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Pairwise Comparisons Based Non-Numerical Ranking

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Abstract. A systematic procedure for deriving *weakly ordered non-numerical rankings* from given sets of data is proposed and analysed. The data are assumed to be collected using the *Pairwise Comparisons* paradigm. The concept of a partially ordered approximation of an arbitrary binary relation is formally defined and some solutions are proposed. The problem of testing and the importance of *indifference* and the power of *weak order extensions* are also discussed.

1. Introduction

A *ranking* or *preference* is usually defined as a weakly ordered relationship between a set of items such that, for any two items, the first is either "less preferred", "more preferred", or "indifferent" to the second one [8]. The ranking is numerical if numbers are used to measure importance and to create the ranking relation. Numerical rankings are usually totally ordered. Various kinds of *global indexes* are popular examples of numerical rankings.

The *Pairwise Comparisons* method is based on the observation that while ranking the importance of *several* objects is often problematic, it is much easier to do when restricted to *two* objects [3]. The problem is then reduced to constructing a global ranking from the set of partially ordered pairs. The method could be traced to the 1785 Marquis de Condorcet paper [17]), it was explicitly mentioned and analysed by Fechner in 1860 [5], made popular by Thurstone in 1927 [26], and was transformed into a kind of semi-formal methodology by Saaty in 1977 (called *AHP*, Analytic Hierarchy Process, see [4, 8, 21]).

At present Pairwise Comparisons are practically identified with the controversial Saaty's AHP. On one hand AHP has respected practical applications, on the other hand it is still considered by many (see

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[4, 9, 15]) as a flawed procedure that produces arbitrary rankings. For more details the reader is referred to [4, 9, 14, 15], we believe that most of the problems mentioned in [4, 9, 14, 15] and others, stem mainly from the following two sources:

1. The final outcome is always expected to be totally ordered (i.e. for all a,b, either $a<b$ or $b>a$),

2. Numbers are used to calculate the final outcome.

Pairwise Comparisons based non-numerical solutions were proposed and discussed in [9, 13, 14]. The model presented in this paper stems from [13], was highly influenced by [6], and is orthogonal to that of [9]. The concept of "consistency", crucial in [9] is not discussed in this paper at all. Algorithms for the automatic construction of a final ranking are the essence of this paper, but they are not discussed in [9]. The model presented below uses no numbers and is entirely based on the concept of partial orders.

This paper is a revised and extended version of [10]. It uses some concepts and ideas presented in [11] that were unknown when [10] was written.

The paper is structured into eight parts, from Sections 2 to 8 and Apendix A. In Section 2 the basic notions of partially ordered relations are recalled. The formal definitions of *ranking*, *ranking problem* and *pairwise comparisons ranking data* are given in Section 3. In Section 4 some solutions to the problem of *partial order approximation of arbitrary relations* are presented, while Section 5 is devoted to *weak order approximations of arbitrary partial orders*. Section 6 discusses the problem of *testing* models like the one presented in this paper. Some *solutions to the ranking problem* (as defined in Section 3) are discussed in Section 7. Section 8 contains some final comments and Appendix A contains the proof of an important theorem from section 4.

2. Total, Weak and Partial Orders

Let X be a *finite* set. A relation $\lhd \subseteq X \times X$ is a (*sharp*) *partial order* if it is irreflexive and transitive, i.e. if $a \lhd b \;\Rightarrow\; \neg(b \lhd a)$ and $a \lhd b \lhd c \;\Rightarrow\; a \lhd c$, for all $a,b,c \in X$. A pair $(X,\lhd)$ is called a *partially ordered set*. We will often identify $(X,\lhd)$ with $\lhd$, when X is known.

We write $a \sim_{\lhd} b$ if $\neg(a \lhd b) \wedge \neg(b \lhd a)$, that is if a and b are either *distinct incompatible* (w.r.t. $\lhd$) or *identical* elements of X. We also write

$$a \approx_{\lhd} b \iff \{x \mid x \sim_{\lhd} a\} = \{x \mid x \sim_{\lhd} b\}.$$

The relation $\approx_{\lhd}$ is an equivalence relation (i.e. it is reflexive, symmetric and transitive) and it is called *the equivalence with respect to* $\lhd$, since if $a \approx_{\lhd} b$, there is nothing in $\lhd$ that can distinquish between a and b (see [6] for details). We always have $a \approx_{\lhd} b \Rightarrow a \sim_{\lhd} b$, and one can show that [6]:

$$a \approx_{\lhd} b \iff \{x \mid a \lhd x\} = \{x \mid b \lhd x\} \;\wedge\; \{x \mid x \lhd a\} = \{x \mid x \lhd b\}$$

A partial order is [6]

- *total* or *linear*, if $\sim_{\lhd}$ is the identity relation, i.e., for all $a,b \in X.\;\; a \lhd b \vee b \lhd a \vee a = b$,

- *weak* or *stratified*, if $a \sim_{\lhd} b \sim_{\lhd} c \Rightarrow a \sim_{\lhd} c$, i.e. if $\sim_{\lhd}$ is an equivalence relation,

Evidently, every total order is weak. Weak orders are often defined in an alternative way, namely [6],

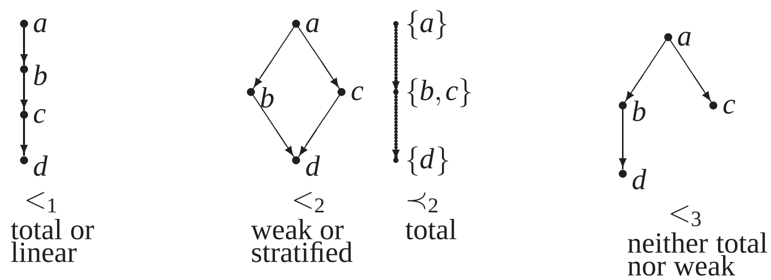


Figure 1: Various types of partial orders (represented as Hasse diagrams). The total order $\prec_2$ represents the weak order $<_2$., The partial order $<_3$ is nether total nor weak.

- a partial order $(X, \lhd)$ is a weak order iff there exists a total order $(Y, \prec)$ and a mapping $\phi : X \to Y$ such that $\forall x, y \in X.\ x \lhd y \iff \phi(x) \prec \phi(y)$.

This definition is illustrated in Figure 1, let $\phi : \{a,b,c,d\} \to \{\{a\},\{b,c\},\{d\}\}$ and $\phi(a) = \{a\}$, $\phi(b) = \phi(c) = \{b,c\}$, $\phi(d) = \{d\}$. Note that for all $x, y \in \{a,b,c,d\}$ we have $x <_2 y \iff \phi(x) \prec_2 \phi(y)$.

Following [6], in this paper $a \lhd b$ is interpreted as "*a is less preferred than b*", and $a \approx_\lhd b$ is interpreted as "*a* and *b* are *indifferent*".

The preferred outcome of any ranking is a total order. For any total order $\lhd$, both $\sim_\lhd$ and $\approx_\lhd$ are just the equality relations. A total order has two natural models, both deeply embedded in the human perception of reality, namely: *time* and *numbers*.

Unfortunately in many cases it is not reasonable to insist that everything can or should be totally ordered. We may not have sufficient knowledge or such a perfect ranking may not even exist [1]. Quite often insisting on a totally ordered ranking results in an artificial and misleading "global index".

Weak (stratified) orders are a very natural generalization of total orders. They allow the modelling of some regular indifference, their interpretation is very simple and intuitive, and they are reluctantly accepted by decision makers. Although not as much as one might expect given the huge theory of such orders (see [6, 8]).

If $\lhd$ is a weak order then $a \approx_\lhd b \iff a \sim_\lhd b$, so indifference means distinct incomparability or identity, and the relation $\lhd$ can be interpreted as a sequence of equivalence classes of $\sim_\lhd$. For the weak order $<_2$ from Figure 3, the equivalence classes of $\sim_{<_2}$ are $\{a\}$, $\{b,c\}$, and $\{d\}$. There are, however, cases where insisting on weak orders may not be reasonable. Those cases will not be discusses in this paper, the reader is referred to [6, 8] for more details.

3. Ranking Problem and Pairwise Comparisons Ranking Data

This section provides a theoretical framework to our approach. We will start with formal definition of the ranking problem and ranking data collected using Pairwise Comparisons paradigm.

Definition 3.1. A *ranking* is just a partial order $Rank = (X, <^{rank})$, where X is the set of objects to be ranked and $<^{rank}$ is a ranking relation. We assume that $<^{rank}$ is a weak or total order. The ranking relation $<^{rank}$ is unknown and the *ranking problem* is to construct $<^{rank}$ on the basis of *ranking data*. □

Definition 3.2. A *pairwise comparisons ranking data* is a pair $PCRD = (X, \mathcal{R})$, where $\mathcal{R}$ is a total function $\mathcal{R} : X \times X \to RV$. The elements of set $RV = \{v_0, v_1, ..., v_k\}$, $k \geq 1$ are called *ranking values*. The value v_0 is interpreted as *indifference,* so we assume $\mathcal{R}(x,x) = v_0$ for all $x \in X$. The values $v_1, ..., v_k$ are interpreted as *preferences*. We assume preferences are totally ordered and $v_k \hookleftarrow v_{k-1} \hookleftarrow ... \hookleftarrow v_1$. The total order $\hookleftarrow$ describes the degree of preference represented by the elements of RV. If $v_i \hookleftarrow v_j$ then v_i represents stronger preference than v_j (for example v_i represents *strongly better* and v_j represents *slightly in favour*). Usually we will write $a\ v_i\ b$ instaed of $\mathcal{R}(a,b) = v_i$, $i = 0, ..., k$.

The function $\mathcal{R}$ is constructed using the *Pairwise Comparisons* paradigm. For each pair $x, y \in X$ the value $\mathcal{R}(x,y)$ is decided based on the analysis of x and y only, independently of the rest of X. □

For example we may define RV (see [9]) as $RV = \{\approx, \sqsubset, \subset, <, \prec\}$, with the following interpretation $a \approx b$: a and b are *indifferent*, $a \sqsubset b$: *slightly in favour* of b, $a \subset b$: *in favour* of b, $a < b$: b is *strongly better*, $a \prec b$: b is *extremely better*. The list $\sqsubset$, $\subset$, $<$, $\prec$ may be shorter or longer, but not empty and not much longer (due to limitations of the human mind [2]). In this case we assume: $a \prec b \implies a < b \implies a \subset b \implies a \sqsubset c$, i.e. $\prec \hookleftarrow < \hookleftarrow \subset \hookleftarrow \sqsubset$.

Given a pairwise comparisons ranking data $PCRD = (X, \mathcal{R})$, with $RV = \{v_0, v_1, ..., v_k\}$, we may define the relations $R_i \subseteq X \times X$, $i = 0, ..., k$ in the following manner:

$$\begin{aligned} xR_0y &\iff \mathcal{R}(x,y) = v_0, \\ xR_ky &\iff \mathcal{R}(x,y) = v_k, \\ xR_iy &\iff \mathcal{R}(x,y) \in \{v_i, v_{i+1}, ..., v_k\}, \ \ i = 1, ..., k-1. \end{aligned}$$

Corollary 3.1.

1. $R_0 \cup R_1 \cup ... \cup R_k = X \times X$
2. $R_k \subseteq R_{k-1} \subseteq ... \subseteq R_1$ and $R_0 \cap R_1 = \emptyset$.
3. $v_i \hookleftarrow v_j \iff R_i \subseteq R_j$, $i, j = 1, ..., k$ □

It is often useful to represent a pairwise comparison ranking data $PCRD$ as a tuple $(X, R_0, R_1, ..., R_k)$ (see [9]) instead of a pair $(X, \mathcal{R})$. Usually we use the same symbol to denote both v_i and R_i. For example $(X, \approx, \sqsubset, \subset, <, \prec)$ is a pairwise comparison ranking data (with the interpretation described above).

We may now describe the *ranking problem* as follows: "*derive the ranking relation* $<^{rank}$ *from a given pairwise comparison ranking data PCRD*". Note that in a general case, *none* of the relations R_i, $i = 1, ..., k$, could be even a partial order. The problem is that X is believed to be partially or weakly ordered by the ranking relation $<^{rank}$ but the data acquisition process may be so influenced by informational noise, imprecision, randomness, or expert ignorance that the collected data $R_1, R_2, ..., R_k$ are only some relations on X. We may say that they give a fuzzy picture of ranking, and to focus it, we must do some pruning and/or extending.

- For a given pairwise comparison ranking data $PCRD = (X, \mathcal{R})$, the ranking relation derived from $PCRD$ will be denoded by $<^{rank}_{PCRD}$, or $<^{rank}_{(X,\mathcal{R})}$.

The tools needed to solve the ranking problem will be presented in the following two sections.

4. Partial Order Approximations of Arbitrary Relations

Let X be a set, and R be a relation on X. The relation R may or may not be a partial order. Our goal is to find a relation $<_R$ on X which could be interpreted as the "best" partial order approximation of R. If R is a partial order then obviously $<_R$ should equal to R.

The problem is, when can a partial order $<_R$ be interpreted as *a partial order approximation* of a given relation R, not to mention the "best" approximation? Approximations of relations (sets, numbers, etc.) are usually defined as follows, a relation R^{up} is an (upper) approximation of R if R^{up} has a desired property and $R \subseteq R^{up}$, or, a relation R^{low} is an (lower) approximation of R if R^{low} has a desired property and $R^{low} \subseteq R$. This idea is behind many *closure* definitions [20] and Pawlak's Rough Sets [19], but in pure form, it does not seem to work for this problem. If R is not a partial order then it is either not transitive or contains a cycle, or both. Making R transitive would grow it up while removing all cycles would shrink it. In this paper we will the concept of a partial order approximation of an arbitrary relation R proposed in [11].

For every relation $R \subseteq X \times X$, let $R^+ = \bigcup_{i=1}^{\infty} R^i$, denote the *transitive closure* of R, $id = \{(x,x) \mid x \in X\}$ denote the identity relation, and let $R^\circ = R \cup id$ denote the *reflexive closure* of R (see [20] for details).

For each relation R and each $a \in X$ we define: $R(a) = \{x \mid aRx\}$ and $R^{-1}(a) = \{x \mid xRa\}$.

For every relation R, define the relations R^{cyc}, $R^\bullet$, $aR^\subset b$ and $a \equiv_R b$ as

- $aR^{cyc}b \iff aR^+b \wedge bR^+a$,
- $aR^\bullet b \iff aRb \wedge \neg(aR^{cyc}b)$,
- $aR^\subset b \iff R^\circ(b) \subset R^\circ(a) \wedge (R^\circ)^{-1}(a) \subset (R^\circ)^{-1}(b)$,
- $a \equiv_R b \iff R(a) = R(b) \wedge R^{-1}(a) = R^{-1}(b)$.

If $aR^{cyc}b$ we will say that a and b belong to some cycle, the relation $R^\bullet$ is called an *acyclic refinement of* R, and $R^\subset$ is called an *inclusion kernel* of R. The relation $\equiv_R$ is an *equivalence relation* (i.e. it is reflexive, symmetric and transitive) and it is called *the equivalence with respect to* R, since if $a \equiv_R b$, *there is nothing in R that can distinguish between a and b.*

Corollary 4.1. If R is a partial order then $R^{cyc} = \emptyset$, $R = R^+ = R^\bullet = R^\subset$ and $\equiv_R = \approx_R$. □

Definition 4.1. ([11])
A partial order $< \subseteq X \times X$ is a *partial order approximation* of a relation $R \subseteq X \times X$ if it satisfies the following three conditions:

1. $a < b \implies aR^+b$,
2. $a < b \implies \neg aR^{cyc}b$ (or, equivalently $a < b \implies \neg bR^+a$),
3. $aR^\subset b \wedge aR^\bullet b \implies a < b$,
4. $a \equiv_R b \implies a \equiv_< b$. □

Since R^+ is the smallest transitive relation containing R (see [20]), and due to informational noise, imprecision, randomness, etc., some parts of R might be missing, it is reasonable to assume that R^+ is the upper bound of $<$. If R is interpreted as an estimation of a ranking, then $aR^{cyc}b$ is interpreted that as far as ranking is concerned, a and b are indifferent, so $aR^{cyc}b \implies (\neg a < b \wedge \neg b < a)$, which is expressed by (2) of the above definition. When $a < b \implies aR^+b$, then $\neg aR^{cyc}b$ can be replaced by $\neg bR^+a$. The condition (3) defines the lower bound (see [11] for more detailed explanation). The condition (4) ensures preservation of the equivalence with respect to R.

Since R is constructed on the basis of pairwise comparisons paradigm, it may happen that aRb makes sense only locally, when the domain is restricted to $\{a,b\}$, and it needs to be pruned in global setting (see [11]). In such cases we may require $a <^{rank} b \implies aRb$, which leads to the following definition.

Definition 4.2. A partial order $< \subseteq X \times X$ is an *inner* partial order approximation of a relation $R \subseteq X \times X$, if it is a partial order approximation of R, and satisfies: $a < b \implies aRb$. □

Every partial order is transitive, acyclic and equal to its inclusion property kernel. An arbitrary relation R may not have these properties but we may try to refine R using transive closure, acyslic refinement and finding inclusion property kernel, in various orders or simultaneously (i.e. using set theory intersection). We will show that there are exactly four partial order approximations that can be derived in this way.

Let us first define the relation $R^{\subset\wedge\bullet}$ as follows: $aR^{\subset\wedge\bullet}b \iff aR^{\subset}b \wedge aR^{\bullet}b$.

The following properties of four "natural" partial order approximations of a given relation R have been proved.

Theorem 4.1.

1. The relations $R^{\subset\wedge\bullet}$, $(R^{\bullet})^{\subset}$, $(R^{\bullet})^{+}$, $(R^{+})^{\bullet}$ are partial order approximations of R.
2. The relations $R^{\subset\wedge\bullet}$ and $(R^{\bullet})^{\subset}$ are inner partial order approximations of R.
3. $R^{\subset\wedge\bullet} \subseteq (R^{\bullet})^{\subset} \subseteq (R^{\bullet})^{+} \subseteq (R^{+})^{\bullet}$.
4. If R is transitive, i.e. $R = R^+$, then $R^{\subset\wedge\bullet} = (R^{\bullet})^{\subset} = (R^{\bullet})^{+} = (R^{+})^{\bullet}$.
5. If R is a partial order, then $R = R^{\subset\wedge\bullet} = (R^{\bullet})^{\subset} = (R^{\bullet})^{+} = (R^{+})^{\bullet}$.
6. If R is acyclic, i.e. $R = R^{\bullet}$, then $R^{\subset} = R^{\subset\wedge\bullet} = (R^{\bullet})^{\subset}$ and $(R^{\bullet})^{+} = (R^{+})^{\bullet}$.
7. If a partial order $<$ is a partial order approximation of R then
   $aR^{\subset\wedge\bullet}b \implies a < b \implies a(R^{+})^{\bullet}b$.
8. $aR^{cyc}b \implies a \equiv_{(R^{+})^{\bullet}} b$.
9. The relations $R^{\subset\wedge\bullet}$, $(R^{\bullet})^{\subset}$, $(R^{\bullet})^{+}$, $(R^{+})^{\bullet}$ are the only partial order approximations of R that can be derived from R by using operations '$\cap$', '$^{\subset}$', '$^{+}$' and '$^{\bullet}$'. □

The proof of this theorem is long and rather technical, so it has been moved to Appendix A.

With an exception of (8), the above theorem is practically self-explanatory. The assertion (8) says that if a and b belong to a cycle in R then they are equivalent with respect to $(R^{+})^{\bullet}$. This indicate that if we have a reason to believe that all cycles result from errors, informational noise, etc., and all elements of a cycle should be interpreted as indifferent, then $(R^{+})^{\bullet}$ is most likely *the best partial order approximation of R*. It does not necessarily mean that $(R^{+})^{\bullet}$ is always the best partial order approximation. It was argued in [11] that each of the relations $R^{\subset\wedge\bullet}$, $(R^{\bullet})^{\subset}$, $(R^{\bullet})^{+}$, $(R^{+})^{\bullet}$ is better approximation than other in some given circumstances. Nevertheless, some experiments made to justify some claims of [9] indicate that often $(R^{+})^{\bullet}$ could be interpreted as the "best" partial order approximation. This appears to be especially true when cycles of R are naturally interpreted as indifference (see Theorem 4.1(8)).

However Theorem 4.1 may still not provide a solution to our approximation problem. Even if R may in general be imprecise, in most cases *some parts* of R describe the precise ranking. For instance if R is the result of expert voting, if all experts agree that aRb, then we may assume that $a <^{rank} b$ (see *Pareto's principle* [8]). In fact we are looking for the solution to the following problem:

Let X be a set, R and $\lhd$ be two relations on X such that $\lhd$ is a partial order and $\lhd \subseteq R$. The relation R may or may not be a partial order. Our goal is to find a relation $<_{(R,\lhd)}$ on X which could be interpreted as the "best" partial order approximation of R satisfying $\lhd \subseteq <_{(R,\lhd)}$. If R is a partial order then obviously $<_{(R,\lhd)}$ equals R.

The relation $\lhd$ is called a *partially ordered kernel* of R ([11]). In general it may happen that $\lhd$ is not included in any partial order approximation discussed in the previous section (Figure 1 in [13] shows the case of $a \lhd b$ and $\neg a(R^{+})^{\bullet}b)$. In general the union of partial orders may not be a partial order at all, however we may use the following lemma.

Lemma 4.1. Let R be a relation, $<_1$ and $<_2$ be partial orders satisfying:

1. $a <_1 b \implies aR^{+}b$, and
2. $a <_2 b \implies aR^{+}b \wedge \neg(bR^{+}a)$.

Then $(<_1 \cup <_2)^{+}$ is the smallest partial order containing $<_1 \cup <_2$.

Proof:
$(<_1 \cup <_2)^{+}$ is evidently the smallest transitive relation containing $<_1 \cup <_2$. It suffices to show that $(<_1 \cup <_2)^{+}$ is irreflexive. Suppose it is not irreflexive, i.e. there exists x_0 such that $x_0(<_1 \cup <_2)^{+}x_0$. This means $x_0Q_1x_1Q_2x_2 \ldots x_{n-1}Q_nx_n$, with $x_n = x_0$, where Q_i is either $<_1$ or $<_2$. Since $<_1$ and $<_2$ are sharp partial orders, then at least one of Q_i's, say Q_k, must be equal to $<_2$. Since $<_1 \subseteq R^{+}$ and $<_2 \subseteq R^{+}$, then for each $i, j \leq n$, we have $x_iR^{+}x_j \wedge x_jR^{+}x_i$. In particular $x_kR^{+}x_{k-1}$, a contradiction as $x_{k-1} <_2 x_k \implies \neg x_{k-1}R^{+}x_k$. Hence $(<_1 \cup <_2)^{+}$ is irreflexive. □

Corollary 4.2. Let $<_R$ be any partial order approximation of R. For each partial order $\lhd \subseteq R$, the relation $(\lhd \cup <_R)^{+}$ is the smallest partial order containing $\lhd \cup <_R$. □

From Corollary 4.2 we may define $<_{(R,\lhd)}$ as $(\lhd \cup <_R)^+$ where $<_R$ is any partial order approximation of R, we consider the best in given circumstances. The relation $<_{(R,\lhd)}$ is usually *not a weak order*. The experiments discussed in Section 6 suggest that setting $<_R = (R^+)^\bullet$ is often a good choice.

We would like to point out that for a given relation R the relations $R^{\subset\wedge\bullet}$, $(R^\bullet)^\subset$, $(R^\bullet)^+$, $(R^+)^\bullet$ are *not* the only relations that satisfy Definition 4.1. They are the only partial order approximations of R that can be derived from R by using operations '$\cap$', '$^\subset$', '$^+$' and '$^\bullet$'; the operations that can be considered "natural" in the process of obtaining a partial order approximation of the relation R.

5. Weak Order Approximations Arbitrary Partial Orders

Let X be a set and let $\lhd$ be a partial order on X. The relation $\lhd$ may or may not be a weak order. We are looking for the "best" weak order extension of $\lhd$. It appears that in this case the solution may not be unique.

Note that weak order extensions reflect the fact that if $x \approx_\lhd y$ than *all reasonable methods* for extending $\lhd$ will have x equivalent to y in the extension since there is nothing in the data that distinguishes between them (for details see [6]), which leads to the definition below (for both weak an total orders).

A weak (or total) order $\lhd^w \subseteq X \times X$ is a *proper weak (or total) order extension of* $\lhd$ if and only if : $(x \lhd y \Rightarrow x \lhd^w y)$ and $(x \approx_\lhd y \Rightarrow x \sim_{\lhd^w} y)$.

If X is finite then for every partial order $\lhd$ its proper weak extension always exists. If $\lhd$ is weak, than its only proper weak extension is $\lhd^w = \lhd$. If $\lhd$ if not weak, there are usually more than one such extensions. Various methods were proposed and discussed in [6] and specially in [7]. For our purposes, the best seems to be the method based on the concept of a *global score function* [6], which is defined as (for every finite set X, $\parallel X \parallel$ denotes its number of elements):

$$g_\lhd(x) = \parallel \{z \mid z \lhd x\} \parallel - \parallel \{z \mid x \lhd z\} \parallel .$$

Given the global score function $g_\lhd(x)$, we define the relation $\lhd_g^w \subseteq X \times X$ as

$$a \lhd_g^w b \iff g_\lhd(a) < g_\lhd(b).$$

Proposition 5.1. ([6])
The relation $\lhd_g^w$ is a proper weak extension of a partial order $\lhd$. $\square$

Some other variations of $g_\lhd$ and their interpretations were analyzed in [13]. From Proposition 5.1 it follows that every finite partial order has a proper weak extension. The well known procedure "topological sorting", popular in scheduling problems, guarantees that every finite partial order has a total extension (Szpilrain Theorem guarantees it for all partial orders [6]), but even finite partial orders usually *do not have* proper total extensions. Note that the total order $\lhd_t$ is a proper total extension of $\lhd$ if and only if the relation $\approx_\lhd$ equals the identity, i.e $a \approx_\lhd b \iff a = b$. For example no weak order has a proper total extension unless it is also already total. This indicates that while *expecting a final ordering to be weak may be reasonable, expecting a final total ordering is often unreasonable*. It may however happen, and often does, that a proper weak extension is a total order, which suggests that *we should stop* *seeking a priori total orderings since weak orders appear to be more natural models of preferences than total orders*.

6. Testing

Suppose we have an algorithm to calculate the ranking relation $<^{rank}_{(X,R)}$ for a given pairwise comparison ranking data $PCRD = (X, R)$. How can we test the results of such algorithm?

Testing means that there are some data and results that are known to be correct, and then the technique is applied to the same data. The differences between the correct results and those obtained by a given technique are used to judge the value of the technique. Hence testing models such as the one presented above is problematic since *it is not obvious what should be tested against*. What are the correct results for a given data? If the object has measurable attributes and there is a precise algorithm to calculate the value, the whole problem disappears. Nevertheless we think we have designed a proper test for these kinds of ranking techniques.

A blindfolded person compared the weights of stones. The person put one stone in their left hand and another in their right, and then decided which of the relations $\approx$, $\sqsubset$, $\subset$, $<$, or $\prec$ (interpreted as described in Section 3) held. The experiment was repeated for the same set of stones by various people; and then again for different stones and different number of stones; and again for various subsets of $\{\sqsubset, \subset, <, \prec\}$. The results of some such experiments are presented in Figure 2 and Figure 3. There were many similar results but with more stones involved, so we presented only the smallest cases. Those experiments have most likely been carried out by the prehistoric man. Our ancestors probably used this technique to decide which stone is better to kill an enemy or an animal.

In this experiment the stones can be weighted using precise scale, so we have the precise results to test against.

The complete analysis of those experiments has not been finished yet. The initial goal of those experiments was to support or to refute the model presented in [9], however those experiments can also be used for the same purpose for the approach presented in this paper.

7. Some Solutions to the Ranking Problem

We will now propose some solutions to the ranking problem defined in Section 3. In this section, for any relation R, $<_R$ denotes a partial order approximation of R (for instance one of the relations $R^{\subset\wedge\bullet}$, $(R^{\bullet})^{\subset}$, $(R^{\bullet})^{+}$, $(R^{+})^{\bullet}$). The final outcome depends on the choice of $<_R$, but this problem will not be discussed in this paper. The choice of $<_R$ depends on what problem the relation R models. As we have previously mentioned, the relation $(R^{+})^{\bullet}$ often seems to be a good choice.

We will consider three distinct cases, starting with the simplest one.

Case 1

We start with the simplest and probably the most common case, $k = 1$. In this case $\mathcal{R}(a,b) = v_0$ means a and b are indifferent, and $\mathcal{R}(a,b) = v_1$ means b is preferred over a. Since $R_0 \cup R_1 = X \times X$, the case is reduced to finding the best weak order approximation of the relation R_1. This case was analysed in the context of Social Choice and Arrows' axioms (see [1, 8]) in [13] and in the context of traditional numerical pairwise comparisons approach (but with Koczkodaj's consistency [15], not the most popular Saaty's consistency [21]) in [14].

In this case, first we need to calculate $<_{R_1}$, the "best" partial order approximation of R (often $(R^+)^\bullet$, but not always), and then to find a proper weak extension of $<_{R_1}$, preferably $(<_{R_1})^w_g$.

- We may then set the ranking relation $<^{rank}_{(X,\mathcal{R})}$ as $(<_{R_1})^w_g$.

In this case we will often write $<^{rank}_{R_1}$ instead of $<^{rank}_{(X,\mathcal{R})}$. The outcome is a weak order, and it may or may not be a total order.

The shape of the function $\mathcal{R}$ that is starting point in the process of creating $<^{rank}_{R_1}$ depends on what kind of preference is used. The stronger preference (for instance *strongly better* instead of *slightly better*) results in a smaller relation R_1 and bigger relation R_0 (which represents indifference). On the other hand, since the data acquisition process is imprecise (due to informational noise, imprecision, randomness, expert ignorance, etc.), the weaker the preference the smaller the confidence and the greater chance of a wrong assessment. That is, the chance that one has an assessment of "a is slightly better than b" when in fact b is better than a or they are indifferent is much larger than the chance that one has an assessment of "a is much better than b", as if there is any doubt one gets an indifferent assessment.

In other words, for stronger preferences we may expect that aR_1b implies $a <^{rank} b$ for all $a,b \in X$, and that R_1 is also a partial order; while for weak preferences we should rather be expecting aR_1b and $\neg(a <^{rank} b)$ for some $a,b \in X$. Which approach is better? Should we insist on finding the data acquisition process with strong discriminatory power? This is usually expensive and the confidence level for the results is rather low. Or, should we apply a discriminatory power for which we have a high confidence level (but which might yield a relatively big indifference relation R_0) and assume that the correction process (i.e. calculating the relation $(<_{R_1})^w_g$) presented above, will correctly identify the relation $<^{rank}$?

We were unable to find much in the literature on this subject for partial orders, so we used the results of some experiments mentioned in the previous section. Figure 2 gives the results of one such experiment.

The stones were weighed and their weights created an increasing total order C, A, E, D, B, exactly the same as $<^{rank}_{<} = <^{rank}_{\subset}$, but different than $<^{rank}_{\sqsubset}$ - the result of using the finest preference. In fact in this case, the most discrimatory preference $\sqsubset$, and the least discrimatory and very crude preference $\prec$ produced the same outcome, different than actual ordering. On the other hand, the medium discrimatory preference $\subset$ and the relatively low preference $<$ produced the correct ranking. The relations $\prec$, $<$, $\subset$ were partial orders included in the correct total ranking, but none of them was even a weak order. The relation $\sqsubset$ was not a partial order and it was not transitive.

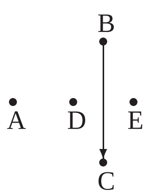
The weights difference among A, D, E were relatively small, so different persons provided different relations $\sqsubset$. For one person the $\sqsubset$-preference beetween A and D depended on which stone was put in

$\prec$	A	B	C	D	E
A	$\approx$	$\approx$	$\approx$	$\approx$	$\approx$
B	$\approx$	$\approx$	$\succ$	$\approx$	$\approx$
C	$\approx$	$\prec$	$\approx$	$\approx$	$\approx$
D	$\approx$	$\approx$	$\approx$	$\approx$	$\approx$
E	$\approx$	$\approx$	$\approx$	$\approx$	$\approx$

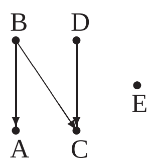
$<$	A	B	C	D	E
A	$\approx$	$<$	$\approx$	$\approx$	$\approx$
B	$>$	$\approx$	$>$	$\approx$	$\approx$
C	$\approx$	$<$	$\approx$	$<$	$\approx$
D	$\approx$	$\approx$	$>$	$\approx$	$\approx$
E	$\approx$	$\approx$	$\approx$	$\approx$	$\approx$

$\subset$	A	B	C	D	E
A	$\approx$	$\subset$	$\approx$	$\subset$	$\approx$
B	$\supset$	$\approx$	$\supset$	$\approx$	$\supset$
C	$\approx$	$\subset$	$\approx$	$\subset$	$\subset$
D	$\supset$	$\approx$	$\supset$	$\approx$	$\approx$
E	$\approx$	$\subset$	$\supset$	$\approx$	$\approx$

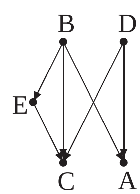
$\sqsubset$	A	B	C	D	E
A	$\approx$	$\sqsubset$	$\sqsupset$	$\sqsupset$	$\sqsubset$
B	$\sqsupset$	$\approx$	$\sqsupset$	$\sqsupset$	$\sqsupset$
C	$\sqsubset$	$\sqsubset$	$\approx$	$\sqsubset$	$\sqsubset$
D	$\sqsubset$	$\sqsubset$	$\sqsupset$	$\approx$	$\sqsupset$
E	$\sqsupset$	$\sqsubset$	$\sqsupset$	$\sqsubset$	$\approx$



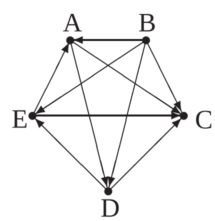
The relation $\prec$



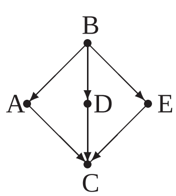
The relation $<$



The relation $\subset$



The relation $\sqsubset$



Identical partial orders $<^{rank}_{\prec}$ and $<^{rank}_{\sqsubset}$

(*Hasse Diagram*)

Identical partial orders $<^{rank}_{<}$ and $<^{rank}_{\subset}$

(*Hasse Diagram*)

Figure 2: Four pairwise comparisons ranking data $(X,\approx,\prec)$, $(X,\approx,<)$, $(X,\approx,\subset)$ and $(X,\approx,\sqsubset)$, acquired for the same set of objects $X = \{A,B,C,D,E\}$, and the ranking relations they generate. Results of one experiment "weighing with hands". The relations $\approx$, $\prec$, $<$, $\subset$, $\sqsubset$ are interpreted as described in Section 3. For each relation $R \in \{\approx,<,\subset,\sqsubset\}$, the ranking $<^{rank}_{R}$ is set as $<^{rank}_{R} = ((R^{+})^{\bullet})^{w}_{g}$. The wrong judgement of $\sqsubset$ is in two grey cells.

which hand. On the other hand the outcomes for $\prec$, $<$ and $\subset$ were the same for all persons.

Many other experiments of "weighing with hands only" resulted in similar outcome, and even though their complete analysis is not finished yet, we have come to the following conclusion (for the case of $k = 1$):

1. The order identification power of weak extension procedures is substantial and *vastly underestimated*.
2. If the ranked set of objects is, by its nature, expected to be totally ordered, the weak extension can detect it, even if the pairwise comparison process is not very precise, and often results in "indifference".
3. It is a *serious error* to attempt to find a *total extension* without going through a weak extension process.
4. In general, *admitting incomparability on the level of pairwise comparisons is better than insisting on an order at any cost*. The latter approach leads to an arbitrary and often incorrect total ordering.
5. Using less fine but more certain preferences is better than finer but uncertain preferences.

We will now consider the second case, where the results of Lemma 4.1 and Corollary 4.2 will be used.

Case 2

For a moment consider again the case $k = 1$ and the pairwaise comparison ranking data (X, R_0, R_1). Even if R_1 may in general be imprecise, in most cases *some parts* of R_1 describe the precise ranking. For instance if R_1 is the result of expert voting, if all experts agree that aR_1b, than we may assume that $a <^{rank} b$ (see *Pareto's principle* [8]). Similarly if a person a is both taller and heavier than b, we would rather say that a is bigger than b, where "bigger" is a calculated ranking relation.

This leads us to the case, where $k = 2$, and the pairwaise comparison ranking data is defined as $PCRD = (X, R_0, R_1, R_2)$, where R_2 is a partial order and $R_2 \subseteq <^{rank}$ (i.e. R_2 is a *partially ordered kernel* of R_1). The mathematics for this case is provided by Lemma 4.1 and Corollary 4.2 from Section 4. To be consistent with the notation used in Section 4, we will write R instead of R_1 and $\lhd$ instead of R_2.

- In this case we set $<^{rank}_{PCRD}$ as $(<_{(R,\lhd)})^{w}_{g}$, and denote it as $<^{rank}_{(R,\lhd)}$.

An experiment illustrating this case is in Figure 3. The stone were weighted and their weights created and increasing total order C, A, E, D, B, G, F. This order was not detected by neither the fine preference $\sqsubset$, nor by "certain" preference $\lhd$, but was correctly detected by combining $\sqsubset$ and $\lhd$, i.e. by the pairwise comparisons ranking data $(X, \approx, \sqsubset, \lhd)$, where $X = \{A, B, C, D, E, F, G\}$.

Note that in most cases deriving a partially ordered kernel $\lhd$ from R is rather easy and natural process, which means a transformation of a pairwise comparisons ranking data $(X, \approx, \sqsubset)$, i.e. Case 1, into $(X, \approx, \sqsubset, \lhd)$, i.e. Case 2, is an easy and natural process as well. Intuitively, a ranking derived from

$(X,\approx,\sqsubset,\lhd)$ should be more accurate than a ranking derived from $(X,\approx,\sqsubset)$. The experiments mentioned in Section 6 seem to confirm this (see Figure 3).

We are now ready to analyse the most general case of any k.

Case 3

For an arbitrary k, a pairwise comparisons ranking data $(X,\mathcal{R})$ can be defined as a tuple $(X,R_0,R_1,\ldots,R_k)$ with $R_k \subseteq R_{k-1} \subseteq \ldots \subseteq R_1$. Without any loss of generality we may assume that R_k is a partial order. If it is not we may construct $<_{R_k}$, its partial order approximation as defined in Section 4, set new R_k as $<_{R_k}$, new R_0 as $R_0 \setminus <_{R_k}$, and new R_i as $R_i \cup <_{R_k}$, for $i = 1,\ldots,k-1$.

Let s be the smallest number such that for all $i \geq s$, R_i are partial orders. If $s = 1$, we just set $<^{rank}_{(X,\mathcal{R})}$ as $(R_1)^w_g$. Otherwise we define the relations $\widehat{R}_1,\widehat{R}_2, \ldots ,\widehat{R}_k$ as follows

$$
\begin{aligned}
\widehat{R}_i &= R_i, \text{ for } i = s, s+1,\ldots,k,\\
\widehat{R}_{s-1} &= (<_{R_{s-1}} \cup \widehat{R}_s)^+,\\
\widehat{R}_{j-1} &= (<_{R_{j-1}\cup\widehat{R}_j} \cup \widehat{R}_j)^+, \text{ for } j = s-1,\ldots,2.
\end{aligned}
$$

where $<_{R_{s-1}}$ is a partial order approximation of R_{s-1} and $<_{R_{j-1}\cup\widehat{R}_j}$ are partial order approximations of $R_{j-1} \cup \widehat{R}_j$ for $j = s-1,\ldots,2$. In particular $\widehat{R}_1 = (<_{R_1\cup\widehat{R}_2})^+$.

Lemma 7.1. 1. For all $i = 1,\ldots,k$, the relations $\widehat{R}_i$ are partial orders.

2. $\widehat{R}_k \subseteq \widehat{R}_{k-1} \subseteq \ldots \subseteq \widehat{R}_1$.

Proof:
(1) The relations $R_s,R_{s+1},\ldots,R_k$ and $\widehat{R}_s = R_s$ are partial orders by the definition. Since $\widehat{R}_s = R_s \subseteq R_{s-1}$, by Collorary 4.2, $\widehat{R}_{s-1}$ is a partial order too. Since each $\widehat{R}_j \subseteq (R_{j-1}\cup\widehat{R}_j)$, by Collorary 4.2 again, $\widehat{R}_{j-1}$ is a partial order for all $j = s-1,\ldots,2$.
(2) From the definition we have $\widehat{R}_k \subseteq \widehat{R}_{k-1} \subseteq \ldots \subseteq \widehat{R}_s$. Since for all relations Q,S, we have $Q \subseteq (S\cup Q)^+$, then clearly $\widehat{R}_s \subseteq \widehat{R}_{s-1} \subseteq \ldots \subseteq \widehat{R}_1$. □

- Since the partial order $\widehat{R}_1$ may not be a weak order, we set $<^{rank}_{(X,\mathcal{R})} = (\widehat{R}_1)^w_g$.

The algorithm presented above is orthogonal to that from [9]. It worked well for the "weighing with hands" experiment. In general the tuple $(X,\widehat{R}_0,\widehat{R}_1,\ldots,\widehat{R}_k)$, where $x\widehat{R}_0 y \iff \neg(x\widehat{R}_1 y)$, may not satisfy the consistency rules proposed in [9], even though for "weighing with hands" experiments it usually does. The algorithm presented above is easy to program, while the method presented in [9] requires human intervention (changing of preferences).

$\sqsubset$	A	B	C	D	E	F	G
A	$\approx$	$\sqsubset$	$\sqsupset$	$\sqsupset$	$\sqsubset$	$\sqsubset$	$\sqsubset$
B	$\sqsupset$	$\approx$	$\sqsupset$	$\sqsupset$	$\sqsupset$	$\sqsubset$	$\sqsubset$
C	$\sqsubset$	$\sqsubset$	$\approx$	$\sqsubset$	$\sqsubset$	$\sqsubset$	$\sqsubset$
D	$\sqsubset$	$\sqsubset$	$\sqsupset$	$\approx$	$\sqsupset$	$\sqsubset$	$\sqsubset$
E	$\sqsupset$	$\sqsubset$	$\sqsupset$	$\sqsubset$	$\approx$	$\sqsubset$	$\sqsubset$
F	$\sqsupset$	$\sqsupset$	$\sqsupset$	$\sqsupset$	$\sqsupset$	$\approx$	$\sqsupset$
G	$\sqsupset$	$\sqsupset$	$\sqsupset$	$\sqsupset$	$\sqsupset$	$\sqsubset$	$\approx$

$\lhd$	A	B	C	D	E	F	G
A	$\approx$	$\lhd$	$\approx$	$\approx$	$\approx$	$\lhd$	$\lhd$
B	$\rhd$	$\approx$	$\rhd$	$\approx$	$\approx$	$\lhd$	$\lhd$
C	$\approx$	$\lhd$	$\approx$	$\lhd$	$\approx$	$\lhd$	$\lhd$
D	$\approx$	$\approx$	$\rhd$	$\approx$	$\approx$	$\lhd$	$\lhd$
E	$\approx$	$\approx$	$\approx$	$\approx$	$\approx$	$\lhd$	$\lhd$
F	$\rhd$	$\rhd$	$\rhd$	$\rhd$	$\rhd$	$\approx$	$\approx$
G	$\rhd$	$\rhd$	$\rhd$	$\rhd$	$\rhd$	$\approx$	$\approx$

$\sqsubset / \lhd$	A	B	C	D	E	F	G
A	$\approx$	$\lhd$	$\sqsupset$	$\sqsupset$	$\sqsubset$	$\lhd$	$\lhd$
B	$\rhd$	$\approx$	$\rhd$	$\sqsupset$	$\sqsupset$	$\lhd$	$\lhd$
C	$\sqsubset$	$\lhd$	$\approx$	$\lhd$	$\sqsubset$	$\lhd$	$\lhd$
D	$\sqsubset$	$\sqsubset$	$\rhd$	$\approx$	$\sqsupset$	$\lhd$	$\lhd$
E	$\sqsupset$	$\sqsubset$	$\sqsupset$	$\sqsubset$	$\approx$	$\lhd$	$\lhd$
F	$\rhd$	$\rhd$	$\rhd$	$\rhd$	$\rhd$	$\approx$	$\sqsupset$
G	$\rhd$	$\rhd$	$\rhd$	$\rhd$	$\rhd$	$\sqsubset$	$\approx$

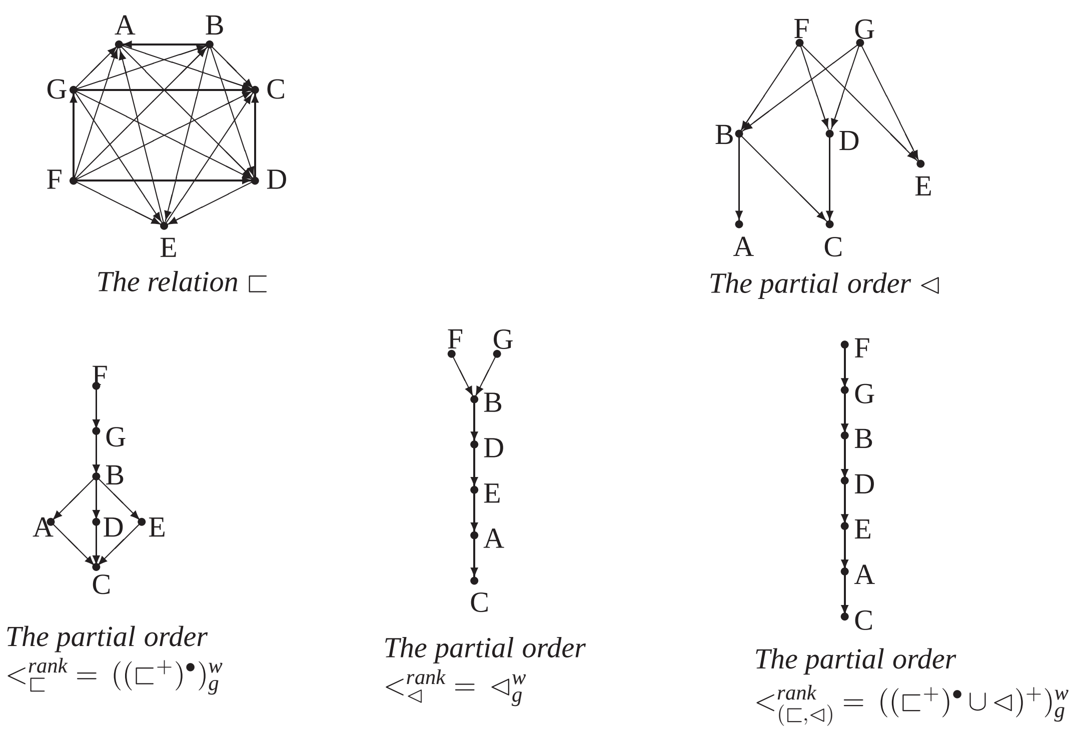


Figure 3: Three pairwise comparisons ranking data $(X,\approx,\sqsubset)$, $(X,\approx,\lhd)$ and $(X,\approx,\sqsubset,\lhd)$, acquired for the same set of objects $X = \{A,B,C,D,E,F,G\}$, and the ranking relations they generate. Results of some experiments "weighing with hands". The relation $\approx$ is indifference, $\sqsubset$ is interpreted as "slightly in favour" and $\lhd$ as "strongly better". We assume $a \lhd b$ implies $a \sqsubset b$. All partial orders are represented as Hasse diagrams.

8. Relationship to Rough Sets

As we have pointed out before, the relationship between the model presented above and the Rough Sets approach is not obvious, and a direct translation into any reasonable Rough Sets settings is problematic. Let us restrict our considerations to the problem of approximating an arbitrary relation by a partial order. How can this be done with Rough Sets? In the spirit of Rough Sets [18, 19], the relations $(R^{\bullet})^{+}$ and $(R^{+})^{\bullet}$ can be seen as upper partial order approximations of R, while $R^{\subset\wedge\bullet}$ or $(R^{\bullet})^{\subset}$ can be seen as lower partial order approximations of R. But this is true in spirit only, as in general case R may be included in neither $(R^{\bullet})^{+}$ nor $(R^{+})^{\bullet}$, and neither $R^{\subset\wedge\bullet}$ nor $(R^{\bullet})^{\subset}$ satisfy all of the properties required from lower (Rough Sets) approximations.

The principles of Rough Rets [18, 19] can be formulated as follows. Let U be a finite and nonempty universum of elements, and let $E \subseteq U \times U$ be an equivalence (i.e. reflexive, symmetric and transitive) relation. For each equivalence relation $E \subseteq U \times U$, $[x]_E$ will denote the equivalence class of E containing x and U/E will denote the set of all equivalence classes of E. The elements of U/E are called elementary sets and they are interpreted as basic observable, measurable, or definable sets. The pair (U,E) is referred to as a Pawlak approximation space. A set $A \subseteq U$ is approximated by two subsets of U, $\underline{A}$ - called the lower approximation of A, and $\overline{A}$ - called the upper approximation of A, where:

$$\underline{A} = \bigcup\{[x]_E \mid x \in U \wedge [x]_E \subseteq A\}, \qquad \overline{A} = \bigcup\{[x]_E \mid x \in U \wedge [x]_E \cap A \neq \emptyset\}.$$

Since every relation is a set of pairs, this approach can be used for relations as well [23]. Unfortunately, in such cases as ours we want approximations to have some specific properties like irreflexivity, transitivity etc., and most of those properties are not closed under the set union operator. As was pointed out in [28], in general one cannot expect approximations to have desired properties (see [28] for details). It is also unclear how to define the relation E for cases such as ours.

However the Rough Sets can also be defined in an orthogonal (sometimes called 'topological') manner [19, 24, 27]. For a given (U,E) we may define $\mathcal{D}(U)$ as the smallest set containing $\emptyset$, all of the elements of U/E and that is closed under set union. Clearly U/E is the set of all components generated by $\mathcal{D}(U)$ [16]. We may start with defining a space as $(U,\mathcal{D})$ where $\mathcal{D}$ is a family of sets that contains $\emptyset$ and for each $x \in U$ there is $A \in \mathcal{D}$ such that $x \in A$ (i.e. $\mathcal{D}$ is a cover of U [20]). We may now define $E_{\mathcal{D}}$ as the equivalence relation generated by the set of all components defined by $\mathcal{D}$ (see for example [16]). Hence both approaches are equivalent [19, 24, 28], however now for each $A \subseteq U$ we have:

$$\underline{A} = \bigcup\{X \mid X \subseteq A \wedge X \in \mathcal{D}\}, \qquad \overline{A} = \bigcap\{X \mid A \subseteq X \wedge X \in \mathcal{D}\}.$$

We can now define $\mathcal{D}$ as a set of relations having the desired properties and then calculate $\underline{R}$ and/or $\overline{R}$ with respect to a given $\mathcal{D}$. Such an approach was proposed and analysed in [28], however it seems to have only limited applications. It assumes that the set $\mathcal{D}$ is closed under both union and intersection, and few properties of relations do this. For instance, transitivity is not closed under union and having a cycle is not closed under intersection. Some properties, like "having exactly one cycle" are preserved by neither union nor intersection. This problem was discussed in [28] and they proposed that perhaps a different $\mathcal{D}$ could be used for the lower and upper approximations. But this solution again seems to have rather limited applications. The approach of [28] assumes additionally that, for the upper approximation there is at least one element of $\mathcal{D}$ that contains R, and, for the lower approximation there exists at least one element of $\mathcal{D}$ that is included in R. These are assumptions that are too strong for cases such as ours. If R contains a cycle, then there is no partial order that contains R! Very often $R \setminus (R^{+})^{\bullet} \neq \emptyset$ and $R \setminus (R^{\bullet})^{+} \neq \emptyset$. A possible solution to this problem involves complicated generalisations of the concepts of both lower and upper approximations and the use of mixed approximations. For example, $R^{\bullet}$ can be interpreted as a lower acyclic approximation of R, and then $(R^{\bullet})^{+}$ can be interpreted as an upper transitive approximation of $R^{\bullet}$. Similarly, R^{+} can be interpreted an upper transitive approximation of R, and $(R^{+})^{\bullet}$ as a lower acyclic approximation of R^{+}. Note that $(R^{\bullet})^{+} \subseteq (R^{+})^{\bullet}$, as expected from the point of view of the Rough Sets paradigm; however, here we are mixing different approximations. The details of this solution are quite long, well beyond the scope of this paper and will be presented elsewhere [12].

9. Final Comments

The concepts of ranking, ranking problem and pairwise comparisons ranking data have been defined and analysed in the setting of partial orders. Some solutions have been presented. No numbers were used whatsoever, which we believe is more fair and objective approach. The importance of the indifference relation and the power of the weak order extension procedure have been emphasised. A method of testing has been proposed. The approach presented in this paper rely heavily on the concept of partial order approximation of an arbitrary relation. A formal definition of this concept has been provided and its properties were analysed. Some relationship to the Rough Sets paradigm has also been discussed.

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Appendix A: A Proof of Theorem 4.1

We will start with stating two simple folklore results. For the completness we will also provide proofs.

Lemma A.1. For every relation R:

1. $bR^{\circ} \subset aR^{\circ} \implies aRb$,
2. $R^{\circ}a \subset R^{\circ}b \implies aRb$.

Proof:
(1) Let $bR^{\circ} \subset aR^{\circ}$. Since $b \in bR^{\circ}$, then $b \in aR^{\circ}$, i.e. $aRb \vee a = b$. But $a = b$ implies $bR^{\circ} = aR^{\circ}$, so aRb.
(2) Dually to (1). □

Lemma A.2. If R is a partial order then the following three statements are equivalent:

1. aRb,
2. $bR^{\circ} \subset aR^{\circ}$,
3. $R^{\circ}a \subset R^{\circ}b$.

Proof:
(2) $\implies$ (1) and (3) $\implies$ (1) follow from Lemma A.1.
(1) $\implies$ (2): Let aRb and $x \in bR^{\circ}$. If $x = b$ then aRb implies $b \in aR^{\circ}$. If $x \neq b$ then aRb and bRx, which implies aRx, i.e. $x \in aR^{\circ}$. Hence $bR^{\circ} \subseteq aR^{\circ}$. But $aRb \implies a \neq b \wedge \neg bRa$, so $a \notin bR^{\circ}$, which means $bR^{\circ} \subset aR^{\circ}$.
(1) $\implies$ (3): Similarly to (1) $\implies$ (2). □

Immediately from appropriate definitions we have.

Corollary A.1.

1. $R^{\subset} \subseteq R$ and $R^{\subset}$ is a partial order.
2. $R^{\bullet} \subseteq R$, $R^{\bullet}$ is acyclic (i.e. also irreflexive), and $aR^{\bullet}b \iff aRb \wedge \neg(bR^{+}a)$. □

For each equivalence relation $E \subseteq X \times X$, $[x]_E$ will denote the equivalence class of E containing x and X/E will donote the set of all equivalence classes of E.

For every relation R, we define R_{id}^{cyc} as: $aR_{id}^{cyc}b \iff aR^{cyc}b \vee a = b$.

Note that R_{id}^{cyc} is an equivalence relation. The following result is well known however it is usually formulated in terms of quasi-orders (pre-orders).

Lemma A.3. (Schröder [25])
For every relation $R \subseteq X \times X$, let $\prec_R \subseteq (X/R_{id}^{cyc}) \times (X/R_{id}^{cyc})$ be the following relation:

$$[x]_{R_{id}^{cyc}} \prec_R [y]_{R_{id}^{cyc}} \iff xR^{+}y \wedge \neg yR^{+}x.$$

The relation $\prec_R$ is a partial order on X/R_{id}^{cyc}. □

We need some properties of the relation $\equiv_R$, for various R. Note that $\equiv_R$ can equivalently be defined as:

$$a \equiv_R b \iff \forall x.\ (xRa \iff xRb) \wedge (aRx \iff bRx).$$

Lemma A.4. For every two relations R and Q: $\quad a \equiv_R b \wedge a \equiv_Q b \implies a \equiv_{R\cap Q} b.$

Proof:
$a \equiv_R b \wedge a \equiv_Q b \implies aR = bR \wedge Ra = Rb \wedge aQ = bQ \wedge Qa = Qb \implies a(R\cap Q) = b(R\cap Q) \wedge (R\cap Q)a = (R\cap Q)b \iff a \equiv_{R\cap Q} b.$ □

Lemma A.5. For every relation R we have:

1. $a \equiv_R b \implies a \equiv_{R^+} b$,
2. $a \equiv_R b \implies a \equiv_{R^\bullet} b$,
3. $a \equiv_R b \implies a \equiv_{R^\subset} b$.

Proof:

(1) $xR^+a \iff xRx_1R\ldots Rx_nRa$. But $a \equiv_R b \implies (x_nRa \Leftrightarrow x_nRb)$, so $xR^+a \iff xRx_1R\ldots Rx_nRb \iff xR^+b$. Similarly we show $aR^+x \iff bR^+x$, hence $a \equiv_{R^+} b$.

(2) Since $xR^\bullet a \iff xRa \wedge \neg aR^+x$ and $xRa \iff xRb$, then $xR^\bullet a \implies xRb$. Suppose bR^+x, i.e. $bRx_1R\ldots x_kRx$. But $bRx_1 \iff aRx_1$, so $bR^+x \iff aR^+x$, a contradiction as $xR^\bullet a \implies \neg aR^+x$. Hence $xR^\bullet a \implies xR^\bullet b$. By replacing a with b, we immediately get $xR^\bullet b \implies xR^\bullet a$, i.e. $xR^\bullet a \iff xR^\bullet b$. In an almost identical manner we show $aR^\bullet x \iff bR^\bullet x$, so $a \equiv_{R^\bullet} b$.

(3) Note that if $a = b$ then clearly $a \equiv_{R^\subset} b$, so assume $a \neq b$.
First we show that $a \equiv_R b$ implies $\forall x.\ R^\circ x \subset R^\circ a \iff R^\circ x \subset R^\circ b$. Suppose $R^\circ x \subset R^\circ a$, i.e. $Rx \cup \{x\} \subset Ra \cup \{a\}$. Since $Ra = Rb$, then $R^\circ x = Rx \cup \{x\} \subseteq Rb \cup \{a\}$.
We now have to consider two cases:
Case 1: $a \in Rb$. Since $Ra = Rb$ then $a \in Ra$, so we have $R^\circ x \subset R^\circ a \cup \{a\} = Ra = Rb \subseteq Rb \cup \{b\} = R^\circ b$, so $R^\circ x \subset R^\circ b$.
Case 2: $a \notin Rb$. First we show that $a \in Rx \implies a \in Rb$. We have $a \in Rx \iff aRx \iff bRx \iff b \in Rx$ and $b \in Rx \subseteq R^\circ x \subset Ra \cup \{a\} \implies bRa \vee a = b$. Since $a \neq b$ then bRa. Because $a \equiv_R b$ the we have $Ra = Rb$ and $aR = bR$, so $bRa \wedge Ra = Rb \implies bRb$, while $bRb \wedge aR = bR \implies aRb$, i.e. $a \in Rb$. This means $a \notin Rb$ implies $a \notin Rb \wedge a \notin Rx$. Hence we have: $R^\circ a = R^\circ a \setminus \{a\} \subset (Rb \cup \{a\}) \setminus \{a\} = Rb \subseteq R^\circ b$, so $R^\circ x \subset R^\circ b$. In this way we have proved $\forall x.\ R^\circ x \subset R^\circ a \implies R^\circ x \subset R^\circ b$. Similarly we prove that $a \equiv_R b$ implies $\forall x.\ xR^\circ \subset aR^\circ \implies xR^\circ \subset bR^\circ$, which means that $a \equiv_R b$ implies

$\forall x.\ (R^\circ x \subset R^\circ a \wedge xR^\circ \subset aR^\circ) \implies (R^\circ x \subset R^\circ b \wedge xR^\circ \subset bR^\circ)$.

By replacing a with b we get an inverse inclusion, so in fact we proved:

$\forall x.\ (R^\circ x \subset R^\circ a \wedge xR^\circ \subset aR^\circ) \iff (R^\circ x \subset R^\circ b \wedge xR^\circ \subset bR^\circ)$,

i.e. $\forall x.\ (xR^\subset a \iff xR^\subset b)$. In almost identical way we can prove $\forall x.\ (aR^\subset x \iff bR^\subset x)$. Hence $a \equiv_{R^\subset} b$. □

We can now prove Theorem 4.1.

Theorem 4.1.

1. The relations $R^{\subset\wedge\bullet}$, $(R^\bullet)^\subset$, $(R^\bullet)^+$, $(R^+)^\bullet$ are partial order approximations of R.
2. The relations $R^{\subset\wedge\bullet}$ and $(R^\bullet)^\subset$ are inner partial order approximations of R.
3. $R^{\subset\wedge\bullet} \subseteq (R^\bullet)^\subset \subseteq (R^\bullet)^+ \subseteq (R^+)^\bullet$.
4. If R is transitive, i.e. $R = R^+$, then $R^{\subset\wedge\bullet} = (R^\bullet)^\subset = (R^\bullet)^+ = (R^+)^\bullet$.
5. If R is a partial order, then $R = R^{\subset\wedge\bullet} = (R^\bullet)^\subset = (R^\bullet)^+ = (R^+)^\bullet$.
6. If R is acyclic, i.e. $R = R^\bullet$, then $R^\subset = R^{\subset\wedge\bullet} = (R^\bullet)^\subset$ and $(R^\bullet)^+ = (R^+)^\bullet$.

7. If a partial order $<$ is a partial order approximation of R then
$aR^{\subset\wedge\bullet} b \implies a < b \implies a(R^{+})^{\bullet} b.$

8. $aR^{cyc}b \implies a \equiv_{(R^{+})^{\bullet}} b.$

9. The relations $R^{\subset\wedge\bullet}$, $(R^{\bullet})^{\subset}$, $(R^{\bullet})^{+}$, $(R^{+})^{\bullet}$ are the only partial order approximations of R that can be derived from R by using operations '$\cap$', '${}^{\subset}$', '${}^{+}$' and '${}^{\bullet}$'.

Proof:
First we show that the relations $R^{\subset\wedge\bullet}$, $(R^{\bullet})^{\subset}$, $(R^{\bullet})^{+}$, $(R^{+})^{\bullet}$ are partial orders. Consider $R^{\subset\wedge\bullet}$. Clearly $aR^{\subset\wedge\bullet} b \iff aR^{\subset} b \wedge aR^{\bullet}b \iff aR^{\subset} b \wedge \neg bR^{+}a$. By Corollary A.1(2) the relation $R^{\subset\wedge\bullet}$ is irreflexive so we need only to prove its transitivity. Suppose that $aR^{\subset\wedge\bullet} b$ and $bR^{\subset\wedge\bullet} c$. This means $aR^{\subset} b$, $bR^{\subset} c$, $\neg bR^{+}a$ and $\neg cR^{+}b$. By Corollary A.1(1), $R^{\subset}$ is transitive, so $aR^{\subset} c$, and by Lemma A.1, aRb, bRc and aRc. Hence we only need to show that $\neg cR^{+}a$. Suppose $cR^{+}a$. Then $cR^{+}a$ and aRb implies $cR^{+}b$, a contradition as $aR^{\subset\wedge\bullet} c$ implies $\neg cR^{+}b$. Therefore $R^{\subset\wedge\bullet}$ is a partial order.
Consider $(R^{\bullet})^{\subset}$. From Corollary A.1(1) it immediately follows that the relation $(R^{\bullet})^{\subset}$ is a partial order.
Consider $(R^{\bullet})^{+}$. By Corollary A.1(2), we have $aR^{\bullet}b \iff aRb \wedge \neg(bR^{+}a)$. The relation $(R^{\bullet})^{+}$ is clearly transitive, we need only to show $\neg(a(R^{\bullet})^{+}a)$ for all $a \in X$. Since $aRb \wedge \neg(bR^{+}a) \implies a \neq b$, then $\neg aR^{\bullet}a$. Suppose $a(R^{\bullet})^{+}a$. Since $\neg aR^{\bullet}a$, this means $aR^{\bullet}b(R^{\bullet})^{+}a$, for some $b \neq a$. But $aR^{\bullet}b \implies aRb$ and $b(R^{\bullet})^{+}a \implies bR^{+}a$, so we have $aRb \wedge bR^{+}a$, contradicting $aR^{\bullet}b$. Hence $\neg(a(R^{\bullet})^{+}a)$, i.e. $(R^{\bullet})^{+}$ is a partial order.
Consider $(R^{+})^{\bullet}$. Notice that $a(R^{+})^{\bullet}b \iff aR^{+}b \wedge \neg bR^{+}a \iff [x]_{R^{cyc}_{id}} \prec_R [y]_{R^{cyc}_{id}}$, where $\prec_R$ is the relation from Lemma A.3. Hence, by Lemma A.3, the relation $(R^{+})^{\bullet}$ is a partial order.

We will now prove (3), i.e. $R^{\subset\wedge\bullet} \subseteq (R^{\bullet})^{\subset} \subseteq (R^{\bullet})^{+} \subseteq (R^{+})^{\bullet}$.
Suppose $aR^{\subset\wedge\bullet}b$, i.e. $aR^{\subset} b \wedge \neg bR^{+}a$. Then aRb and $\neg bR^{+}a$, so $a \in (R^{\bullet})^{\circ}a \cap (R^{\bullet})^{\circ}b$. Assume that $x \in R^{\bullet}a$ and $x \notin R^{\bullet}b$. Since $aR^{\subset\wedge\bullet} b \implies aR^{\subset} b$, then we have $Ra \subset Rb$. But $R^{\bullet}a \subseteq Ra$, so $x \in Rb$. We now have $x \in Rb$ and $x \notin R^{\bullet}b$, i.e. $bR^{+}x$. Since $x \in R^{\bullet}a$ means xRa, then $bR^{+}ax$ and xRa give us $bR^{+}a$, a contradiction as, $aR^{\subset\wedge\bullet} b \implies \neg bR^{+}a$. Hence $R^{\bullet}a \subseteq R^{\bullet}b$. Since $a \neq b$ then $R^{\bullet}a \neq R^{\bullet}b$, so $(R^{\bullet})^{\circ}a \subset (R^{\bullet})^{\circ}b$. Similarly we show $b(R^{\bullet})^{\circ} \subset a(R^{\bullet})^{\circ}$, hence $a(R^{\bullet})^{\subset}b$. Therefore $R^{\subset\wedge\bullet} \subseteq (R^{\bullet})^{\subset}$.
By Lemma A.1 we have $(R^{\bullet})^{\subset} \subseteq R^{\bullet}$, and clearly $R^{\bullet} \subseteq (R^{\bullet})^{+}$, hence $(R^{\bullet})^{\subset} \subseteq (R^{\bullet})^{+}$.
Suppose $a(R^{\bullet})^{+}b$. Recall that $x(R^{+})^{\bullet}y \iff xR^{+}y \wedge \neg yR^{+}x$. By Corollary 3.1(2), we have $R^{\bullet} \subset R$. Hence $a(R^{\bullet})^{+} + b \implies aR^{+}b$. Suppose $bR^{+}a$. Then $aR^{cyc}b$, i.e. $\neg aR^{\bullet}b$, a contradiction. Hence $a(R^{+})^{\bullet}b$, i.e. $(R^{\bullet})^{+} \subseteq (R^{+})^{\bullet}$. Therefore we have proved the assertion (3).

Note that (3) together with the fact that all $R^{\subset\wedge\bullet}$, $(R^{\bullet})^{\subset}$, $(R^{\bullet})^{+}$, $(R^{+})^{\bullet}$ are partial orders imply that $R^{\subset\wedge\bullet}$, $(R^{\bullet})^{\subset}$, $(R^{\bullet})^{+}$, $(R^{+})^{\bullet}$ satisfy (1),(2) and (3) of Definition 4.1. By Lemma A.5, $(R^{\bullet})^{+}$ and $(R^{+})^{\bullet}$ satisfy (4) of Definition 4.1; and by Lemmas A.5 and A.4, $R^{\subset\wedge\bullet}$ and $(R^{\bullet})^{\subset}$ satisfy satisfy (4) of Definition 4.1. Therefore the assertion (1) of the above theorem does hold.

The assertion (1) and Corollary A.1(1,2) yield the assertion (2).

Hence (1), (2) and (3) hold. We will now prove (4). It suffices to show that if $R = R^{+}$ then $(R^{+})^{\bullet} \subseteq R^{\subset\wedge\bullet}$. Note that in this case $a(R^{+})^{\bullet}b \iff aRb \wedge \neg bRa$. If $R = R^{+}$ then $(R^{+})^{\bullet} = R^{\bullet}$, so we only need to show $(R^{+})^{\bullet} \subseteq R^{\subset}$. Let $a(R^{+})^{\bullet}b$. This means $a \neq b$ and $\neg bRa$. Furthermore $\neg bRa$ implies $a \notin bR \wedge b \notin Ra$. Assume $x \in bR^{\circ}$. If $x = b$ then aRb implies $b \in aR$, i.e. $x \in aR^{\circ}$. If $x \neq b$ then $x \in bR^{\circ} \implies bRx$. Since R is transitive $aRb \wedge bRx \implies aRx \implies x \in Ra \implies x \in R^{\circ}a$. Hence $bR^{\circ} \subseteq aR^{\circ}$. Since $a \neq b$ and $a \notin bR$, then $a \notin bR^{\circ}$, which means $bR^{\circ} \subset aR^{\circ}$. Dually we show $R^{\circ}a \subset R^{\circ}b$, i.e. $aR^{\subset} b$, so we have proved (4).

The assertion (5) follows from (4) and Lemma A.2.

If $R = R^{\bullet}$ then clearly $(R^{\bullet})^{\subset} = R^{\subset}$. We also heve $R^{\subset\wedge\bullet} = R^{\subset} \cap R^{\bullet} = R^{\subset} \cap R = R^{\subset}$ as, by Lemma A.1, $R^{\subset} \subseteq R$. From (3) it follows $(R^{\bullet})^{+} \subseteq (R^{+})^{\bullet}$. If $R = R^{\bullet}$, then $(R^{+})^{\bullet} \subseteq R^{+} = (R^{\bullet})^{+}$, i.e. $(R^{\bullet})^{+} = (R^{+})^{\bullet}$, so we have proved (6).

The assertion (7) follows from (1), (3) and Definition 4.1.

The assertion (8) is a consequence of Lemma A.3. Recall that we have
$a \equiv_{(R^+)^{\bullet}} b \iff \{x \mid x(R^+)^{\bullet}a\} = \{x \mid x(R^+)^{\bullet}b\} \wedge \{x \mid a(R^+)^{\bullet}x\} = \{x \mid b(R^+)^{\bullet}x\}$.
If $aR^{cyc}b$ then $[a]_{R^{cyc}_{id}} = [b]_{R^{cyc}_{id}}$. Hence we have
$x(R^+)^{\bullet}a \iff [x]_{R^{cyc}_{id}} \prec_{(R^+)^{\bullet}} [a]_{R^{cyc}_{id}} \iff [x]_{R^{cyc}_{id}} \prec_{(R^+)^{\bullet}} [b]_{R^{cyc}_{id}} \iff x(R^+)^{\bullet}b$,
which means $\{x \mid x(R^+)^{\bullet}a\} = \{x \mid x(R^+)^{\bullet}b\}$. Similarly we can prove $\{x \mid a(R^+)^{\bullet}x\} = \{x \mid b(R^+)^{\bullet}x\}$. Thus the assertion (8) does hold as well.

To show (9) first notice that, $(R^{\subset})^{\bullet} = (R^{\subset})^{+} = R^{\subset}$ (as $R^{\subset}$ is a partial order), $R^{+} \cap R^{\bullet} = (R^{+})^{\bullet}$ (from the definition of acyclic refinement), and $R^{+} \cap R^{\subset} = R^{\subset}$ (since $R^{\subset} \subseteq R \subseteq R^{+}$). Since $R^{+} = (R^{+})^{+}$, from (4) we have $(R^{+})^{\bullet} = ((R^{+})^{+})^{\bullet} = ((R^{+})^{\bullet} = (R^{+})^{\bullet}$. From (1), (3) and (5) it follows that additional applications of '$\cap$', '$^{\subset}$', '$^{+}$' and '$^{\bullet}$' do not produce new relations. □