Chapter 12
Property-Driven Rough Sets Approximations of Relations

Ryszard Janicki*

Abstract. The problem of approximating an arbitrary relation by a relation with desired properties is formally defined and analysed. Two special cases, approximation by partial orders and approximation by equivalence relations are discussed in detail.

Keywords: Approximation of raw data, properties of relations, rough sets, α-approximations, partial order, equivalence relation.

12.1 Introduction

While, in general, sets are just arbitrary collections of arbitrary elements [8], when they are applied in other parts of Mathematics or Science, they usually have some structure and properties. Their elements are usually engaged in complex relationships. While a collection that consists of, say, a white elephant, computer mouse, empty set, and a letter 'a', is a proper set (c.f. [8, 12]), in most applications the sets are more homogenous, as 'sets of integers', 'vertices', 'variables', etc., and quite often they have some very specific structures like 'trees', 'partitions', 'partial orders', etc.

Those structures and properties are essential when it comes to the problem of *approximation* of raw empirical data by appropriate mathematical concepts.

The simplest and most abstract way of modelling complex connections relationships is to use the notion of *relation*.

Ryszard Janicki
Department of Computing and Software,
McMaster University,
Hamilton, ON, L8S 4K1 Canada
e-mail: janicki@mcmaster.ca

* Partially supported by NSERC grant of Canada.

A. Skowron and Z. Suraj (Eds.): Rough Sets and Intelligent Systems, ISRL 42, pp. 333–357.
springerlink.com

The problem we will try to deal with in this chapter can be formulated as follows. We have a set of data that have been obtained in an empirical manner. From the nature of the problem, we know that the set should have some structure and desired properties, for example, it should be partially ordered, or partially ordered for one attribute and partitioned by some equivalence relation for another attribute (so it should be represented by two binary relations) but because the data are empirical it is not. In general case, this might be just an arbitrary set without the desired structure and properties. *What is the 'best' approximation that has the desired structure and properties and how it can be computed?* For the approximation of arbitrary relations by partial orders, this problem was discussed and some solutions were proposed in [6] (within both the standard theory of relations [8, 12] and Rough Sets paradigm [9, 10]).

In this chapter, we will generalise and refine some ideas of [6] to arbitrary relations, and we will illustrate our concepts by showing approximations by two of the most often used kinds of relations, partial orders and equivalence relations.

It appears that the concept of approximation has two different intuitions in Mathematics and Science. The first one stems from the fact that often, empirical numerical data have some errors, so in reality, we seldom have the value x (unless the measurements are expressible in integers) but usually some interval $(x-\varepsilon, x+\varepsilon)$, that is, the lower approximation and the upper approximation. Rough Sets [9, 10] exploit this idea for general sets. The second intuition can be illustrated by *least square approximation* of points in the two-dimensional plane (c.f. [15]). Here we know or assume that the points should be on a straight line and we are trying to find the line that fits the data best. In this case the data have a structure (points in two dimensional plane, that is, a relation that is a function) and should satisfy a desired property (be on the straight line). Note that even if we replace a solution $f(x) = ax + b$ by two lines $f_1(x) = ax + b - \delta$ and $f_2(x) = ax + b + \delta$, where δ is a standard error (c.f. [15]), there is no guarantee that any point resides between $f_1(x)$ and $f_2(x)$. Hence this is not the case of an upper, or lower approximation in the sense of Rough Sets. However this approach assumes that there is a well-defined concept of a *metric* which allows us to minimise the distance, and this concept is not obvious, and often not even possible for non-numerical objects (see for instance [6]).

The approach presented in this chapter is a mixture of both intuitions. There is no metric, but the concept of "minimal distance" is replaced and somehow simulated by a sequence of property-driven lower and/or upper approximations, in the style of Rough Sets.

The chapter is structured as follows. The next section provides basic facts about relations. Section 12.3 recalls the classical Rough Set approach to the approximation of relations. In Section 12.4, the concept of property-driven rough approximation of arbitrary relations is introduced, and the basic definitions are given. The next three sections provide basic theoretical framework for the approach presented. Section 12.5 deals with a single property (as for instance transitivity only), Section 12.6 provides an analysis of a composition of two properties (for instance symmetry and transitivity), and Section 12.7 extends the obtained results to a composition of an arbitrary number of properties. In Section 12.8, we use the ideas presented in previous sections to approximate an arbitrary binary relation by a partial order, and in Section 12.9 to approximate an arbitrary binary relation by an equivalence relation. Section 12.8 refines some results of [6]. The last section contains final comments.

12.2 Relations and Some of Their Basic Classifications

In this section, we recall some fairly known concepts and results that will be used in the following sections [2, 8, 12].

Let X be a set, any $R \subseteq \underbrace{X \times X \times ... \times X}_{n} = \prod_{i=1}^{n} X$ is called an *n-ary relation* (on X).

If $n = 2$, that is, $R \subseteq X \times X$ then R is called a *binary relation* (on X).

Customarily, we will use the generic name *relation* for both n-ary and binary relations and apply the prefixes 'n-ary' and 'binary' only when needed. For the rest of this section, we assume that any relation is a binary relation, that is, a relation $R \subseteq X \times X$. We also will often write aRb to denote $(a,b) \in R$.

Definition 12.1 (Basic Types of Relations). Let $R, <, and \equiv$ be relations on X.

1. $id_X = \{(x,x) \mid x \in X\}$, or just id, is called the identity relation.
2. R is reflexive iff $id \subseteq R$, that is, $(x,x) \in R$ for all $x \in X$.
3. R is irreflexive iff $id \cap R = \emptyset$, that is, $(x,x) \notin R$ for all $x \in X$.
4. R is symmetric iff for all $x,y \in X$, $xRy \Rightarrow yRx$.
5. R is transitive iff for all $x,y,z \in X$, $xRy \wedge yRz \Rightarrow xRz$.
6. A relation $\equiv$ is an equivalence relation iff it is reflexive, symmetric and transitive, that is, $x \equiv x$, $x \equiv y \Rightarrow y \equiv x$, and $x \equiv y \equiv z \Rightarrow x \equiv z$, for all $x,y,z \in X$.
7. A relation $<$ is a (sharp) partial order iff it is irreflexive and transitive, that is, $\neg(x < x)$ and $x < y < z \Rightarrow x < z$ for all $x,y,z \in X$. □

For every equivalence relation $\equiv$ on X and every $x \in X$, the set $[x]_{\equiv} = \{y \mid x \equiv y\}$ denotes an *equivalence class* containing the element x.

We also have $[x]_{\equiv} = [y]_{\equiv}$if and only if $x \equiv y$ (c.f. [2, 8, 12]).

The set of all equivalence classes of an equivalence relation $\equiv$ is denoted as $X/_{\equiv}$, and it is a *partition* of X, that is, the sets from $X/_{\equiv}$ are disjoint and cover the whole X.

For every two relations R,S on X, the relational composition $R \circ S$ is defined as $a(R \circ S)c$if and only if $\exists b \in X.\ aRb \wedge bRc$, for all $a,c \in X$; for every relation R on X, we have $R^{-1} = \{(a,b) \mid (b,a) \in R\}$, and $R^0 = id$, $R^k = \underbrace{R \circ ... \circ R}_{k}$ for $k > 0$.

For every relation R on X, the smallest transitive (reflexive, symmetric, etc.) relation on X containing R is called the *transitive* (*reflexive*, *symmetric*, etc.) *closure* of R (c.f. [2, 12]).

Proposition 12.1 (Explicit Expresions for Closures [2, 12]). *Let R be a relation on X.*

1. $R^{ref} = R \cup id$ *is the reflexive closure of* R.
2. $R^{\overline{sym}} = R \cup R^{-1}$ *is the symmetric closure of* R.
3. $R^{+} = \bigcup_{i=1}^{\infty} R^{i}$ *is the transitive closure of* R.
4. $R^{*} = \bigcup_{i=0}^{\infty} R^{i}$ *is the reflexive-transitive closure of* R. □

Closures correspond to simple upper approximations of relations in the sense of Rough Sets. Concepts corresponding to lower approximations are more complex and less systematic and will be discussed later (in Section 12.8).

12.3 Classical Rough Relations

The principles of Rough Rets [9, 10] can be formulated as follows. Let U be a finite and nonempty universe of elements, and let $E \subseteq U \times U$ be an *equivalence relation*. The elements of $U/_E$ are called elementary sets, and they are interpreted as basic observable, measurable, or definable sets. The pair (U,E) is referred to as a Pawlak approximation space. A set $X \subseteq U$ is approximated by two subsets of U, $\underline{\mathbf{A}}(X)$ - called the lower approximation of X, and $\overline{\mathbf{A}}(X)$ - called the upper approximation of X, where $\underline{\mathbf{A}}(X)$ and $\overline{\mathbf{A}}(X)$ are defined as follows.

Definition 12.2 ([9, 10])

1. $\underline{\mathbf{A}}(X) = \bigcup\{[x]_E \mid x \in U \wedge [x]_E \subseteq X\}$,
2. $\overline{\mathbf{A}}(X) = \bigcup\{[x]_E \mid x \in U \wedge [x]_E \cap X \neq \emptyset\}$. □

Rough set approximations satisfy the following properties:

Proposition 12.2 (Pawlak [10])

1. $X \subseteq Y \Longrightarrow \underline{\mathbf{A}}(X) \subseteq \underline{\mathbf{A}}(Y)$,
2. $\underline{\mathbf{A}}(X \cap Y) = \underline{\mathbf{A}}(X) \cap \underline{\mathbf{A}}(Y)$,
3. $\underline{\mathbf{A}}(X) \subseteq X$,
4. $\underline{\mathbf{A}}(X) = \underline{\mathbf{A}}(\underline{\mathbf{A}}(X))$,
5. $\overline{\mathbf{A}}(X) = \underline{\mathbf{A}}(\overline{\mathbf{A}}(X))$,
6. $X \subseteq Y \Longrightarrow \overline{\mathbf{A}}(X) \subseteq \overline{\mathbf{A}}(Y)$,
7. $\overline{\mathbf{A}}(X \cup Y) = \overline{\mathbf{A}}(X) \cup \overline{\mathbf{A}}(Y)$,
8. $X \subseteq \overline{\mathbf{A}}(X)$,
9. $\overline{\mathbf{A}}(X) = \overline{\mathbf{A}}(\overline{\mathbf{A}}(X))$,
10. $\underline{\mathbf{A}}(X) = \overline{\mathbf{A}}(\underline{\mathbf{A}}(X))$. □

Since every relation is a set of pairs, this approach can be used for relations as well [13]. Unfortunately, in such cases as ours, we want approximations to have some specific properties like irreflexivity, transitivity etc., and most of those properties are not closed under the set union operator. As was pointed out in [17], in general, one cannot expect approximations to have the desired properties (see [17] for details). It is also unclear how to define the relation E for cases such as ours.

janicki@mcmaster.ca

However, the Rough Sets can also be defined in an orthogonal (sometimes called 'topological') manner [10, 14, 16]. For a given (U,E), we may define $\mathcal{D}(U)$ as the smallest set containing $\emptyset$, all of the elements of $U/_E$ and that is closed under set union. Clearly, $U/_E$ is the set of all components generated by $\mathcal{D}(U)$ [8]. We may start with defining a space as $(U,\mathcal{D})$, where $\mathcal{D}$ is a family of sets that contains $\emptyset$, and for each $x \in U$, there is $X \in \mathcal{D}$ such that $x \in X$ (i.e. $\mathcal{D}$ is a cover of U [12]). We may now define $E_{\mathcal{D}}$ as the equivalence relation generated by the set of all components defined by $\mathcal{D}$ (see for example [8]). Hence, both approaches are equivalent [10, 14, 17]; however, now for each $X \subseteq U$, we might use different formulas for $\underline{\mathbf{A}}(X)$ and $\overline{\mathbf{A}}(X)$.

Proposition 12.3 ([10, 14, 17])

1. $\underline{\mathbf{A}}(X) = \bigcup\{Y \mid Y \subseteq X \wedge Y \in \mathcal{D}\}$,
2. $\overline{\mathbf{A}}(X) = \bigcap\{Y \mid X \subseteq Y \wedge Y \in \mathcal{D}\}$. □

We can now define $\mathcal{D}$ as a set of relations having the desired properties and then calculate $\underline{\mathbf{A}}(R)$ and/or $\overline{\mathbf{A}}(R)$ with respect to a given $\mathcal{D}$. Such an approach was proposed and analysed in [17]; however, it seems to have only limited applications. It assumes that the set $\mathcal{D}$ is closed under both union and intersection, and few properties of relations do this. For instance, transitivity is not closed under union and having a cycle is not closed under intersection. Some properties, like 'having exactly one cycle', are preserved by neither union nor intersection. This problem was discussed in [17], and they proposed that perhaps a different $\mathcal{D}$ could be used for the lower and upper approximations. But this solution again seems to have rather limited applications. The approach of [17] assumes additionally that, for the upper approximation there is at least one element of $\mathcal{D}$ that contains R, and, for the lower approximation there exists at least one element of $\mathcal{D}$ that is included in R. These are assumptions that are too strong for many applications (see [6]). If R contains a cycle, then there is no partial order that contains R!

To solve those problems, we need to create a new setting.

12.4 Property-Driven Rough Approximations of Relations

In this section, we will provide a formal basis of our approach.

Let X be a set and $\mathbf{X} = \prod_1^n X = \underbrace{X \times \ldots \times X}_{n}$. We assume that in this section any relation is an n-ary relation and a subset of $\mathbf{X}$.

Definition 12.3. 1. Any first-order predicate (c.f. [3]) α containing one atomic n-ary relational symbol R (which may occur more than once) will be called an **n-ary relational property** (or just a property).

2. Let $R \subseteq \mathbf{X}$. An n-ary relational property α is called a **property of** the n-ary relation R if the symbol R is interpreted as the relation R, all variables of α

are over the set X, and the tuple (X,R) is a model of α, that is, α holds for any assignment (c.f. *[3]*). □

Obvious examples of properties are *transitivity* ($\alpha = (\forall a,b,c.\ \mathsf{R}(a,b) \wedge \mathsf{R}(b,c) \Rightarrow \mathsf{R}(a,c))$), *reflexivity* ($\alpha = (\forall a.\ \mathsf{R}(a,a))$), *symmetry* ($\alpha = (\forall a,b.\ \mathsf{R}(a,b) \Rightarrow \mathsf{R}(b,a))$ etc., for binary relations. Standardly, when it does not cause any confusion, the same symbol is used to denote both R and R. We would like to point out the difference between *a property*, that is, just a statement that may or may not be true and where R is just a symbol, and *a property of R*, a statement that is true for all assignments, and R is a well-defined relation.

Definition 12.4. Let $\mathcal{P}$ be a finite set of *n-ary relational properties*, such that for each $\alpha \in \mathcal{P}$, there is a non-empty relation $Q \subseteq \mathbf{X}$, and α is a *property of Q*.

1. Any element $\alpha \in \mathcal{P}$ is called an **elementary property**.
2. For each elementary property $\alpha \in \mathcal{P}$, $P_\alpha \subseteq 2^{\mathbf{X}}$ is the set of n-ary relations over X that satisfy the property α. □

Definition 12.4 allows $\emptyset \in P_\alpha$, but disallows $P_\alpha = \emptyset$ and $P_\alpha = \{\emptyset\}$.

Even though any property can be called 'elementary', it is assumed that in any concrete case the elemetary properties are 'simple' and 'regular'. They are just atomic parts from which the real more sophisticated properties are built.

Definition 12.5. 1. For every $\alpha \in \mathcal{P}$, P_α is closed under intersection iff for each $R,S \in P_\alpha$, $R \cap S \in P_\alpha$. The set of all $\alpha \in \mathcal{P}$ that are closed under intersection will be denoted by $\mathcal{P}^\cap$.
2. For every $\alpha \in \mathcal{P}$, P_α is closed under union iff for each $R,S \in P_\alpha$, $R \cup S \in P_\alpha$. The set of all $\alpha \in \mathcal{P}$ that are closed under union will be denoted by $\mathcal{P}^\cup$. □

Some examples of properties for binary relations:

- $\alpha =$*transitivity*, or $\alpha =$*partial ordering*, P_α is closed under intersection but not under union,
- $\alpha =$*symmetry*, P_α is closed both under intersection and under union,
- $\alpha =$*having a cycle*, P_α is closed under union but not under intersection.

Assumption 1. *We assume that if* $\alpha \in \mathcal{P}$ *then,* P_α *is either closed under union or it is closed under intersection (or both), that is,* $\mathcal{P} = \mathcal{P}^\cup \cup \mathcal{P}^\cap$. □

This assumption is much weaker than it might appear as this is an assumption only about *elementary* properties, not about composite more sophisticated properties that will be considered later. However, it is absolutely needed as we want to define lower and upper approximations in the style of Proposition 12.3 and want at least one of them to exist.

Definition 12.6. The pair $(\mathbf{X}, \{P_\alpha \mid \alpha \in \mathcal{P}\})$ will be called an **property-driven approximation space for the n-ary relations over X.** □

When $\emptyset \in \{P_\alpha \mid \alpha \in \mathcal{P}\}$ and $\{P_\alpha \mid \alpha \in \mathcal{P}\}$ is a cover then, Definition 12.6 corresponds to the definition of space $(U, \mathcal{D})$ from Section 12.3. In Section 12.3, the elements of $\mathcal{D}$ were used to construct lower and upper approximations $\underline{\mathbf{A}}(X)$ and $\overline{\mathbf{A}}(X)$ (see Proposition 12.3), here our intention is to use the elements of $\{P_\alpha \mid \alpha \in \mathcal{P}\}$ as building bricks of our property-driven lower and upper approximations. However, as opposed to the properties of $\mathcal{D}$, it may happen that $\emptyset \notin \{P_\alpha \mid \alpha \in \mathcal{P}\}$ and that $\{P_\alpha \mid \alpha \in \mathcal{P}\}$ is not a cover.

Intuitively, for every relation R and every property $\alpha \in \mathcal{P}$, we expect an appropriate lower approximation of R to be a subset of R that belongs to P_α, and an appropriate upper approximation of R to be a superset of R that also belongs to P_α. Note that, these are weaker expectations than required from classical rough set approximations where we expect 'the largest subset of R' for lower and 'the smallest superset of R' for upper approximation. However, even this may not always be possible, which leads us to the following definition.

Definition 12.7. Let $R \subseteq \mathbf{X}$ be a non-empty relation and $\alpha \in \mathcal{P}$. We say that:

1. R has α-lower bound if and only if $\exists Q \in P_\alpha.\ Q \subseteq R$,
2. R has α-upper bound if and only if $\exists Q \in P_\alpha.\ R \subseteq Q$.

We also define

3. $lb_\alpha(R) = \{Q \mid Q \in P_\alpha \wedge Q \subseteq R\}$, the set of all α-lower bounds of R, and
4. $ub_\alpha(R) = \{Q \mid Q \in P_\alpha \wedge R \subseteq Q\}$, the set of all α-upper bounds of R. □

Note that, if the relation $\mathbf{X} = \prod_{i=1}^{n} X$ satisfies α, then α-upper bound exists for any $R \subseteq \mathbf{X}$, and if the relation $\emptyset$ satisfies α, then α-lower bound exists for any $R \subseteq \mathbf{X}$.

Some examples for binary relations:

- $\alpha =$*transitivity*, R - any relation, both α-lower bound and α-upper bound do exist,
- $\alpha =$*reflexivity*, R - any relation, α-lower bound exists only when R is already reflexive, α-upper bound does exist,
- $\alpha =$*irreflexivity*, R - any relation, α-lower bound does exist, α-upper bound exists only when R is already irreflexive,
- $\alpha =$*symmetry*, R - any relation, α-lower bound may not exist while α-upper bound does exist,
- $\alpha =$*partial ordering*, R has a cycle, α-lower bound exists but α-upper bound does not exist.
- $\alpha =$*partial ordering*, R - any relation, α-lower bound exists, α-upper bound may not exist,
- $\alpha =$*equivalence*, R - any relation, α-lower bound exists only when R is reflexive, α-upper bound does exist,
- $\alpha =$*having a cycle*, R is a partial order, α-lower bound does not exist, α-upper bound exists,

- α =*having a cycle*, R - any relation, α-lower bound may not exist, α-upper bound exists,
- $\alpha = (R(a,b) \wedge \neg R(c,d))$, R any relation such that $(a,b) \notin R$ and $(c,d) \in R$, neither α-lower bound nor α-upper bound exists.

The remaining auxilliary concepts that are needed to formally define lower and upper approximations that preserve elementary properties of relations are the well known concepts of maximal and minimal elements of families of relations (c.f. [2, 12]).

Definition 12.8. For every family of relations $\mathcal{F} \subseteq 2^{\mathbf{X}}$, we define

1. $min(\mathcal{F}) = \{R \mid \forall Q \in \mathcal{F}. Q \subseteq R \Rightarrow R = Q\}$, the set of all minimal elements of $\mathcal{F}$,
2. $max(\mathcal{F}) = \{R \mid \forall Q \in \mathcal{F}. R \subseteq Q \Rightarrow R = Q\}$, the set of all maximal elements of $\mathcal{F}$.

□

We are now able to provide the two main definitions of our model.

Definition 12.9 (α-lower and α-upper approximations)

1. If R has α-lower bound then we define its α-*lower approximation* as:

$$\underline{\mathbf{A}}_{\alpha}(R) = \bigcap \{Q \mid Q \in max(lb_{\alpha}(R))\}.$$

2. If R has α-upper bound then we define its α-*upper approximation* as:

$$\overline{\mathbf{A}}_{\alpha}(R) = \bigcup \{Q \mid Q \in min(ub_{\alpha}(R))\}.$$

□

If R *does not have* α-lower bound (α-upper bound) then its α-lower approximation (α-upper approximation) *does not exist*. This is the major difference between this model and the standard Rough Sets model. It might happen that neither α-lower approximation nor α-upper approximation exists. Then α should probably not be called an 'elementary' property and it should instead be decomposed into a conjunction of simpler properties. This problem will be discussed in Section 12.6.

To show that Definition 12.9 is sound, we need to prove the following:

(1) if the relation R has a property α, both approximations are reduced to identity,
(2) for every property $\alpha \in \mathcal{P}$, both $\underline{\mathbf{A}}_{\alpha}(R)$ and $\overline{\mathbf{A}}_{\alpha}(R)$ satisfy the property α (if they exist), first of all this is what they were invented for,
(3) when a property α is closed under both union and intersection, and an α-(lower/upper) approximation exists, it should be identical to the standard lower/upper approximation (either that of Definition 12.2 or its equivalent version from Proposition 12.3).

The result below proves the point (1).

Proposition 12.4. *If* $R \in P_{\alpha}$ *then* $\underline{\mathbf{A}}_{\alpha}(R) = \overline{\mathbf{A}}_{\alpha}(R) = R$.

Proof. If $R \in P_\alpha$ then $lb_\alpha(R) = ub_\alpha(R) = \{R\}$. □

The proof of point (2) will be split into two parts.

Proposition 12.5

1. *If* $\alpha \in \mathcal{P}^\cap$ *and R has* α*-lower bound then*
$$\underline{\mathbf{A}}_\alpha(R) = \bigcap\{Q \mid Q \in max(lb_\alpha(R))\} \in P_\alpha.$$
2. *If* $\alpha \in \mathcal{P}^\cup$ *and R has* α*-upper bound then*
$$\overline{\mathbf{A}}_\alpha(R) = \bigcup\{Q \mid Q \in min(ub_\alpha(R))\} \in P_\alpha.$$

Proof
(1) Every element of $max(lb_\alpha(R))$ is in P_α. Since $\alpha \in \mathcal{P}^\cap$, the intersection of all elements of $max(lb_\alpha(R))$ is also in P_α.
(2) Every element of $min(ub_\alpha(R))$ is in P_α. Since $\alpha \in \mathcal{P}^\cup$, the union of all elements of $min(ub_\alpha(R))$ also is in P_α. □

The second part involves new representations of both $\underline{\mathbf{A}}_\alpha(R)$ and $\overline{\mathbf{A}}_\alpha(R)$, more or less in the style of $\underline{\mathbf{A}}(R)$ and $\overline{\mathbf{A}}(R)$ from Proposition 12.3.

Proposition 12.6

1. *If* $\alpha \in \mathcal{P}^\cup$ *and R has* α*-lower bound, then*
$$\underline{\mathbf{A}}_\alpha(R) = \bigcup\{Q \mid Q \in lb_\alpha(R)\} = \bigcup\{Q \mid Q \subseteq R \wedge Q \in P_\alpha\} \in P_\alpha.$$
2. *If* $\alpha \in \mathcal{P}^\cap$ *and R has* α*-upper bound, then*
$$\overline{\mathbf{A}}_\alpha(R) = \bigcap\{Q \mid Q \in ub_\alpha(R)\} = \bigcap\{Q \mid R \subseteq Q \wedge Q \in P_\alpha\} \in P_\alpha.$$

Proof
(1) If $\alpha \in \mathcal{P}^\cup$ and R has α-lower bound, then $max(lb_\alpha(R))$ is a singleton set, that is, $max(lb_\alpha(R)) = \{Q\}$, where $Q = \bigcup\{S \mid S \in lb_\alpha(R)\}$. Every element of $lb_\alpha(R)$ is clearly in P_α. Since $\alpha \in \mathcal{P}^\cup$, the union of all elements of $lb_\alpha(R)$ is in P_α, that is, $Q \in P_\alpha$.
(2) If $\alpha \in \mathcal{P}^\cap$ and R has α-upper bound, then $min(ub_\alpha(R))$ is a singleton set, that is, $min(ub_\alpha(R)) = \{S\}$, where $S = \bigcap\{Q \mid Q \in ub_\alpha(R)\}$. Since $\alpha \in \mathcal{P}^\cap$, the intersection of all elements of $ub_\alpha(R)$ is in P_α, that is, $S \in P_\alpha$. □

The next result shows *when this model is exactly the same as the classical Rough Sets approach to relations* (the version from [16, 17] illustrated by Proposition 12.3). It is a proof of point (3) of the soudness requirements.

Corollary 12.1

1. *If* $\alpha \in \mathcal{P}^\cup \cap \mathcal{P}^\cap$ *and R has* α*-lower bound, then* $\underline{\mathbf{A}}_\alpha(R) = \underline{\mathbf{A}}(R)$, *and,*
2. *if* $\alpha \in \mathcal{P}^\cup \cap \mathcal{P}^\cap$ *and R has* α*-upper bound, then* $\overline{\mathbf{A}}_\alpha(R) = \overline{\mathbf{A}}(R)$,

where $\underline{\mathbf{A}}(R)$ *and* $\overline{\mathbf{A}}(R)$ *are classical upper and lower rough approximations over the space* $(X \times X, \{P_\alpha \mid \alpha \in \mathcal{P}\})$, *as defined in Proposition 12.3.*

Proof (1) From the second equality in Proposition 12.6(1).
(2) From the second equality in Proposition 12.6(2). □

In this section, we defined lower and upper approximations that provide desired relational properties. In the next section, we will discuss major properties of these approximations.

12.5 Properties of α-Approximations

In this section, we will show that the operational and compositional properties of α-lower and α-upper approximations are pretty close (but not identical) to those of standard rough set approximations as presented in Proposition 12.2. We start with the properties of α-lower approximation (compare with Proposition 12.2(1–4) for standard rough set lower approximation).

Proposition 12.7. *If* $R,Q \subseteq \mathbf{X}$ *have* α*-lower bound then:*

1. $R \subseteq Q \Longrightarrow \underline{\mathbf{A}}_\alpha(R) \subseteq \underline{\mathbf{A}}_\alpha(Q)$,
2. $\underline{\mathbf{A}}_\alpha(R) \subseteq R$,
3. $\underline{\mathbf{A}}_\alpha(R) = \underline{\mathbf{A}}_\alpha(\underline{\mathbf{A}}_\alpha(R))$,
4. $\underline{\mathbf{A}}_\alpha(R\cap Q) = \underline{\mathbf{A}}_\alpha(\underline{\mathbf{A}}_\alpha(R)\cap \underline{\mathbf{A}}_\alpha(Q))$,
5. *if* $\alpha \in \mathcal{P}^\cap$ *then* $\underline{\mathbf{A}}_\alpha(R\cap Q) = \underline{\mathbf{A}}_\alpha(R)\cap \underline{\mathbf{A}}_\alpha(Q)$,
6. *if* R *has* α*-upper bound then* $\overline{\mathbf{A}}_\alpha(R) = \underline{\mathbf{A}}_\alpha(\overline{\mathbf{A}}_\alpha(R))$.

Proof.
(1) Since $R \subseteq Q \Longrightarrow lb_\alpha(R) \subseteq lb_\alpha(Q) \Longrightarrow max(lb_\alpha(R)) \subseteq lb_\alpha(Q)$, then for each $S \in max(lb_\alpha(R))$, there is $S' \in max(lb_\alpha(Q))$ such that $S \subseteq S'$; and intersection preserves inclusion.
(2) Since $S \in lb_\alpha(R) \Longrightarrow S \subseteq R$, and intersection preserves inclusion.
(3) From Proposition 12.4 because $\underline{\mathbf{A}}_\alpha(R) \in P_\alpha$.
(4) By (1) we have $\underline{\mathbf{A}}_\alpha(R\cap Q) \subseteq \underline{\mathbf{A}}_\alpha(R)$ and $\underline{\mathbf{A}}_\alpha(R\cap Q) \subseteq \underline{\mathbf{A}}_\alpha(Q)$, so $\underline{\mathbf{A}}_\alpha(R\cap Q) \subseteq \underline{\mathbf{A}}_\alpha(R)\cap\underline{\mathbf{A}}_\alpha(Q)$. Hence, by (2) and (3) $\underline{\mathbf{A}}_\alpha(R\cap Q) \subseteq \underline{\mathbf{A}}_\alpha(\underline{\mathbf{A}}_\alpha(R)\cap\underline{\mathbf{A}}_\alpha(Q))$.
By the definition, we have $\underline{\mathbf{A}}_\alpha(\underline{\mathbf{A}}_\alpha(R)\cap\underline{\mathbf{A}}_\alpha(Q)) = \bigcap\{S \mid S \in max(lb_\alpha(\underline{\mathbf{A}}_\alpha(Q)\cap\underline{\mathbf{A}}_\alpha(Q)))\}$. Let $T \in lb_\alpha(\underline{\mathbf{A}}_\alpha(Q)\cap\underline{\mathbf{A}}_\alpha(Q)))$. This means $T \in P_\alpha \wedge T \subseteq \underline{\mathbf{A}}_\alpha(R)\cap\underline{\mathbf{A}}_\alpha(Q)$; hence, $T \in P_\alpha \wedge T \subseteq R \wedge T \subseteq Q$, that is, $T \in P_\alpha \wedge T \subseteq R\cap Q$. Therefore, $T \in lb_\alpha(R\cap Q)$. In this way, we proved that $lb_\alpha(\underline{\mathbf{A}}_\alpha(Q)\cap\underline{\mathbf{A}}_\alpha(Q)) \subseteq lb_\alpha(R\cap Q)$. Hence,
$max(lb_\alpha(\underline{\mathbf{A}}_\alpha(Q)\cap\underline{\mathbf{A}}_\alpha(Q))) \subseteq lb_\alpha(X\cap Q)$, that is, for each $S \in max(lb_\alpha(\underline{\mathbf{A}}_\alpha(Q)\cap\underline{\mathbf{A}}_\alpha(Y)))$, there exists $S' \in max(lb_\alpha(R\cap Q))$, such that $S \subseteq S'$. Since intersection preserves inclusion, this means that $\underline{\mathbf{A}}_\alpha(\underline{\mathbf{A}}_\alpha(R)\cap\underline{\mathbf{A}}_\alpha(Q)) \subseteq \underline{\mathbf{A}}_\alpha(R\cap Q)$.
(5) By (4) of this proposition, we have $\underline{\mathbf{A}}_\alpha(R\cap Q) = \underline{\mathbf{A}}_\alpha(\underline{\mathbf{A}}_\alpha(R)\cap\underline{\mathbf{A}}_\alpha(Q))$. If $\alpha \in \mathcal{P}^\cap$ then, $\underline{\mathbf{A}}_\alpha(R)\cap\underline{\mathbf{A}}_\alpha(Q) \in P_\alpha$, so by Proposition 12.4, we have
$\underline{\mathbf{A}}_\alpha(R)\cap\underline{\mathbf{A}}_\alpha(Q) = \underline{\mathbf{A}}_\alpha(\underline{\mathbf{A}}_\alpha(R)\cap\underline{\mathbf{A}}_\alpha(Q))$.

(6) If R has α-upper bound then $\overline{\mathbf{A}}_\alpha(R) \in P_\alpha$ so from Proposition 12.4, it follows that,
$\overline{\mathbf{A}}_\alpha(X) = \underline{\mathbf{A}}_\alpha(\overline{\mathbf{A}}_\alpha(X))$. □

The difference from the classical case is that intersection splits into two cases and mixing lower with upper α-approximation is conditional.

We will now present the properties of α-upper approximation (compare with Proposition 12.2(5–10) for standard rough set upper approximation).

Proposition 12.8. *If $R, Q \subseteq \mathbf{X}$ have α-upper bound then*

1. $R \subseteq Q \Longrightarrow \overline{\mathbf{A}}_\alpha(R) \subseteq \overline{\mathbf{A}}_\alpha(Q)$,
2. $R \subseteq \overline{\mathbf{A}}_\alpha(R)$,
3. $\overline{\mathbf{A}}_\alpha(R) = \overline{\mathbf{A}}_\alpha(\overline{\mathbf{A}}_\alpha(R))$,
4. $\overline{\mathbf{A}}_\alpha(R \cup Q) = \overline{\mathbf{A}}_\alpha(\overline{\mathbf{A}}_\alpha(R) \cup \overline{\mathbf{A}}_\alpha(Q))$,
5. *if* $\alpha \in \mathcal{P}^\cup$ *then* $\overline{\mathbf{A}}_\alpha(R \cup Q) = \overline{\mathbf{A}}_\alpha(R) \cup \overline{\mathbf{A}}_\alpha(Q)$,
6. *if R has α-lower bound then* $\underline{\mathbf{A}}_\alpha(R) = \overline{\mathbf{A}}_\alpha(\underline{\mathbf{A}}_\alpha(R))$.

Proof.
(1) Since $R \subseteq Q \Longrightarrow ub_\alpha(Q) \subseteq ub_\alpha(R) \Longrightarrow min(ub_\alpha(Q)) \subseteq ub_\alpha(R)$, then for each $S' \in min(ub_\alpha(Q))$ there is $S \in min(ub_\alpha(R))$ such that $S \subseteq S'$; and union preserves inclusion.
(2) Since $S \in ub_\alpha(R) \Longrightarrow R \subseteq S$, and union preserves inclusion.
(3) From Proposition 12.4 because $\overline{\mathbf{A}}_\alpha(R) \in P_\alpha$.
(4) By (1) we have $\overline{\mathbf{A}}_\alpha(R) \subseteq \overline{\mathbf{A}}_\alpha(R \cup Q)$ and $\overline{\mathbf{A}}_\alpha(Q) \subseteq \overline{\mathbf{A}}_\alpha(R \cup Q)$, so $\overline{\mathbf{A}}_\alpha(R) \cup \overline{\mathbf{A}}_\alpha(Q) \subseteq \overline{\mathbf{A}}_\alpha(R \cup Q)$. Hence, by (2) and (3) $\overline{\mathbf{A}}_\alpha(\overline{\mathbf{A}}_\alpha(R) \cup \overline{\mathbf{A}}_\alpha(Q)) \subseteq \overline{\mathbf{A}}_\alpha(R \cup Q)$.
Since $R \subseteq \overline{\mathbf{A}}_\alpha(R)$ and $Q \subseteq \overline{\mathbf{A}}_\alpha(Q)$ then, $R \cup Q \subseteq \overline{\mathbf{A}}_\alpha(R) \cup \overline{\mathbf{A}}_\alpha(Q)$, i.e.
$up_\alpha(\overline{\mathbf{A}}_\alpha(R) \cup \overline{\mathbf{A}}_\alpha(Q) \subseteq up_\alpha(R \cup Q)$, and consequently, $min(up_\alpha(\overline{\mathbf{A}}_\alpha(R) \cup \overline{\mathbf{A}}_\alpha(Q)) \subseteq up_\alpha(R \cup Q)$. Hence, for each $S' \in min(up_\alpha(\overline{\mathbf{A}}_\alpha(R) \cup \overline{\mathbf{A}}_\alpha(Q))$, there exists
$S \in min(up_\alpha(R \cup Q))$ such that $S \subseteq S'$. Since union preserves inclusion, we obtained
$\overline{\mathbf{A}}_\alpha(R \cup Q) \subseteq \overline{\mathbf{A}}_\alpha(\overline{\mathbf{A}}_\alpha(R) \cup \overline{\mathbf{A}}_\alpha(Q))$.
(5) By (4) of this proposition, $\overline{\mathbf{A}}_\alpha(R \cup Q) = \overline{\mathbf{A}}_\alpha(\overline{\mathbf{A}}_\alpha(R) \cup \overline{\mathbf{A}}_\alpha(Q))$. If $\alpha \in \mathcal{P}^\cup$ then
$\overline{\mathbf{A}}_\alpha(R) \cup \overline{\mathbf{A}}_\alpha(Q) \in P_\alpha$, so by Proposition 12.4 we have
$\overline{\mathbf{A}}_\alpha(R) \cup \overline{\mathbf{A}}_\alpha(Q) = \overline{\mathbf{A}}_\alpha(\overline{\mathbf{A}}_\alpha(R) \cup \overline{\mathbf{A}}_\alpha(Q))$.
(6) If R has α-lower bound then, $\underline{\mathbf{A}}_\alpha(R) \in P_\alpha$ so by Proposition 12.4 we have,
$\underline{\mathbf{A}}_\alpha(X) = \overline{\mathbf{A}}_\alpha(\underline{\mathbf{A}}_\alpha(X))$. □

Here the difference from the classical case is that union splits into two cases and mixing upper with lower α-approximation is conditional.

12.6 Composite Properties

Most of the interesting properties are composite properties. For example, a binary relation can be made a partial order by applying transitive closure first and making the outcome acyclic later, or in the opposite order (see Section 12.8 and [6]), or a relation can be made an equivalence relation by applying relexive, symmetric and transitive closures in this order (see Section 12.9). In this section, we will propose a framework for doing this kind of compositions in a systematic way.

In principle, we will try to solve the following problem. Suppose we have two properties α and β, but we are really interested in the property $\alpha \wedge \beta$. Under what circumstances do the approximations $\underline{\mathbf{A}}_\alpha(\underline{\mathbf{A}}_\beta(R))$, $\overline{\mathbf{A}}_\alpha(\underline{\mathbf{A}}_\beta(R))$, $\underline{\mathbf{A}}_\alpha(\overline{\mathbf{A}}_\beta(R))$ and $\overline{\mathbf{A}}_\alpha(\overline{\mathbf{A}}_\beta(R))$ exist and satisfy the property $\alpha \wedge \beta$? What is the relationship between $\underline{\mathbf{A}}_\alpha(\overline{\mathbf{A}}_\beta(R))$ and $\overline{\mathbf{A}}_\beta(\underline{\mathbf{A}}_\alpha(R))$, $\underline{\mathbf{A}}_\alpha(\underline{\mathbf{A}}_\beta(R))$ and $\underline{\mathbf{A}}_\beta(\underline{\mathbf{A}}_\alpha(R))$, etc.? What about the relationship between the approximations $\underline{\mathbf{A}}_{\alpha\wedge\beta}(R)$ and $\underline{\mathbf{A}}_\alpha(\underline{\mathbf{A}}_\beta(R))$, and between $\overline{\mathbf{A}}_{\alpha\wedge\beta}(R)$ and $\overline{\mathbf{A}}_\alpha(\overline{\mathbf{A}}_\beta(R))$?

We will restrict our attention to the conjuction operator '$\wedge$' only. The other two basic operators of propositional logic, conjuction '$\vee$', and negation '$\neg$', will not be discussed. Adding them to this model is an open research problem. However, most of the popular properties in Science and also in Mathematics are defined as conjuctions of two or more basic properties.

Hence we start with the following definition.

Definition 12.10. $\mathcal{P}^\wedge = \{\alpha_1 \wedge \ldots \wedge \alpha_k \mid k \geq 1 \text{ and } \alpha_i \in \mathcal{P} \text{ for } i = 1,\ldots,k\}$.
The elements of $\mathcal{P}^\wedge$ are called **composite properties**. □

Propositions 12.5 and 12.6 guarantee that if either $\underline{\mathbf{A}}_\alpha(R)$ or $\overline{\mathbf{A}}_\alpha(R)$ exists, it satisfies the property α. But if R has a property β different from α, neither $\underline{\mathbf{A}}_\alpha(R)$ nor $\overline{\mathbf{A}}_\alpha(R)$ may satisfy β. For example if R is transitive, its symmetric closure is symmetric, but may not be transitive any longer [12].

Definition 12.11. Let $\alpha, \beta \in \mathcal{P}^\wedge$.

1. We say that α **l-preserves** β iff
   for every $R \in P_\beta$, if R has α-lower bound then $\underline{\mathbf{A}}_\alpha(R) \in P_\beta$,
2. We say that α **u-preserves** β iff
   if R has α-upper bound then $\overline{\mathbf{A}}_\alpha(R) \in P_\beta$. □

The result below validates the above definition.

Proposition 12.9. *Let* $\alpha, \beta \in \mathcal{P}^\wedge$. *Then we have:*

1. *if* α *l-preserves* β, R *has* β*-lower bound and* $\underline{\mathbf{A}}_\beta(R)$ *has* α*-lower bound, then*
   $S = \underline{\mathbf{A}}_\alpha(\underline{\mathbf{A}}_\beta(R)) \in P_\alpha \cap P_\beta$, *that is, S satisfies the property* $\alpha \wedge \beta$.
2. *if* α *l-preserves* β, R *has* β*-upper bound and* $\overline{\mathbf{A}}_\beta(R)$ *has* α*-lower bound, then*
   $S = \underline{\mathbf{A}}_\alpha(\overline{\mathbf{A}}_\beta(R)) \in P_\alpha \cap P_\beta$, *that is, S satisfies the property* $\alpha \wedge \beta$.
3. *if* α *u-preserves* β, R *has* β*-lower bound and* $\underline{\mathbf{A}}_\beta(R)$ *has* α*-upper bound, then*
   $S = \overline{\mathbf{A}}_\alpha(\underline{\mathbf{A}}_\beta(R)) \in P_\alpha \cap P_\beta$, *that is, S satisfies the property* $\alpha \wedge \beta$.
4. *if* α *u-preserves* β, R *has* β*-upper bound and* $\overline{\mathbf{A}}_\beta(R)$ *has* α*-upper bound, then*
   $S = \overline{\mathbf{A}}_\alpha(\overline{\mathbf{A}}_\beta(R)) \in P_\alpha \cap P_\beta$, *that is, S satisfies the property* $\alpha \wedge \beta$.

Proof. (1) Since R has β-lower bound, by Proposition 12.5(1) - if $\beta \in \mathcal{P}^{\cap}$, or Proposition 12.6(1) - if $\beta \in \mathcal{P}^{\cup}$, then $Q = \underline{\mathbf{A}}_{\beta}(R) \in P_{\beta}$. Since α l-preserves β and $Q = \underline{\mathbf{A}}_{\beta}(R)$ has α-lower bound, then by Definition 12.11(1), $S = \underline{\mathbf{A}}_{\alpha}(Q) \in P_{\beta}$. Again, since $Q = \underline{\mathbf{A}}_{\beta}(R)$ has α-lower bound, by Proposition 12.5(1) - if $\alpha \in \mathcal{P}^{\cap}$, or Proposition 12.6(1) - if $\alpha \in \mathcal{P}^{\cup}$, then $S = \underline{\mathbf{A}}_{\alpha}(Q) \in P_{\alpha}$. Hence $S = \underline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(R)) \in P_{\alpha} \cap P_{\beta}$.
(2), (3) and (4) are carried out similarly as (1). □

In general there are no specific relationships between $\underline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(R))$ and $\underline{\mathbf{A}}_{\beta}(\underline{\mathbf{A}}_{\alpha}(R))$, or between $\overline{\mathbf{A}}_{\alpha}(\overline{\mathbf{A}}_{\beta}(R))$ and $\overline{\mathbf{A}}_{\beta}(\overline{\mathbf{A}}_{\alpha}(R))$. The approximation $\underline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(R))$ may exist but $\underline{\mathbf{A}}_{\beta}(\underline{\mathbf{A}}_{\alpha}(R))$ may not, and similarly for upper approximations. Even if they both exist, they may be equal, not equal, one included into another - or not, etc., some examples will be discussed in Sections 12.8 and 12.9. However, there is a very specific relationship between $\underline{\mathbf{A}}_{\alpha}(\overline{\mathbf{A}}_{\beta}(R))$ and $\overline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(R))$.

Proposition 12.10. *Let* $\alpha, \beta \in \mathcal{P}^{\wedge}$, *and*

- α *u-preserves* β *and* β *l-preserves* α,
- R *has* α*-upper bound and* β*-lower bound,*
- $\overline{\mathbf{A}}_{\alpha}(R)$ *has* β*-lower bound, and*
- $\underline{\mathbf{A}}_{\beta}(R)$ *has* α*-lower bound*

then

1. $\overline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(R)) \in P_{\alpha} \cap P_{\beta}$ *and* $\underline{\mathbf{A}}_{\beta}(\overline{\mathbf{A}}_{\alpha}(R)) \in P_{\alpha} \cap P_{\beta}$.
2. $\overline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(R)) \subseteq \underline{\mathbf{A}}_{\beta}(\overline{\mathbf{A}}_{\alpha}(R))$.

Proof. (1) By Proposition 12.9(2) and 12.9(3).
(2) By Proposition 12.8(2), $R \subseteq \overline{\mathbf{A}}_{\alpha}(R)$, so $\underline{\mathbf{A}}_{\beta}(R) \subseteq \underline{\mathbf{A}}_{\beta}(\overline{\mathbf{A}}_{\alpha}(R))$, and $\overline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(R)) \subseteq \overline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(\overline{\mathbf{A}}_{\alpha}(R)))$. By (1) of this proposition, $\underline{\mathbf{A}}_{\beta}(\overline{\mathbf{A}}_{\alpha}(R)) \in P_{\alpha}$, so by Proposition 12.4, $\overline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(\overline{\mathbf{A}}_{\alpha}(R))) = \underline{\mathbf{A}}_{\beta}(\overline{\mathbf{A}}_{\alpha}(R))$. Therefore $\overline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(R)) \subseteq \underline{\mathbf{A}}_{\beta}(\overline{\mathbf{A}}_{\alpha}(R))$. □

Note that, the above result is consistent with Propositions 12.7(6) and 12.8(6). We would like to point out that, in general, there is *no* inclusion-type relationship between R and $\underline{\mathbf{A}}_{\alpha}(\overline{\mathbf{A}}_{\beta}(R))$ and similary for R and $\overline{\mathbf{A}}_{\beta}(\underline{\mathbf{A}}_{\alpha}(R)$.

Suppose that both $\underline{\mathbf{A}}_{(\alpha\wedge\beta)}(R)$ and $\underline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(R))$ (or $\overline{\mathbf{A}}_{(\alpha\wedge\beta)}(R)$ and $\overline{\mathbf{A}}_{\alpha}(\overline{\mathbf{A}}_{\beta}(R))$) exist and $\underline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(R)) \in P_{(\alpha\wedge\beta)}$ (or $\overline{\mathbf{A}}_{\alpha}(\overline{\mathbf{A}}_{\beta}(R)) \in P_{(\alpha\wedge\beta)}$). Which one is a better approximation of R?

Proposition 12.11. *Assume that* α, β *belong to* $\mathcal{P}^{\wedge}$.

1. *If* R *has* β*-lower bound and* $(\alpha\wedge\beta)$*-lower bound, and* $\underline{\mathbf{A}}_{\beta}(R)$ *has* α*-lower bound, then:* $\underline{\mathbf{A}}_{(\alpha\wedge\beta)}(R) \subseteq \underline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(R)) \subseteq R$,
2. *If* R *has* β*-upper bound and* $(\alpha\wedge\beta)$*-upper bound, and* $\overline{\mathbf{A}}_{\beta}(R)$ *has* α*-upper bound, then:* $R \subseteq \overline{\mathbf{A}}_{\alpha}(\overline{\mathbf{A}}_{\beta}(R)) \subseteq \overline{\mathbf{A}}_{(\alpha\wedge\beta)}(R)$.
3. $\underline{\mathbf{A}}_{(\alpha\wedge\beta)}(R)$, $\overline{\mathbf{A}}_{(\alpha\wedge\beta)}(R)$, $\underline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(R))$, $\overline{\mathbf{A}}_{\alpha}(\overline{\mathbf{A}}_{\beta}(R)) \in P_{(\alpha\wedge\beta)} = P_{\alpha} \cap P_{\beta}$.

Proof. (1) Since obviously $lb_{(\alpha\wedge\beta)}(R) \subseteq lb_{\beta}(R)$ then $\underline{\mathbf{A}}_{(\alpha\wedge\beta)}(R) \subseteq \underline{\mathbf{A}}_{\beta}(R)$. Hence $\underline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{(\alpha\wedge\beta)}(R)) \subseteq \underline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(R))$. Since $\underline{\mathbf{A}}_{(\alpha\wedge\beta)}(R) \in P_{\alpha}$, then due to Proposition 12.4, $\underline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{(\alpha\wedge\beta)}(R)) = \underline{\mathbf{A}}_{(\alpha\wedge\beta)}(R)$. From Proposition 12.7(2) we have $\underline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(R)) \subseteq R$, which ends the proof of (1).
(2) Since obviously $ub_{(\alpha\wedge\beta)}(R) \subseteq ub_{\beta}(R)$ then $min(ub_{(\alpha\wedge\beta)}(R)) \subseteq ub_{\beta}(R)$. This means $\overline{\mathbf{A}}_{\beta}(R) \subseteq \overline{\mathbf{A}}_{(\alpha\wedge\beta)}(R)$. Hence $\overline{\mathbf{A}}_{\alpha}(\overline{\mathbf{A}}_{\beta}(R)) \subseteq \overline{\mathbf{A}}_{\alpha}(\overline{\mathbf{A}}_{(\alpha\wedge\beta)}(R))$. Since $\overline{\mathbf{A}}_{(\alpha\wedge\beta)}(R) \in P_{\alpha}$, then due to Proposition 12.4, $\overline{\mathbf{A}}_{\alpha}(\overline{\mathbf{A}}_{(\alpha\wedge\beta)}(R)) = \overline{\mathbf{A}}_{(\alpha\wedge\beta)}(R)$. From Proposition 12.8(2) we have $R \subseteq \overline{\mathbf{A}}_{\alpha}(\overline{\mathbf{A}}_{\beta}(R))$, which ends the proof of (2).
(3) $\underline{\mathbf{A}}_{(\alpha\wedge\beta)}(R)$ and $\overline{\mathbf{A}}_{(\alpha\wedge\beta)}(R)$ belong to $P_{(\alpha\wedge\beta)}$ by either Proposition 12.5 or Proposition 12.6. $\underline{\mathbf{A}}_{\alpha}(\underline{\mathbf{A}}_{\beta}(R)) \in P_{\alpha} \cap P_{\beta}$ by Proposition 12.9(1), and $\overline{\mathbf{A}}_{\alpha}(\overline{\mathbf{A}}_{\beta}(R)) \in P_{\alpha} \cap P_{\beta}$ by Proposition 12.9(4). $\square$

Proposition 12.11 suggests an important technique for the design of approximation schema. It in principle says that using a complex predicate as a property usually results in a *worse* approximation than when the property is decomposed into simpler ones, and then we approximate a given relation over all these simpler properties. This means that before starting an approximation process we should think carefully how the given property could be decomposed into the simpler (and more regular with respect to the theory presented) ones.

12.7 Mixed Approximations

Many properties may have the form like $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_k$, and in general only some sequences of α_i-approximations could lead to the desired result. In this section we will provide some framework to deal with this problem.

We adopt the following convention, we will often write $\mathbf{A}^{(0)}_{\alpha}(R)$ instead of $\underline{\mathbf{A}}_{\alpha}(R)$ and $\mathbf{A}^{(1)}_{\alpha}(R)$ instead of $\overline{\mathbf{A}}_{\alpha}(R)$.

Definition 12.12. A sequence $s = (\alpha_1, i_1)(\alpha_2, i_2)\ldots(\alpha_k, i_k)$, where $k \geq 1$, $\alpha_j \in \mathcal{P}$ and $i_j \in \{0,1\}$ for $j = 1,\ldots,k$, is a **proper approximation schedule** of a given relation $R \subseteq \mathbf{X}$, iff the following conditions are satisfied

1. $\alpha_i \neq \alpha_{i+1}$, for $i = 1,\ldots,k-1$, and
2. the **mixed approximation** $\mathbf{A}^{s}(R)$, defined as

$$\mathbf{A}^{s}(R) = \mathbf{A}^{(i_1)}_{\alpha_1}(\mathbf{A}^{(i_2)}_{\alpha_2}(\ldots(\mathbf{A}^{(i_k)}_{\alpha_k}(R))\ldots))$$

   does exist and $\mathbf{A}^{s}(R) \in \mathcal{P}_{(\alpha_1\wedge\ldots\wedge\alpha_k)}$.

A conjunction $\pi(s) = \alpha_1 \wedge \alpha_2 \wedge \ldots. \wedge \alpha_k$ is the **composite property generated by** the sequence s. $\square$

We will also write $\alpha^{(0)}$ instead of $(\alpha,0)$, $\alpha^{(1)}$ instead of $(\alpha,1)$, 'α-0 bound' instead of 'α-lower bound', 'α-1 bound' instead of 'α-upper bound', '0-preserves' instead of 'l-preserves', and '1-preserves' instead of 'u-preserves'.

Proposition 12.12. *Let $R \subseteq \mathbf{X}$ and $s = \alpha_1^{(i_1)}\alpha_2^{(i_2)}....\alpha_k^{(i_k)}$ be a sequence with $\alpha_i \neq \alpha_{i+1}$, for $i = 1,...,k-1$. Define subsequences of s as follows: $s_k = \alpha_k^{(i_k)}$, $s_{k-1} = \alpha_{k-1}^{(i_{k-1})} s_k$,..., $s_2 = \alpha_2^{(i_2)} s_3$, $s_1 = \alpha_1^{(i_1)} s_2 = s$.*
The sequence $s = \alpha_1^{(i_1)}\alpha_2^{(i_2)}....\alpha_k^{(i_k)}$ is a proper approximation schedule of the relation R iff the following conditions are satisfied:

1. *R has α_k-i_k bound,*
2. *for each $j = k-1,...,1$, α_j i_j-preserves $\pi(s_{j+1})$ and $\mathbf{A}^{s_{j+1}}(R)$ has α_j-i_j bound.*

Proof. By induction on the length of s, using Propositions 12.3, 12.6 and 12.9. □

While Definition 12.12 is not constructive, Proposition 12.12 suggests a recursive algorithm that can be used to compute $\mathbf{A}^s(R)$.

12.8 Approximations by Partial Orders

The theory proposed above can be applied to any composite property and any relation. We will now apply it to the approximation of arbitrary binary relations by partial orders. This section is a refined version of the results initially presented in [6].

We start with defining two operations on binary relations that will later be used to construct partial order approximations.

Let X be a set and $R \subseteq X \times X$ be any relation.

Definition 12.13 ([6]). Let $R \subseteq X \times X$.

1. The relation $R^\bullet$, the **acyclic refinement** of R, is defined as follows:

$$aR^\bullet b \text{ if and only if } aRb \wedge \neg(aR^{cyc}b),$$

   where $aR^{cyc}b$if and only if $aR^+b \wedge bR^+a$, that is, $aR^{cyc}b$ means a and b belong to some cycle.
2. The relation $R^\subset$, the **inclusion property kernel** of R, is defined as:

$$aR^\subset b \text{ if and only if } bR^{ref} \subset aR^{ref} \wedge R^{ref}a \subset R^{ref}b,$$

   where R^{ref} is a reflexive closure of R, and for every relation $S \subseteq X \times X$,
   $Sa = \{x \mid xSa\}$ and $aS = \{x \mid aSx\}$.
3. The relation $R^{\subset\wedge\bullet}$ is defined as follows: $aR^{\subset\wedge\bullet}b$if and only if $aR^\subset b \wedge aR^\bullet b$. □

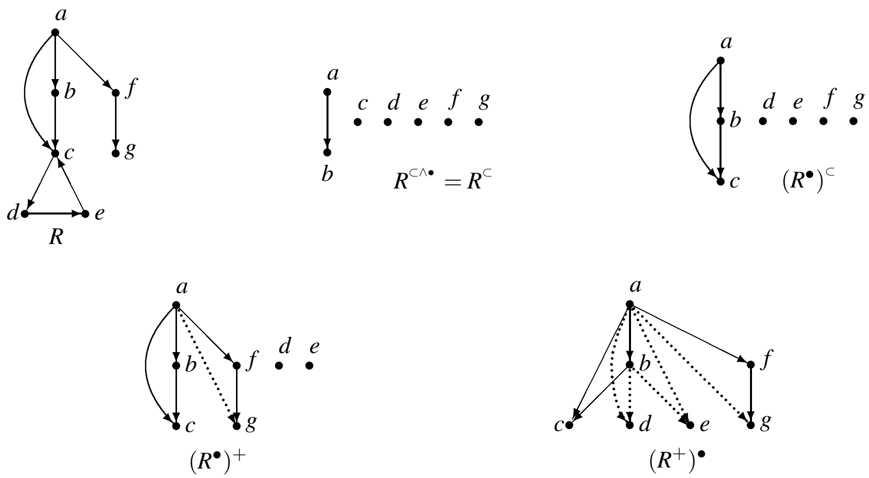

Fig. 12.1. **[6]** An example of a relation R, its partial order approximations $R^{\subset\wedge\bullet}$, $R^{\subset}$, $(R^{\bullet})^{\subset}$, $(R^{\bullet})^{+}$ and $(R^{+})^{\bullet}$. Dotted lines in $(R^{\bullet})^{+}$ and $(R^{+})^{\bullet}$ indicate the relationship that is not in R and was added by the transitivity operation. In general, we only have $R^{\subset\wedge\bullet} \subseteq R^{\subset}$ and it might happen that $R^{\subset\wedge\bullet} \neq R^{\subset}$ (c.f. [6]).

The word 'kernel' is often used as an antonym of 'closure'. While 'closure' is defined as the least superset having a desired property, the word 'kernel' is often used to name the greatest subset having a desired property. The 'inclusion property kernel' is a kernel in this sense. While $R^{\bullet}$ is an acyclic subset of R, it is not a kernel, as the greatest acyclic subrelation usually does not exist. Hence, the name 'refinement' was proposed and used (see [6]).

Theorem 12.1 ([6]). *For every relation $R \subseteq X \times X$, we have.*

1. *The relations $R^{\subset\wedge\bullet}$, $R^{\subset}$, $(R^{\bullet})^{\subset}$, $(R^{\bullet})^{+}$, and $(R^{+})^{\bullet}$ are partial orders, and can be considered as partial order approximations of R.*
2. $R^{\subset\wedge\bullet} \subseteq (R^{\bullet})^{\subset} \subseteq (R^{\bullet})^{+} \subseteq (R^{+})^{\bullet}$.
3. $R^{\subset\wedge\bullet} \subseteq (R^{\subset})^{\bullet} = R^{\subset} \subseteq R$. □

The statements like $(R^{\bullet})^{\subset}$ should be read as follows, find the acyclic refinement of R first and then find the inclusion property kernel of $R^{\bullet}$.

The relations $R^{\subset\wedge\bullet}$, $R^{\subset}$, $(R^{\bullet})^{\subset}$, $(R^{\bullet})^{+}$, $(R^{+})^{\bullet}$ and Theorem 12.1 are illustrated by an example in Figure 12.1 (from [6]). A formal definition of a partial order approximation of arbitrary binary relations has been given and justified in [6] and all the relations $R^{\subset\wedge\bullet}$, $R^{\subset}$, $(R^{\bullet})^{\subset}$, $(R^{\bullet})^{+}$, $(R^{+})^{\bullet}$ satisfy that definition. The idea of using the relation $(R^{+})^{\bullet}$ as a partial order approximation came from Schröder (1985) and was initially formulated in terms of quasi-orders (see [6]). The other approximations, to our knowledge, originated from [6] and its conference predecessor. There is no universal inclusion-type relationship between $R^{\subset}$ and $(R^{\bullet})^{\subset}, (R^{\bullet})^{+}, (R^{+})^{\bullet}$, and considering $R^{\subset}$ alone as a partial order approximation of R is a little bit controversial (see an example from Figure 1 of [6]), but justifiable in some cases (see [6]).

Partial order approximations of arbitrary binary relations play a crucial role in the theory of non-numerical rankings based on the pairwise comparisons paradigm [4, 7].

Let $\mathcal{P}$ be the following set of properties *over the relation R*, $\mathcal{P} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, where:

- $\alpha_1 \stackrel{df}{=} \forall a,b \in X.\ bR^{ref} \subset aR^{ref} \wedge R^{ref}a \subset R^{ref}b$, that is, $\alpha_1 =$ *inclusion property*,
- $\alpha_2 \stackrel{df}{=} \forall a,b,c \in X.\ aRb \wedge bRa \Longrightarrow aRc$, that is, $\alpha_2 =$ *transitivity*,
- $\alpha_3 \stackrel{df}{=} \forall a,b \in X.\ \neg(aR^{cyc}b)$, that is $\alpha_3 =$ *acyclicity*,
- $\alpha_4 \stackrel{df}{=} \alpha_1 \wedge \alpha_3$,
- $\alpha_5 \stackrel{df}{=} (\forall a \in X.\ \neg(aRa)) \wedge \alpha_2$, that is, $\alpha_5 =$ *partial ordering*.

Consider the property-driven rough set approximation space

$$(X \times X, \{P_\alpha \mid \alpha \in \mathcal{P}\}).$$

Directly from the definitions we may conclude that an arbitrary relation R has (see [6], Sections 3 and 4, for details):

- α_1-lower bound, but may not have α_1-upper bound,
- α_2-lower bound and α_2-upper bound,
- α_3-lower bound, but may not have α_3-upper bound,
- α_4-lower bound, but may not have α_4-upper bound,
- α_5-lower bound, but may not have α_5-upper bound,

We have here $\mathcal{P}^\cap = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, $\mathcal{P}^\cup = \emptyset$.

It turns out that all the partially ordered approximations $R^\subset$, $R^{\subset\wedge\bullet}$, $(R^\bullet)^\subset$, $(R^\bullet)^+$ and $(R^+)^\bullet$ from Theorem 12.1 can be obtained naturally using the Rough Sets approach proposed in the previous sections. First, we will show that all operations involved are either lower or upper α-approximations.

Proposition 12.13 ([6])

1. $R^\subset = \underline{\mathbf{A}}_{\alpha_1}(R)$,
2. $R^+ = \overline{\mathbf{A}}_{\alpha_2}(R)$,
3. $R^\bullet = \underline{\mathbf{A}}_{\alpha_3}(R)$,
4. $R^{\subset\wedge\bullet} = \underline{\mathbf{A}}_{\alpha_4}(R) = \underline{\mathbf{A}}_{\alpha_1\wedge\alpha_3}(R)$,

Proof (1) Since R has α_1-lower bound and $lb_{\alpha_1}(R) = \{R^\subset\}$, for the details of the latter see [6], Sections 3 and 4.
(2) Since $\alpha_2 \in \mathcal{P}^\cap$ and R has α_2-upper bound we can use Proposition 12.6(2), which says that $\overline{\mathbf{A}}_{\alpha_2}(R)$ is the smallest transitive relation containing R, that is, R^+ (c.f. [12]).
(3) First note that R has α_3-lower bound. By the definition, we have

$\underline{\mathbf{A}}_{\alpha_3}(R) = \bigcap\{Q \mid Q \in max(lb_{\alpha_3}(R))\}$. Let $aR^{cyc}b$. This means $a = a_1Ra_2R\ldots Ra_{k-1}Ra_k = b$, where $i \neq j \Rightarrow a_i \neq a_j$. Let $Q \in lb_{\alpha_1}(R)$. Note that, $Q \in max(lb_{\alpha_1}(R))$ if and only if there is a_r such that $(a_{r-1}, a_r) \notin Q$ but $a = a_0Qa_1 \ldots Qa_{r-1}$ and $a_rQ\ldots Qa_k$. Hence $R^{\bullet} = \bigcap\{Q \mid Q \in max(lb_{\alpha_3}(R))\} = \underline{\mathbf{A}}_{\alpha_3}(R)$.
(4) Since R has α_4-lower bound, $\underline{\mathbf{A}}_{\alpha_4}(R)$ exists and can be constructed. We have $\underline{\mathbf{A}}_{\alpha_4}(R) = \bigcap\{Q \mid Q \in max(lb_{\alpha_1\wedge\alpha_3}(R))\} = \bigcap\{Q \mid Q \in max(\{R^{\subset} \cap S \mid S \in lb_{\alpha_3}(R)\})\} = R^{\subset} \cap \bigcap\{Q \mid Q \in max(lb_{\alpha_3}(R))\}$. From (3) it follows $\bigcap\{Q \mid Q \in max(lb_{\alpha_3}(R))\} = R^{\bullet}$, so $R^{\subset\wedge\bullet} = R^{\subset} \cap R^{\bullet} = \underline{\mathbf{A}}_{\alpha_1\wedge\alpha_3}(R) = \underline{\mathbf{A}}_{\alpha_4}(R)$. □

Now we will show that compositions of appropriate approximations are allowed.

Proposition 12.14

1. α_1 *l-preserves* α_3.
2. α_2 *u-preserves* α_3.
3. α_3 *l-preserves* α_1 *and* α_3 *l-preserves* α_2.

Proof (1) By Theorem 12.1(1), if R is acyclic, $R^{\subset}$ is acyclic as well, and clearly R has α_1-lower bound ($\emptyset$ for example).
(2) By Theorem 12.1(1), transitivity preserves acyclity and of course R has α_2-lower bound ($\emptyset$ for example).
(3) By Theorem 12.1(1), acyclic refinement preserves both transitivity and inclusion property kernel, and of course R has α_2-lower bound (for example $\emptyset$ and $\mathcal{X}$).
□

The main result of this section can now be formulated as follows.

Proposition 12.15 ([6])

1. $(R^{\bullet})^{\subset} = \underline{\mathbf{A}}_{\alpha_1}(\underline{\mathbf{A}}_{\alpha_3}(R))$,
2. $(R^{\subset})^{\bullet} = \underline{\mathbf{A}}_{\alpha_3}(\underline{\mathbf{A}}_{\alpha_1}(R)) = R^{\subset}$, *so* $\underline{\mathbf{A}}_{\alpha_3}(\underline{\mathbf{A}}_{\alpha_1}(R)) = \underline{\mathbf{A}}_{\alpha_1}(R)$,
3. $(R^{\bullet})^{+} = \overline{\mathbf{A}}_{\alpha_2}(\underline{\mathbf{A}}_{\alpha_3}(R))$,
4. $(R^{+})^{\bullet} = \underline{\mathbf{A}}_{\alpha_3}(\overline{\mathbf{A}}_{\alpha_2}(R))$.

Proof From Propositions 12.13 and 12.14, and, for (2), from Theorem 12.1(3). □

Proposition 12.15 illustrates well the basic properties of property-driven rough set approximations of binary relations by partial orders. Below we provide some observations.

- Property α_5 does not appear in Proposition 12.15 at all. It is actually a rather useless property. No upper bound exists in a general case, and the relation $\underline{\mathbf{A}}_{\alpha_5}(R)$ is usually not very interesting (c.f. [6]). Property α_5 (being a partial order) is just too strong to be efficiently handled as a whole. We can get much better results when we treat the components of α_5, for instance acyclicity and transitivity, separately and then compose the results obtained.

- We have $R^{\subset\wedge\bullet} = \underline{\mathbf{A}}_{\alpha_4}(R) = \underline{\mathbf{A}}_{\alpha_1\wedge\alpha_3}(R) \subseteq (R^{\bullet})^{\subset} = \underline{\mathbf{A}}_{\alpha_1}(\underline{\mathbf{A}}_{\alpha_3}(R))$, and $R^{\subset\wedge\bullet} = \underline{\mathbf{A}}_{\alpha_1\wedge\alpha_3}(R) \subseteq \underline{\mathbf{A}}_{\alpha_3}(\underline{\mathbf{A}}_{\alpha_1}(R)) = (R^{\subset})^{\bullet} = R^{\subset}$ which illustrates Proposition 12.11(1).
- In general, $\underline{\mathbf{A}}_{\alpha_1}(\underline{\mathbf{A}}_{\alpha_3}(R)) = (R^{\bullet})^{\subset}$ and $\underline{\mathbf{A}}_{\alpha_3}(\underline{\mathbf{A}}_{\alpha_1}(R)) = (R^{\subset})^{\bullet} = R^{\subset}$ are *not equal*.
- We also have $(R^{\bullet})^{+} = \overline{\mathbf{A}}_{\alpha_2}(\underline{\mathbf{A}}_{\alpha_3}(R)) \subseteq \underline{\mathbf{A}}_{\alpha_3}(\overline{\mathbf{A}}_{\alpha_2}(R)) = (R^{+})^{\bullet}$. which illustrates Proposition 12.10(2).

While the approximations of arbitrary relations by partial orders, motivated by pairwise comparisons non-numerical ranking [4, 7], were initially defined in terms of the standard theory of relations, their Rough Sets versions better explain those definitions. The Rough Sets versions provide formal motivation and explanation in places where in the classical versions were just 'gut feelings'.

12.9 Approximations by Equivalence Relations

In this section we will apply the theory of property-driven rough approximations to the approximation of arbitrary binary relations by equivalence relations.

We will start with the classical well-known result.

Proposition 12.16 (Folklore, c.f. [2, 8, 12])
For every relation $R \subseteq X \times X$, the relations $((R^{ref})^{\overline{sym}})^{+} = ((R^{\overline{sym}})^{ref})^{+} = ((R^{\overline{sym}})^{+})^{ref}$ are equivalence relations, and $R \subseteq ((R^{ref})^{\overline{sym}})^{+}$. □

However, in general, the relation $((R^{ref})^{+})^{\overline{sym}} = ((R^{+})^{ref})^{\overline{sym}} = ((R^{+})^{\overline{sym}})^{ref}$ may not be an equivalence relation. The simplest example is probably $X = \{a,b,c\}$ and $R = \{(a,c),(b,c)\}$. Then, $((R^{ref})^{+})^{\overline{sym}} = \{(a,c),(b,c),(c,a),(c,b),(a,a),(b,b),(c,c)\}$, and this relation is not transitive as we have bRc and cRa, but not bRa.

Symmetric closure is not the only method for enforcing symmetry. We can also used the idea of 'kernel' for this task.

Definition 12.14. Let $R \subseteq X \times X$, we define

$$R^{\underline{sym}} = \{(a,b) \mid (a,b) \in R \wedge (b,a) \in R\}.$$

The relation $R^{\underline{sym}}$ will be called a **symmetric kernel** of R. □

It turns out that the symmetric kernel preserves transitivity and can also be used as a tool to construct an approximation by an equivalence relation.

Proposition 12.17. *For every $R \subseteq X \times X$, we have:*

1. *$R^{\underline{sym}}$ is symmetric and $R^{\underline{sym}} \subseteq R$.*
2. *$R^{\underline{sym}} = \bigcup\{Q \mid Q \subseteq R$ and Q is symmetric $\}$.*

3. $((R^{ref})^{\underline{sym}})^{+} = ((R^{\underline{sym}})^{ref})^{+} = ((R^{\underline{sym}})^{+})^{ref}$.
4. $((R^{ref})^{+})^{\underline{sym}} = ((R^{+})^{ref})^{\underline{sym}} = ((R^{+})^{\underline{sym}})^{ref}$.
5. $((R^{ref})^{\underline{sym}})^{+}$ *is an equivalence relation.*
6. $((R^{ref})^{+})^{\underline{sym}}$ *is an equivalence relation.*

Proof. (1) From the definition.
(2) It suffices to show that if $Q \subseteq R$ and Q is symmetric, then $Q \subseteq R^{\underline{sym}}$. Let $(a,b) \in Q$, so $(a,b) \in R$. If $(b,a) \notin R$ then $(b,a) \notin Q$, and, since Q is symmetric, $(a,b) \notin Q$. Hence $(b,a) \in Q$. Since $Q \subseteq R$, $(b,a) \in R$. But if $(a,b) \in R$ and $(b,a) \in R$, then $(a,b) \in R^{\underline{sym}}$. Hence $Q \subseteq R^{\underline{sym}}$.
(3) and (4). Reflexive closure is just adding the identity relation. It does not interfer with either transitive closure or symmetric kernel operation.
(5) Since transitive closure preserves symmetry (c.f. [12]).
(6) It suffices to show that symmetric kernel preserves transitivity. Define $Q = (R^{ref})^{+}$. Clearly Q is transitive. Suppose that $Q^{\underline{sym}}$ is not. This means thare are $a,b,c \in X$ such that $aQ^{\underline{sym}}b$ and $bQ^{\underline{sym}}c$ but $\neg(aQ^{\underline{sym}}c)$. By Definition 12.14, $Q^{\underline{sym}} \subseteq Q$, so aQb and bQc. The relation Q is transitive, so we also have aQc. Since $\neg(aQ^{\underline{sym}}c)$, then, from Definition 12.14, $\neg(cQ^{\underline{sym}}a)$. However, as $Q^{\underline{sym}}$ is symmetric, $aQ^{\underline{sym}}b$ and $bQ^{\underline{sym}}c$ means that we also have $bQ^{\underline{sym}}a$ and $cQ^{\underline{sym}}b$, and consequently bQa and cQb. Since Q is transitive, then have cQa. But aQc and cQa implies that $aQ^{\underline{sym}}c$ and $cQ^{\underline{sym}}a$, contradicting $\neg(aQ^{\underline{sym}}c)$. Hence $Q^{\underline{sym}}$ is transitive. $\square$

The relations $(R^{\underline{sym}})^{+}$, $(R^{+})^{\underline{sym}}$, $(R^{\overline{sym}})^{+}$ and Proposition 12.17 are illustrated by an example in Figure 12.2. While $R \subseteq (R^{\overline{sym}})^{+}$, there is no universal inclusion relationship between R and neither $(R^{\underline{sym}})^{+}$ nor $(R^{+})^{\underline{sym}}$.
Let $\mathcal{P}$ be the following set of properties *over the relation R*, $\mathcal{P} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$, where:

- $\beta_1 \stackrel{df}{=} \forall a \in X.\ aRa$, that is, $\beta_1 =$ *reflexivity*,
- $\beta_2 \stackrel{df}{=} \forall a,b \in X.\ aRb \Rightarrow bRa$, that is, $\beta_2 =$ *symmetry*,
- $\beta_3 \stackrel{df}{=} \forall a,b,c \in X.\ aRb \wedge bRa \Longrightarrow aRc$, that is, $\beta_3 =$ *transitivity*,
- $\beta_4 \stackrel{df}{=} \beta_1 \wedge \beta_2 \wedge \beta_3$, that is, $\beta_3 =$ *equivalence relation.*

Consider the property-driven rough set approximation space

$$(X \times X, \{P_\alpha \mid \alpha \in \mathcal{P}\}).$$

Directly from the definitions, we may conclude that an arbitrary relation
$R \subseteq X \times X$ has:

- β_1-upper bound, but may not have β_1-lower bound, β_1-lower bound exists only when R is already reflexive,
- β_2-upper bound, but may not have β_2-lower bound, β_2-lower bound exists only when $R \cap Id \neq \emptyset$,
- β_3-lower bound and β_3-upper bound,

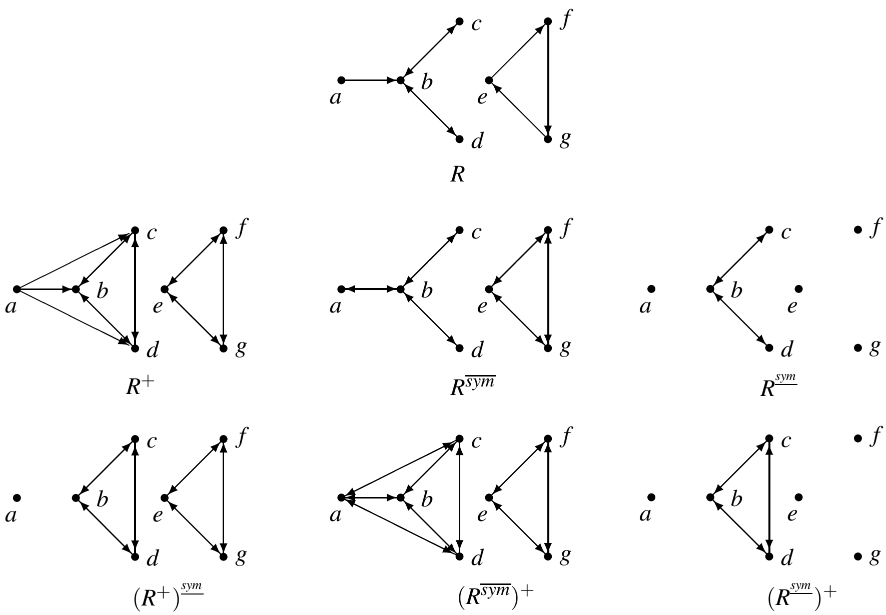

Fig. 12.2. An example of a relation R and the results of applications of transitive closure, symmetric closure and symmetric kernel, in various orders. If R is reflexive then, the relations $(R^+)^{\underline{sym}}$, $(R^{\overline{sym}})^+$ and $(R^{\underline{sym}})^+$ are equivalence relations. In this figure $a \bullet\!\!\longleftrightarrow\!\!\bullet\, b$ means $a \bullet\!\!\longrightarrow\!\!\bullet\, b$ and $a \bullet\!\!\longleftarrow\!\!\bullet\, b$.

- β_4-upper bound, but may not have β_4-lower bound, β_4-lower bound exists only when R is reflexive,

We have here $\mathcal{P}^{\cap} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ and $\mathcal{P}^{\cup} = \{\beta_1, \beta_2\}$.

It turns out that all the approximations by equivalence relations from Propositions 12.16 and 12.17 can naturally be obtained using the Rough Sets approach proposed in the previous sections. First, we will show that all operations involved are either lower or upper α-approximations.

Proposition 12.18. *Let R be a relation on X*

1. $R^{ref} = \overline{\mathbf{A}}_{\beta_1}(R)$.
2. $R^{\overline{sym}} = \overline{\mathbf{A}}_{\beta_2}(R)$.
3. $R^{\underline{sym}} = \underline{\mathbf{A}}_{\beta_2}(R)$.
4. $R^+ = \overline{\mathbf{A}}_{\beta_3}(R)$.

Proof. (1) Clearly R has β_1-upper bound. We have $ub_{\beta_1}(R) = \{Q \mid R^{ref} \subseteq Q\}$, Hence, $min(ub_{\beta_1}(R)) = \{R^{ref}\}$.

(2) Since $\beta_2 \in \mathcal{P}^{\cap}$ and R has β_2-upper bound we can use Proposition 12.6(2), which says that $\overline{\mathbf{A}}_{\beta_2}(R)$ is the smallest symmetric relation containing R, that is, $R^{\overline{sym}}$ (c.f. [12]).
(3) Since $\beta_2 \in \mathcal{P}^{\cup}$ and R, has β_2-lower bound we can use Proposition 12.6(1), which says that $\overline{\mathbf{A}}_{\beta_2}(R)$ is the greatest symmetric relation included R. From Proposition 12.17(2), it is $R^{\underline{sym}}$.
(4) Since $\beta_3 \in \mathcal{P}^{\cap}$ and R has β_3-upper bound we can use Proposition 12.6(2), which says that $\overline{\mathbf{A}}_{\beta_3}(R)$ is the smallest transitive relation containing R, that is, R^+ (c.f. [12]). □

We will now show that the compositions of appropriate approximations are allowed.

Proposition 12.19

1. *β_2 l-preserves and u-preserves β_1.*
2. *β_3 u-preserves β_1.*
3. *β_2 l-preserves β_3 but β_2 does* not *u-preserve β_3.*
4. *β_3 u-preserves β_2.*

Proof (1) Clearly if R is reflexive, then $R^{\underline{sym}}$ and $R^{\overline{sym}}$ are also reflexive. This and Proposition 12.18(2) and (2) prove this assertion.
(2) If R is reflexive, then R^+ is also reflexive. Hence, by Proposition 12.18(4) β_3 u-preserves β_1.
(3) If R is transitive, then by Proposition 12.17(6), $R^{\underline{sym}}$ is also transitive. Hence, by Proposition 12.18(3), β_2 l-preserves β_3. However, $R^{\overline{sym}}$ may not be transitive, as the example after Proposition 12.16 shows. Hence by Proposition 12.18(2), β_2 does not u-preserve β_3.
(4) If R is symmetric, then R^+ is symmetric too. Hence, by Proposition 12.18(4), β_3 u-preserves β_2. □

We can now present the main result of this section.

Proposition 12.20. *Let R be a relation on X.*

1. $((R^{ref})^{\overline{sym}})^+ = \overline{\mathbf{A}}_{\beta_3}(\overline{\mathbf{A}}_{\beta_2}(\overline{\mathbf{A}}_{\beta_1}(R)))$.
2. $((R^{ref})^{\underline{sym}})^+ = \overline{\mathbf{A}}_{\beta_3}(\underline{\mathbf{A}}_{\beta_2}(\overline{\mathbf{A}}_{\beta_1}(R)))$.
3. $((R^{ref})^+)^{\underline{sym}} = \underline{\mathbf{A}}_{\beta_2}(\overline{\mathbf{A}}_{\beta_3}(\overline{\mathbf{A}}_{\beta_1}(R)))$.
4. $((R^{ref})^{\underline{sym}})^+ \subseteq ((R^{ref})^+)^{\underline{sym}}$.

Proof. (1), (2) and (3) A simple consequence of Propositions 12.18 and 12.19.
(4) From Proposition 12.10(2) we have
$\overline{\mathbf{A}}_{\beta_3}(\underline{\mathbf{A}}_{\beta_2}(\overline{\mathbf{A}}_{\beta_1}(R))) \subseteq \underline{\mathbf{A}}_{\beta_2}(\overline{\mathbf{A}}_{\beta_3}(\overline{\mathbf{A}}_{\beta_1}(R)))$. □

Proposition 12.20 illustrates well the basic properties of property-driven rough set approximations of binary relations by equivalence relations. Below, we provide some observations.

janicki@mcmaster.ca

- Property β_4 does not appear in Proposition 12.20 at all. It is actually a rather useless property. Quite often $\underline{\mathbf{A}}_{\beta_4}(R) = \emptyset$ and $\overline{\mathbf{A}}_{\beta_4}(R) = \mathbf{X}$, which is not very helpful. The property β_4 (being an equivalence relation) is just too strong to be efficiently handled as a whole. We can get much better results when we treat the components of β_4, reflexivity, symmetry and transitivity, separately and then compose them together in appropriate manner.
- The assertion $((R^{ref})^{\underline{sym}})^+ \subseteq ((R^{ref})^+)^{\underline{sym}}$ can of course be proved independently, without using Rough sets, but Proposition 12.10(2) makes this (otherwise not obvious) proof trivial.
- We have applied the reflexive closure first, but in fact, it can be applied as the second or third as well (see Propositions 12.16, 12.17(3) and 12.17(4)).

Standardly, $((R^{ref})^{\overline{sym}})^+$ is considered as the only approximation of R by an equivalence relation [12]. In the Rough sets approach, it is natural to think of both upper and lower approximations, which in this case leads to the design of $((R^{ref})^{\underline{sym}})^+$ and $((R^{ref})^+)^{\underline{sym}}$ approximations.

12.10 Final Comment

The approach presented in this chapter is called *property-driven* as its main purpose is to find an approximation, either lower or upper, that satisfies a given predicate, called a property. It could be seen as a substantial extension of the ideas presented for relations in [16, 17] and specially recently in [6]. Both this chapter and [6] were motivated by problems occurring when non-numerical ranking is constructed from empirical data [4, 7]. When thinking in terms of *properties*, very often either only lower or only upper approximation does make sense, and quite often *neither of them if the property is too sophisticated*. This lead us to the idea of composite and mixed approximations.

Proposition 12.11 might be the most useful result of this chapter as it indicates how properties should be dealt with to get the best approximations.

We would like to point out that all the assumptions from Section 12.4, especially Assumption 1, relate only to elementary properties. The requirements for composite properties are indirect and so much weaker.

We believe the schedules can often be interpreted as a simulation of 'property-driven non-numerical metrics', and that finding a good schedule means finding a good approximation. But finding a good schedule appears to be more art than science, as our experience with partial orders and equivalence relations indicates.

In general, for a proper schedule s, we usually have $R \setminus \mathbf{A}^s(R) \neq \emptyset$, $\mathbf{A}^s(R) \setminus R \neq \emptyset$ and $R \cap \mathbf{A}^s(R) \neq \emptyset$. The formal definition of the 'best' proper schedule is an open problem. However, we believe that any rule, if proposed, could only be treated as a guide, as the problem seems to be very domain related.

In this chapter, we deal only with single n-ary relations and the composite properties are of the form $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_k$. A natural extension of the presented theory would be allowing composite properties with the operators of conjunction and negation as well. Another natural extension would be to allow properties with more than one relational symbol, that is, an extension to the relational systems (a tuple $(X, R_1, ..., R_k)$, where X is a set and $R_1, ..., R_k$ are relations on X, c.f. [2]), as suggested in [5]. For the former extension, we see some technical problems. For the latter extension, while the extension of general theory is not problematic, one just needs to follow [5], we expect plenty of technical problems for particular applications. While in theory any abstract data type (as defined for example in [1]), can be represented by a relational structure, it is seldom done in practice, as much of the intution is then lost. From the applications point of view, an extension of the ideas presented here to (at least some of) abstract algebras [2], would be very helpful.

Acknowledgements. The author is grateful to the anonymous referees, who pointed out many typos and suggested a better way to organise the materials of this chapter, and would like to thank Andrzej Skowron and Zbigniew Suraj for their support and encouragement during the work on this chapter.

References

1. Aho, A.V., Hopcroft, J.E., Ullman, J.D.: Data Structures and Algorithms. Addison-Wesley (1983)
2. Burris, S., Sankappanavar, H.P.: A Course in Universal Algebra. Springer (1981)
3. Huth, M.R.A., Ryan, M.D.: Logic in Computer Science. Cambridge University Press (2000)
4. Janicki, R.: Pairwise comparisons based non-numerical ranking. Fundamenta Informaticae 94(2), 197–217 (2009)
5. Janicki, R.: On Rough Sets with Structures and Properties. In: Sakai, H., Chakraborty, M.K., Hassanien, A.E., Ślęzak, D., Zhu, W. (eds.) RSFDGrC 2009. LNCS(LNAI), vol. 5908, pp. 109–116. Springer, Heidelberg (2009)
6. Janicki, R.: Approximations of arbitrary binary relations by partial orders. Classical and rough set models. In: Peters et al. [11], pp. 17–38
7. Janicki, R., Zhai, Y.: On a pairwise comparison based consistent non-numerical ranking. Logic Journal of IGPL (2011), doi:10.1093/jigpal/jzr018
8. Kuratowski, K., Mostowski, A.: Set Theory, 2nd edn. North-Holland (1976)
9. Pawlak, Z.: Rought Sets. International Journal of Computer and Information Sciences 34, 557–590 (1982)
10. Pawlak, Z.: Rough Sets: Theoretical Aspects of Reasoning about Data. System Theory, Knowledge Engineering and Problem Solving, vol. 9, Kluwer Academic Publishers, Dordrecht (1991)
11. Peters, J.F., Skowron, A., Chan, C.-C., Grzymala-Busse, J.W., Ziarko, W.P. (eds.): Transactions on Rough Sets XIII. LNCS, vol. 6499. Springer, Heidelberg (2011)
12. Rosen, K.H.: Discrete Mathematics and Its Applications. McGraw-Hill, New York (1999)
13. Skowron, A., Stepaniuk, J.: Approximation of relations. In: Ziarko, W. (ed.) Rough Sets, Fuzzy Sets and Knowledge Discovery, Workshops in Computing, pp. 161–166. Springer & British Computer Society, London, Berlin (1994)

14. Skowron, A., Stepaniuk, J.: Tolarence approximation spaces. Fundamenta Informaticae 27, 245–253 (1996)
15. Stewart, J.: Calculus. Concepts and Contexts. Brooks/Cole (1997)
16. Yao, Y.Y.: Two views of the theory of rough sets in finite universes. International Journal of Approximate Reasoning 15, 291–317 (1996)
17. Yao, Y.Y., Wang, T.: On Rough Relations: An Alternative Formulation. In: Zhong, N., Skowron, A., Ohsuga, S. (eds.) RSFDGrC 1999. LNCS (LNAI), vol. 1711, pp. 82–91. Springer, Heidelberg (1999)