, <_3^Q, <_4^Q\} = hist_1^Q \cup hist_2^Q$. Hence, as far as operation semantics is concerned, the net N_Q can be regarded as a true representation of the program Q.

Note that there are inhibitor nets for which all their observations are interval orders. The net N_0 from Fig. 6 is such a net. For $m_0 = \{s_1, s_2\}$ and $m_f = \{s_4, s_5\}$, we have $\mathsf{FIS}_{N_0}(m_0\to m_f) = \{BaBcEaBbEbEc, BaBcEaBbEcEb, BcBaEaBbEbEc, BaBaEaBbEcEb\}$ so $\mathsf{IO}_{N_0}(m_0\to m_f) = \{<_4^Q\}$, while $\mathsf{FS}_{N_0}(m_0\to m_f) = \mathsf{FSS}_{N_0}(m_0\to m_f) = \emptyset$, and $\mathsf{TO}_{N_0}(m_0\to m_f) = \mathsf{SO}_{N_0}(m_0\to m_f) = \emptyset$.

10.3. Trace and comtrace semantics

One of the disadvantages of any operational semantics is that does not recognize equivalent executions, so they cannot identify concurrent histories. This is the role of traces, comtraces, and interval traces. In this subsection we briefly recall basic concepts and results of trace and comtrace semantics of elementary inhibitor nets, as described in [22,28].

The trace semantics alone is not particularly interesting as, since it is derived from the firing sequence semantics, it can be reduced to standard trace semantics of elementary nets with self-loops [5,10]. However it sets the basis for interval trace semantics. The interval trace semantics, which is our goal, must be a consistent extension of comtrace semantics.

Let $N = (P, T, F, I, m_0)$ be an inhibitor net. We define the (*trace*) *independency* relation $ind_N^{tr} \subseteq T \times T$ as (cf. [22]):

$$(a,b) \in ind_N^{tr} \iff [(a^\bullet \cup {}^\bullet a) \cap (b^\bullet \cup {}^\bullet b) = \emptyset] \wedge$$
$$[a^\circ \cap ({}^\bullet b \cup b^\bullet) = \emptyset] \wedge [b^\circ \cap ({}^\bullet a \cup a^\bullet) = \emptyset].$$

The trace alphabet is (T, ind_N^{tr}), and the set of all traces defining behaviors that start from the marking m and end at the marking m' is defined as

$$\mathsf{Tr}_N(m\to m') = \{[x]_{ind_N^{tr}} \mid x \in \mathsf{FS}_N(m\to m')\}.$$

The *comtrace semantics* is standardly derived from the firing step sequence semantics [22,27]. In this case we define the following relations $sim_N, ser_N \subseteq T \times T$:

$$(a,b) \in sim_N \iff (a^\bullet \cup {}^\bullet a) \cap (b^\bullet \cup {}^\bullet b) = \emptyset \wedge (a^\circ \cap {}^\bullet b) \cup (b^\circ \cap {}^\bullet a) = \emptyset,$$
$$(a,b) \in ser_N \iff (a,b) \in sim_N \wedge a^\bullet \cap b^\circ = \emptyset.$$

The comtrace alphabet here is (T, sim_N, ser_N), and the set of all comtraces defining behaviors that start from the marking m and end at the marking m' is defined as

$$\mathsf{ComTr}_N(m\to m') = \{[x]_{(sim_n, ser_N)} \mid x \in \mathsf{FSS}_N(m\to m')\}.$$

The following two results validate all the above definitions of this subsection.

Proposition 49 *(Follows from [22,24,28]). For each elementary inhibitor net N_Q and for all markings $m, m' \subseteq P$:*

1. $x \in \mathsf{FS}_N(m\to m') \iff [x]_{ind_N^{tr}} \subseteq \mathsf{FS}_N(m\to m')$,
2. $x \in \mathsf{FSS}_N(m\to m') \iff [x]_{(sim_N, ser_N)} \subseteq \mathsf{FSS}_N(m\to m')$. $\blacksquare$

For the net N_Q from Fig. 6 we have

$$sim_{N_P} = \{(a,c), (c,a)\},$$
$$ser_{N_P} = \{(c,a)\}.$$

i.e. $(a,c) \notin ser_{N_P}$, $\mathsf{ComTr}_{N_P}(\{s_1,s_2\}\rightarrow\{s_4,s_5\}) = \{\mathbf{y_1},\mathbf{y_2}\}$, where $\mathbf{y_1} = \{\{a\}\{b\}\{c\}\}$, $\mathbf{y_2} = \{\{c\}\{a\}\{b\},\{a,c\}\{b\}\}$. When step sequences are interpreted as stratified orders, the comtrace $\mathbf{y_1}$ represents the set $\{<_1^Q\}$ and the comtrace $\mathbf{y_2}$ represents the set $\{<_2^Q,<_3^Q\}$, where $<_1^Q,<_2^Q,<_3^Q$ are the partial orders from Fig. 5.

The *process semantics* (in the sense of [37,41]) has been proposed in [22] and substantially refined in [27]. It was proven that the process semantics and comtrace semantics are equivalent to some extent. The process semantics will not be discussed in this paper, the details can be found in [18,28].

10.4. Interval trace semantics and interval order structure semantics

Since interval traces are just a special kind of general traces, we will just modify the standard trace semantics of inhibitor nets. The main difference is to define the independency relation on BE-transitions instead of transitions, and use firing interval sequences instead of firing sequences. Intuitively, the *interval trace semantics* of the net N_Q of Fig. 6, where $\{a,b,c\}$ are transitions, is just the (slightly modified) *trace semantics* of the net N_Q^1 (with $\{Ba,Ea,Bb,Eb,Bc,Ec\}$ as transitions) with appropriate independency relation. We just need to define this new ind_N relation on BE-transitions and show its validity.

Let $N=(P,T,F,I,m_0)$ be an inhibitor net, and let $\mathscr{T}=\{Bt \mid t\in T\}\cup\{Et \mid t\in T\}$.

We may define the BE*-net independency* relation $ind_N^{net}\subseteq\mathscr{T}\times\mathscr{T}$ as by just mimicking the trace independency relation ind_N^{tr}, but in $\mathscr{T}\times\mathscr{T}$ instead of $T\times T$.

Definition 50. For all distinct $\alpha,\beta\in\mathscr{T}$:

$$(\alpha,\beta)\in ind_N^{net} \iff [(\alpha^\bullet\cup{}^\bullet\alpha)\cap(\beta^\bullet\cup{}^\bullet\beta)=\emptyset]\wedge$$
$$[\alpha^\circ\cap({}^\bullet\beta\cup\beta^\bullet)=\emptyset]\wedge[\beta^\circ\cap({}^\bullet\alpha\cup\alpha^\bullet)=\emptyset].\quad\blacksquare$$

Note that, because for each $t\in T$, ${}^\bullet Et=\{t\}$, and $Et^\circ=\emptyset$, $(Ea,Eb)\in ind_N^{net}$ for all $a,b\in T$. Unfortunately the relation ind_N^{net} usually does not satisfy Definition 12(2), as if $a^\bullet\cap{}^\bullet b\neq\emptyset$ (i.e. a and b are in a conflict [41,42]) then $Ba^\bullet\cap{}^\bullet Bb\neq\emptyset$ too. We will show that we need only a simple and obvious extension of ind_N^{net}.

We define the (*interval trace*) *independency* relation $ind_N\subseteq\mathscr{T}\times\mathscr{T}$ as follows.

Definition 51. For all distinct $a,b\in T$:

1. $(Ba,Bb)\in ind_N\wedge(Ea,Eb)\in ind_N$
2. $(Ba,Eb)\in ind_N \iff [(Ba^\bullet\cup{}^\bullet Ba)\cap(Eb^\bullet\cup{}^\bullet Eb)=\emptyset]\wedge$ $[Ba^\circ\cap({}^\bullet Eb\cup Eb^\bullet)=\emptyset]\wedge[Eb^\circ\cap({}^\bullet Ba\cup Ba^\bullet)=\emptyset]$.

The interval trace alphabet is $(\mathscr{T},ind_N)$. $\blacksquare$

In other words, $ind_N=ind_N^{net}\cup\{(Ba,Bb)\mid a,b\in T\wedge a\neq b\}$. Obviously the relation ind_N satisfies Definition 12(2). The next result shows that Definition 12(1) is satisfied too.

Lemma 52. *For each $t\in T$, $(Bt,Et)\notin ind_N$ and $(Et,Bt)\notin ind_N$.*

Proof. Since $Bt^\bullet\cap{}^\bullet Et=\{t\}$, for each $t\in T$. $\square$

Hence $(\mathscr{T},ind_N)$ is indeed the interval trace alphabet.

The following two results validate Definition 51. The first result, most likely the most crucial, states that if our interests are restricted to interval firing sequences only, we may identify ind_N with ind_N^{net}. This means that the property $(Ba,Bb)\in ind_N$, for *all* $a,b\in T$, which is natural for general interval traces, but a little bit artificial for elementary inhibitor nets, does not introduce undesired and nonexistent behaviors, so it does not cause any harm.

Proposition 53. *For all markings $m,m'\subseteq P$ and all $x\in\mathsf{FIS}_N(m\rightarrow m')$:*

$$[x]_{ind_N}=[x]_{ind_N^{net}}.$$

Proof. Since $ind_N^{net}\subseteq ind_N$, then $[x]_{ind_N^{net}}\subseteq[x]_{ind_N}$. Consider $x_1BaBbx_2\in[x]_{ind_N^{net}}$ and $(Ba,Bb)\notin ind_N^{net}$. By Proposition 49(1), $[x]_{ind_N^{net}}\subseteq\mathsf{FIS}_N(m\rightarrow m')$, so we have $Bb\neq Ba$ and $m[\![x_1\rangle\!\rangle m_{x_1}[\![Ba\rangle\!\rangle m_{x_1Ba}[\![Bb\rangle\!\rangle m_{x_1BaBb}[\![x_2\rangle\!\rangle m'$, where $m_{x_1Ba}=(m_{x_1}\setminus{}^\bullet Ba)\cup\{a\}$ and $m_{x_1BaBb}=(m_{x_1Ba}\setminus{}^\bullet Bb)\cup\{b\}$. $(Ba,Bb)\notin ind_N^{net}$ and $Ba\neq Bb$ imply $Ba^\bullet\cap Bb^\bullet\neq\emptyset$, or $Ba^\circ\cap{}^\bullet Bb\neq\emptyset$, or $Ba^\circ\cap Bb^\bullet\neq\emptyset$, or $Bb^\circ \cap {}^\bullet Ba \neq \emptyset$, or $Bb^\circ \cap Ba^\bullet \neq \emptyset$. If $Ba^\bullet \cap Bb^\bullet \neq \emptyset$ then Bb is not enabled at m_{x_1Ba}, a contradiction as $x_1BaBbx_2 \in \mathsf{FIS}_N(m\to m')$. Let $r \in Ba^\circ \cap {}^\bullet Bb$, i.e. $r \in m_{x_1Ba}$. But $m_{x_1} = (m_{x_1Ba} \setminus \{a\}) \cup {}^\bullet Ba$, so $r \in Ba^\circ \cap m_{x_1}$, i.e. Ba is not enabled at m_{x_1}, a contradiction. Since $Ba^\circ = a^\circ \cup (a^\circ)^\bullet$ and $Bb^\bullet = \{b\}$, then $Ba^\circ \cap Bb^\bullet \neq \emptyset$ means $b \in (a^\circ)^\bullet$, i.e. $a^\circ \cap {}^\bullet b \neq \emptyset$, which implies $Ba^\circ \cap {}^\bullet Bb \neq \emptyset$, which was already considered. If $Bb^\circ \cap Ba^\bullet \neq \emptyset$, then $Bb^\circ \cap m_{x_1Ba} \neq \emptyset$ so Bb is not enabled at m_{x_1Ba}, a contradiction again. Consider $Bb^\circ \cap {}^\bullet Ba \neq \emptyset$. We will show that this case implies $Bb^\circ \cap Ba^\bullet \neq \emptyset$. Since $Bb^\circ = b^\circ \cup (b^\circ)^\bullet$ and ${}^\bullet Ba = {}^\bullet a$, then $b^\circ \cap {}^\bullet a \neq \emptyset$. Let $p \in b^\circ \cap {}^\bullet a$, so $a \in p^\bullet$, and $a \in (b^\circ)^\bullet$, which means, $Bb^\circ \cap Ba^\bullet \neq \emptyset$, as $Ba^\bullet = \{a\}$. □

The second result is an equivalent of Proposition 49 and it states that interval traces produced by applying the relation ind_N are consistent with the concept of firing interval sequences.

Proposition 54. *For each elementary inhibitor net N_Q and for all markings $m, m' \subseteq P$:*

$$x \in \mathsf{FIS}_N(m\to m') \iff [x]_{ind_N} \subseteq \mathsf{FIS}_N(m\to m').$$

Proof. A simple consequence of Propositions 49(1) and 53. □

When we have the relation ind_N, then for each firing sequence x, the interval trace $[x]_{ind_N}$ describes a behavior (concurrent history) of the inhibitor net N.

The set of all interval traces defining behaviors that start from the marking m and end at the marking m' is defined as

$$\mathsf{IntTr}_N(m\to m') = \{[x]_{ind_N} \mid x \in \mathsf{FIS}_N(m\to m')\}.$$

Proposition 54 validates the above definition of $\mathsf{IntTr}_N(m\to m')$.

Since every interval trace uniquely defines an interval order structure, we may define the set of all interval order structures defining behaviors that start from the marking m and end at the marking m' as

$$\mathsf{IOS}_N(m\to m') = \{S^{[x]_{ind_N}} \mid [x]_{ind_N} \in \mathsf{IntTr}_N(m\to m')\}.$$

By Theorem 26 we can also write $\mathsf{IOS}_N(m\to m') = \{S^x \mid x \in \mathsf{FIS}_N(m\to m')\}$. At this point we have completed defining interval trace semantics for elementary nets with inhibitor arcs, as

- $\mathsf{FIS}_N(m\to m')$ provides all valid interval order observations,
- $\mathsf{IntTr}_N(m\to m')$ or $\mathsf{IOS}_N(m\to m')$ provide all valid concurrent histories.

For the net N_Q from Fig. 6 we have $ind_{N_Q} = ind^Q \cup ind^{unused}$, where ind^Q is from Fig. 5, and $ind^{unused} = \{(Ba, Bb), (Ea, Eb), (Bb, Ba), (Eb, Ea), (Ba, Eb), (Eb, Ba)\}$. However for each $m, m' \subseteq P$ and every $x \in \mathsf{FIS}_{N_Q}(m\to m')$, we have $[x]_{in^Q} = [x]_{ind_{N_Q}}$, as the pairs $(Ba, Bb) \in ind_{N_Q}$, $(Ea, Eb) \in ind_{N_Q}$, and $(Ba, Eb) \in ind_{N_Q}$ are *never used*, in any interval firing sequence. Hence the relation ind_{N_Q} is bigger than needed. This is the price paid for having ind_N derived entirely from the static structure of the net N. We have the same situation when traces or comtraces are applied for Petri nets (not necessary with inhibitor arcs), the relations ind, or sim and ser derived from nets, almost always have unused part [22,33].

For the net N_Q we also have, with $m_0 = \{s_1, s_2\}$ and $m_f = \{s_4, s_5\}$,

- $\mathsf{IntTr}_{N_Q}(m_0\to m_f) = \{\mathbf{x}_1, \mathbf{x}_2\}$, where $\mathbf{x}_1 = \{BaEaBbEbBcEc\}$, i.e. the interval sequence representation of $hist_1^Q$, and $\mathbf{x}_2$ is the interval trace discussed in Section 9, i.e. the interval sequence representation of $hist_1^Q$.
- $\mathsf{IOS}_{N_Q}(m_0\to m_f) = \{S_1^Q, S_2^Q\}$, where $S_1^Q = (\{a,b,c\}, <_1^Q, <_1^{Q\frown})$, $S_2^Q = (\{a,b,c\}, \prec_2^Q, \sqsubset_2^Q)$, and $<_1^Q$, $\prec_2^Q$ and $\sqsubset_2^Q$ are these from Fig. 5. The interval order structure S_1^Q represents uniquely the concurrent history $hist_1^Q$, while S_2^Q represents uniquely the concurrent history $hist_2^Q$.

Since, as we already discussed it in Section 10.2,

- $\mathsf{FIS}_{N_Q}(m_0\to m_f) = \mathbf{x}_1 \cup \mathbf{x}_2$, and
- $\mathsf{IO}_{N_Q}(m_0\to m_f) = \{<_1^Q, <_2^Q, <_3^Q, <_4^Q\} = hist_1^Q \cup hist_2^Q$.

the net N_Q is a true net model of the program Q.

10.5. Relationship to ST-traces model

Our approach to Petri net semantics and the ST-traces of Vogler's model from [49] have two important factors in common. The first being that in both models the system states are represented by marking plus transitions that are currently being executed, and the second being that in both models firing sequences are prefixes of interval sequences built from the beginnings and endings of transitions.

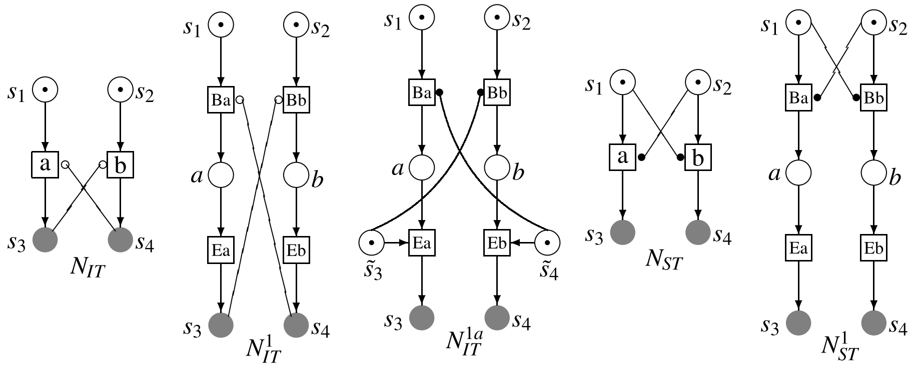


Fig. 7. An example of an inhibitor and activator nets that are equivalent w.r.t. firing sequence and firing step sequence semantics but not equivalent w.r.t. interval firing sequence semantics. We have $\mathsf{FIS}_{N_{IT}}(m_0 \to m_f) = \{BaBbEaEb, BbBaEaEb, BaBbEbEa, BbBaEbEa\}$ and $\mathsf{FIS}_{N_{ST}}(m_0 \to m_f) = \emptyset$.

In our model the system states are represented by extended markings, as in the model of [49] by ST-markings (first proposed in [47]). For a given net (with or without inhibitor arcs), an *ST-marking* is a pair (m_{ST}, c_{ST}), where $m_{ST} \subseteq P$ is a current marking of N and $c_{ST} \subseteq T$ is the set of *currently firing* transitions. Formally, for every extended marking m, the pair $m_{ST}(m) = (m \cap P, m \cap T)$, is an ST-marking.

An extended firing sequence τ such that $m[\![\tau\rangle\!\rangle m'$, can also be interpreted as a *ST-trace* from the ST-marking $m_{ST}(m)$ to the ST-marking $m_{ST}(m')$. Despite the name *trace*,[5] ST-traces are just *sequences* (more precisely, prefixes of interval sequences) of the elements of $\mathscr{T}$, i.e. Bt's and Et's, not sets of (equivalent w.r.t. some rules) sequences as Mazurkiewicz or interval traces. The ST-traces originate from [45] and are a primary tool in defining ST-bisimulation, which also was first proposed in [45]. The ST-bisimulation was developed and used in many papers, one of the latest being [46] which contains an extensive bibliography of this subject. In [49], ST-traces were used to define operational semantics of nets with read/activator arcs, like in our model extended firing sequences are used to define operational interval semantics of nets with inhibitor arcs.

There are also some important differences. Firstly, our model deals only with inhibitor arcs, whereas the model of [49] deals only with read (or activator) arcs. This does matter despite the fact that *it is universally assumed that for elementary nets, inhibitor arcs can be equivalently represented by read/activator arcs from complement places,*[6] *and vice versa.* For the net N^{1a}_{IT} from Fig. 7, the places s_3 and $\tilde{s}_3$, and s_4 and $\tilde{s}_4$ are complementary and the inhibitor net N^1_{IT} from Fig. 7 is considered behaviorally equivalent to the activator net N^{1a}_{ST}. Similarly for nets N_{IT} and N_{ST}, the places s_1 and s_3, and s_2 and s_4 are complementary and the nets N_{IT} and N_{ST} are considered behaviorally equivalent. This assumption is crucial for the results of [4,5,22,27,35] and most likely many others. It is also not questioned in [49] when the obtained results are compared to that of [22]. When operational semantics is defined in terms of sequences or step sequences, this relationship is natural, well defined and well supported by intuition.

Fig. 7 indicates that *this simple construction based on complementary places does not work when operational semantics is expressed in terms of interval sequences* (including ST-traces). Consider the net N_{IT}: its interval sequence semantics is just a firing sequence semantics of the net N^1_{IT}, which means that for the initial marking $m_0 = \{s_1, s_2\}$ and the final marking $m_f = \{s_3, s_4\}$, we have: $\mathsf{FIS}_{N_{IT}}(m_0 \to m_f) = \mathsf{FS}_{N^1_{IT}}(m_0 \to m_f) = \{BaBbEaEb, BbBaEaEb, BaBbEbEa, BbBaEbEa\} = \{a, b\}^{BE}$, i.e just a step $\{a, b\}$. The net N^{1a}_{ST} has been derived from N^1_{IT} by adding appropriate complement places and replacing inhibitor arcs with activator arcs. Firing sequences for elementary nets with activator arcs have not been defined yet, but this is just a simple modification of the definition for inhibitor nets. Let $A \subseteq P \times T$ be a set of *activator* arcs, and for $x \in P \cup T$, let $x^a = \{y \mid (x, y) \in A \cup A^{-1}\}$. Then we have just defined that a transition t is *enabled* at marking m if ${}^\bullet t \cup t^a \subseteq m$ and $t^\bullet \cap M = \emptyset$, and left the remaining part for inhibitor nets. Now we have $\mathsf{FS}_{N^1_{ST}}(m_0 \to m_f) = \mathsf{FS}_{N^1_{IT}}(m_0 \to m_f) = \mathsf{FIS}_{N_{IT}}(m_0 \to m_f) = \{a, b\}^{BE}$.

For the net N_{ST} (which is identical to the net N_3 from Figure 1 in [49]), the situation is much different. For this particular net, the set of ST-traces it generates is the same as the set of firing sequences generated by the net N^1_{ST}, and clearly $\mathsf{FS}_{N^1_{ST}}(m_0 \to m_f) = \emptyset$!

Interval firing sequences for nets like N^1_{ST} has not been formally defined in this paper, but using the same 'holding token' idea and reasoning we used for inhibitor nets in Section 10.2, would result in adopting firing sequences of the net N^1_{ST} as interval firing sequences of the net N_{ST}, i.e. $\mathsf{FIS}_{N_{ST}}(m_0 \to m_f) = \mathsf{FS}_{N^1_{ST}}(m_0 \to m_f)$. But this would mean that with respect to interval firing sequences semantics (and ST-traces semantics) *the inhibitor net N_{IT} and the activator net N_{ST} are not equivalent*, as $\mathsf{FIS}_{N_{IT}}(m_0 \to m_f) = \{a, b\}^{BE} \neq \mathsf{FIS}_{N_{ST}}(m_0 \to m_f) = \emptyset$.

[5] The word "trace" has a few different meanings in computer science. In this paper, we use the word "trace" in the sense of [10], while for ST-traces, the word "trace" has the same meaning as in for example Hoare's CSP [15].

[6] Places $p, q \in P$ are *complementary* (cf. [42]) (p is a complement of q and vice versa) if $p \neq q$, ${}^\bullet p = q^\bullet$ and $p^\bullet = {}^\bullet q$, and $|m_0 \cap \{p, q\}| = 1$.

Inhibitor nets are not explicitly considered in [49], and it is assumed that they can be represented by appropriate activator nets via complementary places. The nets N_{IT} and N_{ST} are considered equivalent but only the semantics of N_{ST} is formally defined. The net N_{ST} is used as an example of the difference between step sequence semantics proposed in [22] and step sequence semantics that can be derived from ST-traces. In the model proposed in [22], which uses firing step sequences as defined in Section 10.1.1, the net N_{IT} *can fire* the step $\{a, b\}$ in m_0, while in the model of [49], the net N_{ST} *can not fire* the step $\{a, b\}$ in m_0.

When current system states are represented by both places and the transitions that are currently being executed, the relationship between inhibitor places and activator places is *different and much more complex* than when current system states are represented by places only. When a transition a is a part of a current system state then ${}^\bullet a \cap a^\bullet = \emptyset$, so both inhibitor and activator places in ${}^\bullet a$ have been deactivated and both inhibitor and activator places in $a^\bullet$ remain unchanged. This means the transition b is enabled in the extended marking $\{a, s_2\}$ in the net N_{IT}, but the transition b is *not* enabled in the ST-marking $(\{a\}, \{s_2\})$ in the net N_{ST}.

The other main difference between [49] and this paper is that neither a counterpart of independency relation nor Mazurkiewicz or interval trace are considered in the former. Moreover, the relational structures of [49], called *spc-orders*, are different than our interval or stratified order structures. A triple $(X, \prec, \sqsubset)$ is an spc-order if both $\prec$ and $\sqsubset$ are partial orders and additionally $a \prec b \Rightarrow a \sqsubset b$ and $a \prec b \sqsubset c \Rightarrow a \sqsubset c$. The spc-orders comprise equivalent ST-traces, but they are not concurrent histories as defined in Section 8.

11. Final comment

We have introduced the concept of interval traces, a special kind of Mazurkiewicz traces, that can provide an abstract semantics of concurrent systems when the operational semantics involves interval orders.

It was proven that interval traces can model interval order structures in the same manner as classical Mazurkiewicz traces can model partial orders [34] and comtraces can model stratified order structures [22].

The concept and theory of interval traces stems from three sources: classical traces, comtraces, and the representation theorem of Abraham, Ben-David, and Magidor ([1], Theorem 21 in this paper). Like comtraces, interval traces are generated by two relations *sim* and *ser* on a given set of events, and the interpretation of these relations is the same as for comtraces. However, comtraces are sets of step sequences of event occurrences, interval traces are just sets of ordinary sequences (like classical traces) but beginnings and ends of event occurrences. Like in classical traces, the structure of interval traces is generated by a single independency relation *ind*, but defined for the beginnings and ends. Technically, interval traces are just a special case of classical traces that are defined on the set of beginnings and ends of events. The representation theorem of Abraham, Ben-David, and Magidor allows representing interval order structures by appropriate partial orders of beginnings and ends. We have shown that the partial order generated by a given interval trace uniquely defines an interval order structure via the Abraham, Ben-David, and Magidor theorem. Moreover this partial order is the least partial order representation of the derived interval order structure.

We have also shown how interval traces can be used to describe an abstract interval order semantics of elementary nets with inhibitor arcs. This new semantics is consistent with, and an extension of, comtrace semantics from [22,28]. By comparing our model of inhibitor nets with Vogler's model of activator nets [49], we have shown that the standard transformation of one type of nets into another via complementary places (cf. [4,5,22,27,35]) does not work for operational semantics with interval sequences (including ST-traces of [49]).

For both Mazurkiewicz traces and comtraces an equivalent pure process semantics (in a sense of [37]) have been constructed [10,22,27]. For interval traces only some initial results have recently been published [4]. The mutex relation, proposed in [29] and used in [19] for generalized comtraces, has not been applied to our model yet.

Acknowledgments

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Appendix A. Proofs of Propositions 14 and 44

To prove Propositions 14 and 44 we need to provide some results on *sequence representations of interval orders*

Suppose $<$ is a finite interval order. By Theorem 4 we know that there exists at least one total order representation $\lhd$, which can further be represented as an appropriate sequence. However, neither Theorem 4, nor any of its known proofs (cf. [11,12,21]), provides an effective method of constructing *all* such total order representations.

In this section we will show how to construct all total order representations of a given finite interval order. The construction, which will be based on the concept of *principal order* [12,21], will be needed to show completeness of our definition of interval traces.

Definition 55 *([12,21])*. Let $<$ be a partial order on X (of any kind, no restrictions).

1. A set $A \subseteq X$ is a **maximal antichain** of $<$ if and only if
$$(\forall a, b \in A.\, a \frown_{<} b \vee a = b) \wedge (\forall a \notin A. \exists b \in A.\, a < b \vee b < a).$$
The set of all maximal antichains of $<$ will be denoted by $\mathscr{A}_{<}$.
2. The relation $\ll_{<} \subseteq \mathscr{A}_{<} \times \mathscr{A}_{<}$, defined as
$$A \ll_{<} B \iff A \neq B \wedge (\forall a \in A \setminus B. \forall b \in B \setminus A.\, a < b)$$
is called a **principal order** of $<$ (*see [21] for more details*). ■

We will now show that principal orders are always partial orders of maximal antichains and we can always recover the partial order $<$ from its principal order $\ll_{<}$.

Proposition 56 *([21]). Let $<$ be any partial order on X.*

1. *The relation $\ll_{<}$ is a partial order.*
2. *For all $a, b \in X$:*
$$a < b \iff a \neq b \wedge (\forall A, B \in \mathscr{A}_{<}.\, a \in A \wedge b \in B \implies A \ll_{<} B). \quad ■$$

Maximal antichains and principal orders are also convenient tools for classifying partial orders.

Theorem 57 *([12,21]). A partial order $<$ is an interval order if and only if its principal order $\ll_{<}$ is a total order (of maximal antichains).* ■

As a simple consequence of Theorem 57 we have the following corollary.

Corollary 58. *A partial order $<$ is stratified if and only if all maximal antichains are equivalence classes of $\frown_{<}$.* ■

When $\ll_{<}$ is a total order, it can be represented as an appropriate sequence of antichains of $<$. *We will identify this sequence representation with the total order $\ll_{<}$ and write $\ll_{<} = A_1 \dots A_n$.* Note that A_i's are different antichains, i.e. $A_i \subseteq A_j$ if $i = j$, hence we have $A_i \ll_{<} A_j$ iff i is smaller than j.

For example for $<_3$ of Fig. 1 and $<_4^Q$ of Fig. 5, we have
$$\ll_{<_3} = \{a\}\{b,c\}\{c,d\} \text{ and } \ll_{<_4^Q} = \{a,c\}\{b,c\}.$$
Both $\ll_{<_3}$ and $\ll_{<_4^Q}$ are total orders of *maximal antichains*.

Let $<$ be an *interval order* over the finite set X (here we assume nothing about X) and let $\ll_{<} = A_1 \dots A_n$ be its principal order represented as a sequence of antichains, and let $\mathscr{X} = \{Ba \mid a \in X\} \cup \{Ea \mid a \in X\}$.

For each $a \in X$, we define:
$$\mathit{first}_{<}(a) = A_i \text{ if } a \in A_i \text{ and either } i = 1 \text{ or } a \notin A_{i-1}, \text{ and}$$
$$\mathit{last}_{<}(a) = A_i \text{ if } a \in A_i \text{ and either } i = n \text{ or } a \notin A_{i+1}.$$

For example for $<_3$ of Fig. 1 and $<_4^Q$ of Fig. 5, $\mathit{first}_{<_3}(a) = \mathit{last}_{<_3}(a) = \{a\}$, $\mathit{first}_{<_3}(c) = \{b,c\}$, $\mathit{last}_{<_3}(c) = \{c,d\}$, $\mathit{first}_{<_4^Q}(a) = \mathit{last}_{<_4^Q}(a) = \{a\}$, $\mathit{first}_{<_4^Q}(c) = \{a,c\}$, $\mathit{last}_{<_4^Q}(c) = \{b,c\}$.

For each A_i, we define:
$$B_{<}(A_i) = \{Ba \mid \mathit{first}_{<}(a) = A_i\},$$
$$E_{<}(A_i) = \{Ea \mid \mathit{last}_{<}(a) = A_i\}.$$

For example $B_{<_3}(\{b,c\}) = \{Bb, Bc\}$, $E_{<_3}(\{b,c\}) = \{Eb\}$, $B_{<_3}(\{c,d\}) = \{Bd\}$, and $E_{<_3}(\{c,d\}) = \{Ec, Ed\}$.

Also, for every set X, let $\mathit{perm}(X)$ denote the set of all permutations of the elements of X.

For example $\mathit{perm}(\{a,b,c\}) = \{abc, acb, bac, bca, cab, cba\}$.

We are now able to provide a constructive definition of all total representations of a given interval order.

Definition 59.

1. A set of sequences $ISR(<) \subseteq \mathscr{X}$ defined as:

$$ISR(<) = perm(B_<(A_1))perm(E_<(A_1))\dots perm(B_<(A_n))perm(E_<(A_n))$$

is called the set of all **interval sequence representations** of the interval order $<$.

2. A set of total orders $TO(<) \subseteq \mathscr{X} \times \mathscr{X}$ defined as

$$TO(<) = \{\lhd_x \mid x \in ISR(<)\}$$

is called the set of all **total order representations** of the interval order $<$. ■

For example for $<_3$ of Fig. 1 and $<_4^Q$ of Fig. 5 we have

$$IRS(<_3) = \left\{ \begin{array}{l} BaEaBbBcEbBdEcEd, BaEaBbBcEbBdEdEc, \\ BaEaBcBbEbBdEdEc, BaEaBcBbEbBdEcEd \end{array} \right\}, \text{ and}$$

$IRS(<_4^Q) = \{BaBcEaBbEbEc, BaBcEaBbEcEb, BcBaEaBbEbEc, BcBaEaBbEcEb\}$.

The following result justifies Definition 59.

Theorem 60. *Let X be a finite set, $<$ be an interval order over X, and $\prec$ be a total order over $\mathscr{X}$. The following two properties are equivalent:*

1. *for each $a \in X$, $Ba \prec Ea$ and for all $a, b \in X$, $a < b \iff Ea \prec Bb$,*
2. *$\prec \in TO(<)$.*

Proof. (2)⇒(1) Suppose that $\prec \in TO(<)$ and let $x \in ISR(<)$ be such that $\prec = \lhd_x$. Let $a \in X$. Note that either $first_<(a) = last_<(a)$ or $first_<(a) \ll_< last_<(a)$. From Definition 59(1), it immediately follows that $Ba \lhd_x Ea$, i.e. $Ba \prec Ea$.

Now suppose $a < b$. Since $a \in last_<(a)$ and $b \in first_<(b)$, from Proposition 56(2), we have $last_<(a) \ll_< first_<(b)$. But since $\ll_<$ is a total order of maximal antichains, $last_<(a) \ll_< first_<(b)$ if and only if $x = \dots Ea \dots Bb \dots$, so $a < b \iff Ea \lhd_x Bb \iff Ea \prec Bb$.

(1)⇒(2) Suppose that each $a \in X$, $Ba \prec Ea$ and for all $a, b \in X$, $a < b \iff Ea \prec Bb$. Let x be such that $\prec = \lhd_x$. We just have to show that $x \in ISR(<)$. Suppose $x \notin ISR(<)$. For every $y \in ISR(<)$ we can write $x = v\,x_1$, $y = v\,y_1$. Let y_0 be such element of $ISR(<)$ that the length of prefix v is maximal.

We have to consider four cases:

Case 1. $x = u\;Ba\;u_x$, $y_0 = u\;Bb\;u_{y_0}$. Hence $x = u\;Ba\;v_1\;Bb\;v_2$ and $y_0 = u\;Bb\;z_1\;Ba\;z_2$. Suppose $z_1 = s\;Ec\;s_1$, i.e. $y_0 = u\;Bb\;s\;Ec\;s_1\;Ba\;z_2$, which means $Ec \lhd_{y_0} Ba$ i.e. $c < a$. Since Ec does not appear in u, we also have $x = u\;Ba\;t\;Ec\;t_1$, which means $Ba \lhd_x Ec$, or $Ba \prec Ec$ i.e. $\neg(c < a)$, a contradiction. This means $z_1 = Ba_1 \dots Ba_m$, so $y_0 = v\;BbBa_1 \dots Ba_m Ba\;z_2$. But $y_0 \in IRS(<)$, so from Definition 59(2) we have that $y_1 = vBaBbBa_1 \dots Ba_m\;z_2 \in IRS(<)$, so u is not maximal, as uBa is a prefix of both x and y_1. Therefore the Case 1 cannot happen.

Case 2. $x = u\;Ea\;u_x$, $y_0 = u\;Eb\;u_{y_0}$. Hence $x = u\;Ea\;v_1\;Eb\;v_2$ and $y_0 = u\;Eb\;z_1\;Ea\;z_2$. Suppose $z_1 = s\;Bc\;s_1$, i.e. $y_0 = u\;Eb\;s\;Bc\;s_1\;Ea\;z_2$, which means $Bc \lhd_{y_0} Ea$ i.e. $\neg(a < c)$. Since Bc does not appear in u, we also have $x = u\;Ea\;t\;Bc\;t_1$, which means $Ea \lhd_x Bc$, or $Ea \prec Bc$ i.e. $a < c$, a contradiction. This means $z_1 = Ea_1 \dots Ea_m$, so $y_0 = v\;EbEa_1 \dots Ea_m Ea\;z_2$. But $y_0 \in IRS(<)$, so from Definition 59(2) we have that $y_1 = vEaEbEa_1 \dots Ea_m\;z_2 \in IRS(<)$, so u is not maximal, as uEa is a prefix of both x and y_1. Therefore the Case 2 also cannot happen.

Case 3. $x = u\;Ba\;u_x$, $y_0 = u\;Eb\;u_{y_0}$. Hence $x = u\;Ba\;v_1\;Eb\;v_2$, which means $Ba \lhd_x Eb$, or $Ba \prec Eb$, i.e. $\neg(b < a)$, and $y_0 = u\;Eb\;z_1\;Ba\;z_2$, which means $Eb \lhd_{y_0} Ba$. i.e. $b < a$, a contradiction, so the Case 3 is not valid.

Case 4. $x = u\;Ea\;u_x$, $y_0 = u\;Bb\;u_{y_0}$. Hence $x = u\;Ea\;v_1\;Bb\;v_2$, which means $Ea \lhd_x Bb$, or $Ea \prec Bb$, i.e. $a < b$, and $y_0 = u\;Bb\;z_1\;Ea\;z_2$, which means $Bb \lhd_{y_0} Ea$. i.e. $\neg(a < b)$, a contradiction, so the Case 4 is not valid too. □

The important fact is that the set $TO(<)$ contains *all* (up to name isomorphism) total representations of an interval order $<$, and the set $IRS(<)$ contains *all* sequence representations of $<$.

We will now apply the obtained results to interval sequences. Let Σ be a set of events, $\mathscr{E} = \{Ba \mid a \in \Sigma\} \cup \{Ea \mid a \in \Sigma\}$, and $\mathsf{InSeq}(\mathscr{E}^*)$ be the set of all sequences over $\mathscr{E}$ that define interval orders (see Definition 6(1)).

Proposition 61. *For every $x \in \mathsf{InSeq}(\mathscr{E}^*)$, we have $\widehat{x} \in ISR(\blacktriangleleft_x)$.*

Proof. By Corollary 7 we have $\lhd_x = \lhd_{\widehat{x}}$ and $\blacktriangleleft_x = \blacktriangleleft_{\widehat{x}}$, and from Theorem 60 it follows that $\lhd_x \in TO(\blacktriangleleft_x)$. Hence, $\lhd_{\widehat{x}} \in TO(\blacktriangleleft_{\widehat{x}}) = \{\lhd_z \mid z \in ISR(\blacktriangleleft_{\widehat{x}})\}$, which means $\widehat{x} \in ISR(\blacktriangleleft_{\widehat{x}}) = ISR(\blacktriangleleft_x)$. □

Proposition 61 allows us to provide very short and simple proofs of Propositions 14 and 44.

Proposition 14. *Let* $(\mathscr{E}, ind)$ *be an interval trace alphabet, and* $x \in \mathsf{InSeq}(\mathscr{E}^*)$*, then for each* $y \in \mathsf{InSeq}(\mathscr{E}^*)$

$$\blacktriangleleft_x = \blacktriangleleft_y \implies x \equiv_{ind} y.$$

Proof. Since $\blacktriangleleft_x = \blacktriangleleft_y$ then clearly $ISR(\blacktriangleleft_x) = ISR(\blacktriangleleft_y)$. Hence, by Proposition 61, $\widehat{x}, \widehat{y} \in ISR(\blacktriangleleft_x)$. Two elements of ISR differ only by permutations of subsequences that consist each only of begins or only of ends (see Definition 59(1)). Hence, by Definition 12(2), if $\widehat{x}, \widehat{y} \in ISR(\blacktriangleleft_x)$, then $x \equiv_{ind} y$. $\square$

Proposition 44. *If* $x \in \mathsf{FIS}_N(m \to m')$*, then for every* $y \in \mathscr{T}^*$*, if* $\blacktriangleleft_x = \blacktriangleleft_y$ *then* $y \in \mathsf{FIS}_N(m \to m')$*.*

Proof. Since $\blacktriangleleft_x = \blacktriangleleft_y$ then clearly $ISR(\blacktriangleleft_x) = ISR(\blacktriangleleft_y)$. Hence, by Proposition 61, $\widehat{x}, \widehat{y} \in ISR(\blacktriangleleft_x)$. We just need to show that if $\{\widehat{x}, \widehat{y}\} \subseteq ISR(\blacktriangleleft_x)$ then $x \in \mathsf{FIS}_N(m \to m')$ implies $y \in \mathsf{FIS}_N(m \to m')$.

All elements of $ISR(\blacktriangleleft_x)$ satisfy a pattern given by Definition 59(2). Assume $\ll_{\blacktriangleleft_x} = A_1 \ldots A_m$. Hence $\widehat{x} = u_1 v_1 \ldots u_n v_n$ and $\widehat{y} = s_1 t_1 \ldots s_n t_n$, where for all $i = 1, \ldots, n$, $u_i, s_i \in perm(B_{\blacktriangleleft_x}(A_i))$ and $v_i, t_i \in perm(E_{\blacktriangleleft_x}(A_i))$.

Assume that $m[\![\widetilde{u}_1\rangle\!\rangle m_1^1 [\![\widetilde{v}_1\rangle\!\rangle m_1^2 \ldots m_{n-1}^2 [\![\widetilde{u}_n\rangle\!\rangle m_n^1 [\![\widetilde{v}_n\rangle\!\rangle m'$. We need prove that $m[\![\widetilde{s}_1\rangle\!\rangle m_1^1 [\![\widetilde{t}_1\rangle\!\rangle m_1^2 \ldots m_{n-1}^2 [\![\widetilde{s}_n\rangle\!\rangle m_n^1 [\![\widetilde{t}_n\rangle\!\rangle m'$ also hold.

Since both u_1 and s_1 belong to $perm(B_{\blacktriangleleft_x}(A_1))$, from Definition 41 we have that $m[\![\widetilde{u}_1\rangle\!\rangle m_1^1$ implies $m[\![\widetilde{s}_1\rangle\!\rangle m_1^1$. Similarly both v_1 and t_1 belong to $perm(E_{\blacktriangleleft_x}(A_1))$, so from Definition 41 we have that $m_1^1 [\![\widetilde{v}_1\rangle\!\rangle m_1^2$ implies $m_1^1 [\![\widetilde{t}_1\rangle\!\rangle m_1^2$. Repeating this reasoning $n-1$ times we obtain $m[\![\widetilde{s}_1\rangle\!\rangle m_1^1 [\![\widetilde{t}_1\rangle\!\rangle m_1^2 \ldots m_{n-1}^2 [\![\widetilde{s}_n\rangle\!\rangle m_n^1 [\![\widetilde{t}_n\rangle\!\rangle m'$, i.e. $m[\![y\rangle\!\rangle m'$, i.e. $y \in \mathsf{FIS}_N(m \to m')$. $\square$

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