Temporal Logics Part 1

- Overview of Temporal Logics
- Propositional Liner-Time Temporal Logics

Specification of Reactive and Distributed Systems

- **Reactive Systeme**: No clear input-output relation
  - Operating systems
  - Embedded systems
  - Network protocols
- Specification techniques: **Temporal logics**
  - Rich choice of temporal logics — multiple classification criteria
  - Some important logics are (polynomial-time) decidable — **Model checking**

Modal Logics

- Original philosophical motivation: Express different *modalities*:

  The proposition "Napoleon was victorious at Waterloo"

  – is false in this world,

  – but could be true in another world.

- Typical modal operators:

  – "*possibly*": $\Diamond p$ — "it is imaginable that p holds",

  – "*necessarily*": $\Box p$ — "it is not imaginable that p doesn't hold",

- Kripke (1963): "**possible world semantics**" (orig. Kanger 1957)

Temporal Logics

- Prior (1955): **Tense Logic** — notation still customary today

  – instead of $\Diamond p$ now temporally: $\mathbf{F}p$ — "p will eventually be true"

  – instead of $\Box p$ now temporally: $\mathbf{G}p$ — "p will always be true"

- Two kinds of applications: Temporal logics are used

  – in AI, to let programs reason about the world,

  – in software technology, to let the world reason about programs

- Pnueli (1977): "**The Temporal Logic of Programs**":

  Argues for using temporal logics as tool for specification and verification, in particular for *reactive systems* such as operating systems and network protocols

Propositional Logics versus First-order Predicate Logics

- **Temporal Propositional Logics**:
  - Classical junktors: $\wedge$, $\vee$, $\neg$
  - Temporal operators: $\mathbf{F}$, $\mathbf{G}$

- Extension to **temporal predicate logics**
  - variable, constant, function and predicate symbols as usual
  - uninterpreted / partially nterpreted / fully interpreted
  - local/global variables
  - sometimes **restrictions on permitted formulae**
    with respect to the interaction between quantifiers and temporal operators, e.g.:

$$(\forall\, y : \mathbf{G}(P(y))) \Leftrightarrow (\mathbf{G}(\forall\, y : P(y)))$$

    "Formula of Barcan" — "highly undecidable" logics

Endogeneous Time versus Exogeneous Time

- **Endogeneous time concept** for *global* argumentation
  - temporal operators are interpreted in *a single* universe
  - *global view* on a distributed program

- **Exogeneous time concept** allows *compositional* argumentation
  - The same formula can be used to refer to different distributed program fragments
  - More modular approach:
    * Specify and verify subprograms
    * Integrate proofs into global view

Linear Time versus Branching Time

This distinction is mainly semantic, but also reflected in syntax

- **Linear Time**:
  - At any point only *one* possible future
- **Branching Time**:
  - At any point *multiple* possible futures

Both approaches are used in software technology

Other Time Aspects

- **Time Points versus Time Intervals**
  - Some properties are easier to formulate using intervals.
- **Discrete Time versus Continuous Time**
  - Continuous (or dense) Time first considered in philosophy
  - Possible application in real time systems
- **Future Only versus Also Past**
  - Philosophiscal approaches: Past at least as important as future
  - Software: Frequently only future
  - Past operators are frequently useful in compositional specifications.

Classification of Temporal Logics — Summary

- *Propositional logics* — first-order predicate logics
- *Endogeneous time* (*global*) — exogeneous time (compositional)
- *Linear time* — *branching time*
- *Time points* — time intervals
- *Discrete time* — continuous time
- *Future* — *also past*

Linear-Time Temporal Logics

- Starting with propositional logics only
- Syntax: Adding temporal operators such as $\mathbf{F}$, $\mathbf{G}$, $\mathbf{X}$, $\mathbf{U}$, ...
- Semantics: Linear time structures
- Starting with future operators only; later also past
- Applications: Safety- and liveness properties

Basis for Semantics of Linear-Time TL: Time Structures

The underlying structure of time is

- in general: a totally ordered set $(S, <)$,
- in the following: the natural numbers $(\mathbb{N}, <)$.

Then:

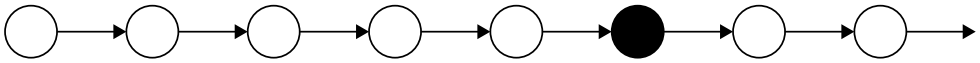
- time is discrete,
- time has a beginning without past,
- time has an infinite future.

Definition: A **linear time structure** is a triple $M = (S, x, L)$ with

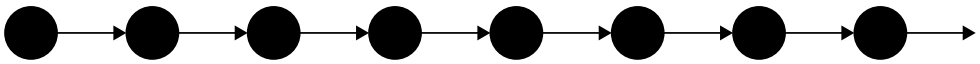
- a *state set* S,
- a *state sequence* $x : \mathbb{N} \to S$ — *timeline, computation sequence, trace, history, run*,
- a state marking $L : S \to \mathbb{P}(AP)$ that maps each state s to the set of those atomic propositions that are true in s

Temporal Operators of Linear-Time Propositional Logic

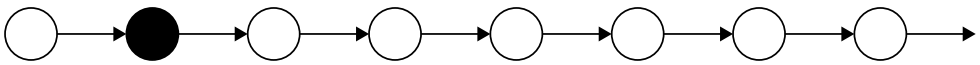
- $\mathbf{F}p$ — "eventually p"



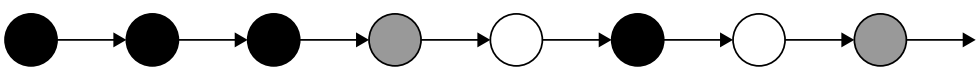
- $\mathbf{G}p$ — "always p"



- $\mathbf{X}p$ — "in the next state p"



- $p\,\mathbf{U}\,q$ — "eventually q, and until then p" (*until*)



Propositional Linear-Time Temporal Logic — Syntax

Definition: The set of formulae of **propositional linear-time temporal logic** is smallest set generated by the following rules:

- every atomic proposition $P : AP$ is a formula;
- if p and q are formulae, then $p \wedge q$ and $\neg\, p$ are formulae, too;
- if p and q are formulae, then $p\,\mathbf{U}\,q$ and $\mathbf{X}p$ formulae, too.

Abbreviations:

$$
\begin{array}{rcl|rcl}
p \vee q & :\equiv & \neg\,(\neg\, p \wedge \neg\, q) & \mathbf{F}p & :\equiv & \mathit{true}\,\mathbf{U}\,p \\
p \Rightarrow q & :\equiv & \neg\, p \vee q & \mathbf{G}p & :\equiv & \neg\,\mathbf{F}\neg\, p \\
p \Leftrightarrow q & :\equiv & (p \Rightarrow q) \wedge (q \Rightarrow p) & \mathbf{F}^{\infty}p & :\equiv & \mathbf{GF}p\text{— “unfinitely often”} \\
\mathit{true} & :\equiv & p \vee \neg\, p & \mathbf{G}^{\infty}p & :\equiv & \mathbf{FG}p\text{— “almost everywhere”} \\
\mathit{false} & :\equiv & \neg\, \mathit{true} & p\,\mathbf{B}\,q & :\equiv & \neg\,((\neg\, p)\,\mathbf{U}\,q)\text{— “}p\text{ before }q\text{”}
\end{array}
$$

Proposition Linear-Time Temporal Logic — Semantics

Notation:

- “$M, x \models p$” means “formula p holds in time structure M for time line x”
- When M is clear from the context, we just write “$x \models p$”
- If $x = s_0, s_1, s_2, s_3, \ldots$, then $x^i = s_i, s_{i+1}, s_{i+2}, \ldots$

Inductive **definition** of $\models$: Let $M = (S, x, L)$ be given.

- $x \models P$ iff $P \in L(s_0)$ for atomic propositions $P : AP$
- $x \models p \wedge q$ iff $x \models p$ and $x \models q$, and
  $x \models \neg\, p$ iff not $x \models p$;
- $x \models (p\,\mathbf{U}\,q)$ iff there is a $j : \mathbb{N}$ such that $x^j \models q$ and for all $k < j$ we have $x^k \models p$;
- $x \models \mathbf{X}p$ iff $x^1 \models p$.

PLTL — Direct Semantics of the Derived Temporal Operators

- $x \models \mathbf{F} q$ iff there is a $j : \mathbb{N}$ such that $x^j \models q$
- $x \models \mathbf{G} q$ iff for all $j : \mathbb{N}$ we have $x^j \models q$
- $x \models (p \, \mathbf{B} \, q)$ iff for all $j : \mathbb{N}$ we have that $x^j \models q$ implies that there is a $k < j$ such that $x^k \models p$
- $x \models \mathbf{F}^\infty q$ iff for all $k : \mathbb{N}$ there is a $j \geq k$ such that $x^j \models q$
- $x \models \mathbf{G}^\infty q$ iff there is a $k : \mathbb{N}$ such that for all $j > k$ we have $x^j \models q$

Exercis

Satisfiability, Validity

Definition:

- A PLTL formula p is **satisfiable** iff there is a linear time structure $M = (S, x, L)$ such that $M, x \models p$

  M is then a **model** of p.
- A PLTL formula p is **valid**, written $\models p$, iff for all linear time structures $M = (S, x, L)$ we have $M, x \models p$

Note: p is valid iff $\neg\, p$ is not satisfiable.

Examples:

- $p \Rightarrow \mathbf{F} q$
- $\mathbf{G}(p \Rightarrow \mathbf{F} q)$
- $\mathbf{G}(p \Rightarrow \mathbf{F} q) \Rightarrow (p \Rightarrow \mathbf{F} q)$
- $(p \Rightarrow \mathbf{F} q) \Rightarrow \mathbf{G}(p \Rightarrow \mathbf{F} q)$

Important Valid Formulae

$\models \mathbf{G}\neg\, p \Leftrightarrow \neg\, \mathbf{F}p$ $\quad$ $\models \mathbf{G}^{\infty}\neg\, p \Leftrightarrow \neg\, \mathbf{F}^{\infty}p$ $\quad$ $\models \mathbf{X}\neg\, p \Leftrightarrow \neg\, \mathbf{X}p$

$\models \mathbf{F}\neg\, p \Leftrightarrow \neg\, \mathbf{G}p$ $\quad$ $\models \mathbf{F}^{\infty}\neg\, p \Leftrightarrow \neg\, \mathbf{G}^{\infty}p$ $\quad$ $\models ((\neg\, p)\,\mathbf{U}\, q) \Leftrightarrow \neg\, (p\,\mathbf{B}\, q)$

Idempotencies

$\models \mathbf{F}\mathbf{F}p \Leftrightarrow \mathbf{F}p$

$\models \mathbf{G}\mathbf{G}p \Leftrightarrow \mathbf{G}p$

$\models \mathbf{F}^{\infty}\mathbf{F}^{\infty}p \Leftrightarrow \mathbf{F}^{\infty}p$

$\models \mathbf{G}^{\infty}\mathbf{G}^{\infty}p \Leftrightarrow \mathbf{G}^{\infty}p$

Implications

$\models p \Rightarrow \mathbf{F}p$ $\quad$ $\models \mathbf{X}p \Rightarrow \mathbf{F}p$ $\quad$ $\models \mathbf{G}p \Rightarrow \mathbf{F}p$ $\quad$ $\models p\,\mathbf{U}\, q \Rightarrow \mathbf{F}q$

$\models \mathbf{G}p \Rightarrow p$ $\quad$ $\models \mathbf{G}p \Rightarrow \mathbf{X}p$ $\quad$ $\models \mathbf{G}p \Rightarrow \mathbf{X}\mathbf{G}p$ $\quad$ $\models \mathbf{G}^{\infty}q \Rightarrow \mathbf{F}^{\infty}q$

$\models \mathbf{X}\mathbf{F}p \Leftrightarrow \mathbf{F}\mathbf{X}p$ $\quad$ $\models \mathbf{X}\mathbf{G}p \Leftrightarrow \mathbf{G}\mathbf{X}p$ $\quad$ $\models ((\mathbf{X}p)\,\mathbf{U}\,(\mathbf{X}q)) \Leftrightarrow \mathbf{X}(p\,\mathbf{U}\, q)$

$\models \mathbf{F}^{\infty}p \Leftrightarrow \mathbf{X}\mathbf{F}^{\infty}p \Leftrightarrow \mathbf{F}\mathbf{F}^{\infty}p \Leftrightarrow \mathbf{G}\mathbf{F}^{\infty}p \Leftrightarrow \mathbf{F}^{\infty}\mathbf{F}^{\infty}p \Leftrightarrow \mathbf{G}^{\infty}\mathbf{F}^{\infty}p$

$\models \mathbf{G}^{\infty}p \Leftrightarrow \mathbf{X}\mathbf{G}^{\infty}p \Leftrightarrow \mathbf{F}\mathbf{G}^{\infty}p \Leftrightarrow \mathbf{G}\mathbf{G}^{\infty}p \Leftrightarrow \mathbf{F}^{\infty}\mathbf{G}^{\infty}p \Leftrightarrow \mathbf{G}^{\infty}\mathbf{G}^{\infty}p$

Interplay between Junctors and Temporal Operators

$\models \mathbf{F}(p \vee q) \Leftrightarrow (\mathbf{F}p \vee \mathbf{F}q)$

$\models \mathbf{F}^{\infty}(p \vee q) \Leftrightarrow (\mathbf{F}^{\infty}p \vee \mathbf{F}^{\infty}q)$

$\models p\,\mathbf{U}\,(q \vee r) \Leftrightarrow (p\,\mathbf{U}\, q \vee p\,\mathbf{U}\, r)$

$\models \mathbf{G}(p \wedge q) \Leftrightarrow (\mathbf{G}p \wedge \mathbf{G}q)$

$\models \mathbf{G}^{\infty}(p \wedge q) \Leftrightarrow (\mathbf{G}^{\infty}p \wedge \mathbf{G}^{\infty}q)$

$\models (p \wedge q)\,\mathbf{U}\, r \Leftrightarrow (p\,\mathbf{U}\, r \wedge q\,\mathbf{U}\, r)$

$\models \mathbf{X}(p \vee q) \Leftrightarrow (\mathbf{X}p \vee \mathbf{X}q)$

$\models \mathbf{X}(p \wedge q) \Leftrightarrow (\mathbf{X}p \wedge \mathbf{X}q)$

$\models \mathbf{X}(p \Rightarrow q) \Leftrightarrow (\mathbf{X}p \Rightarrow \mathbf{X}q)$

$\models \mathbf{X}(p \Leftrightarrow q) \Leftrightarrow (\mathbf{X}p \Leftrightarrow \mathbf{X}q)$

$\models (\mathbf{G}p \vee \mathbf{G}q) \Rightarrow \mathbf{G}(p \vee q)$

$\models (\mathbf{G}^{\infty}p \vee \mathbf{G}^{\infty}q) \Rightarrow \mathbf{G}^{\infty}(p \vee q)$

$\models ((p\,\mathbf{U}\, r) \vee (q\,\mathbf{U}\, r)) \Rightarrow ((p \vee q)\,\mathbf{U}\, r)$

$\models \mathbf{F}(p \wedge q) \Rightarrow \mathbf{F}p \wedge \mathbf{F}q$

$\models \mathbf{F}^{\infty}(p \wedge q) \Rightarrow \mathbf{F}^{\infty}p \wedge \mathbf{F}^{\infty}q$

$\models (p\,\mathbf{U}\,(q \wedge r)) \Rightarrow ((p\,\mathbf{U}\, q) \wedge (p\,\mathbf{U}\, r))$

Monotony and Fixpoint Characterisations

$\models \mathbf{G}(p \Rightarrow q) \Rightarrow (\mathbf{F}p \Rightarrow \mathbf{F}q)$ $\quad$ $\models \mathbf{G}(p \Rightarrow q) \Rightarrow (\mathbf{F}^\infty p \Rightarrow \mathbf{F}^\infty q)$

$\models \mathbf{G}(p \Rightarrow q) \Rightarrow (\mathbf{G}p \Rightarrow \mathbf{G}q)$ $\quad$ $\models \mathbf{G}(p \Rightarrow q) \Rightarrow (\mathbf{G}^\infty p \Rightarrow \mathbf{G}^\infty q)$

$\models \mathbf{G}(p \Rightarrow q) \Rightarrow ((p\,\mathbf{U}\,r) \Rightarrow (q\,\mathbf{U}\,r))$ $\quad$ $\models \mathbf{G}(p \Rightarrow q) \Rightarrow ((r\,\mathbf{U}\,p) \Rightarrow (r\,\mathbf{U}\,q))$

$\models \mathbf{G}(p \Rightarrow q) \Rightarrow (\mathbf{X}p \Rightarrow \mathbf{X}q)$

Fixpoint Characterisations:

$\models \mathbf{F}p \Leftrightarrow p \vee \mathbf{X}\mathbf{F}p$ $\quad$ $\models (p\,\mathbf{U}\,q) \Leftrightarrow q \vee (p \wedge \mathbf{X}(p\,\mathbf{U}\,q))$

$\models \mathbf{G}p \Leftrightarrow p \wedge \mathbf{X}\mathbf{G}p$ $\quad$ $\models (p\,\mathbf{B}\,q) \Leftrightarrow \neg\, q \wedge (p \vee \mathbf{X}(p\,\mathbf{B}\,q))$

} Exercise!

Variants of the Basic Temporal Operators

- $p\,\mathbf{U}\,q$, until now, is known as "*strong until*":

  There is a future state q, and until then p.

- Alternative notations: $p\,\mathbf{U}_s\,q$ or $p\,\mathbf{U}_\exists\,q$.

- *Weak until* $p\,\mathbf{U}_w\,q$ or $p\,\mathbf{U}_\forall\,q$:

  p holds as long as q does not hold — if necessary, forever.

- $x \models p\,\mathbf{U}_\forall q$ iff for all $j : \mathbb{N}$ we have $x^j \models p$ as far as for all $k \leq j$ we have $x^k \models \neg\, q$.

We have:

- $\models p\,\mathbf{U}_\exists q \Leftrightarrow p\,\mathbf{U}_\forall q \wedge \mathbf{F}q$

- $\models p\,\mathbf{U}_\forall q \Leftrightarrow (p\,\mathbf{U}_\exists q \vee \mathbf{G}p) \Leftrightarrow (p\,\mathbf{U}_\exists q \vee \mathbf{G}(p \wedge \neg\, q))$

Variants of the Basic Temporal Operators (ctd.)

Does the future contain the present? — Variants:

- $\mathbf{F}^{\geq} p$: eventually, now or in the future, p — this is $\mathbf{F} p$ from before
- $\mathbf{F}^{>} p$: eventually *after now* p.

We have:

- $\models \mathbf{F}^{>} p \Leftrightarrow \mathbf{X}\mathbf{F}^{\geq} p$
- $\models \mathbf{F}^{\geq} p \Leftrightarrow p \vee \mathbf{F}^{>} p$

Analogously: $p \,\mathbf{U}^{>} q \Leftrightarrow \mathbf{X}(p \,\mathbf{U}\, q)$.

Since $\mathit{false}\,\mathbf{U}^{>} q \Leftrightarrow \mathbf{X}(\mathit{false}\,\mathbf{U}\, q) \Leftrightarrow \mathbf{X} q$, all other basic temporal operators can be generated from *the single* operator $\mathbf{U}^{>}$.

Past

Until now, all operators are future-related — explicitly:

- $\mathbf{F}^{+} p$ — "in the future, eventually p"
- $\mathbf{G}^{+} p$ — "in the future, always p"
- $\mathbf{X}^{+} p$ — "in the next state p"
- $p \,\mathbf{U}^{+} q$ — "in the future, eventually q, and until then p"

Purely future-oriented propositional linear-time temporal logic — *Propositional Linear-time Temporal Logic / Future*: PLTLF

Corresponding past-oriented operators (originally P, H, and S for *since*):

- $\mathbf{F}^{-} p$ — "in the past at some point p"
- $\mathbf{G}^{-} p$ — "in the past, always p"
- $\mathbf{X}^{-}_{\exists} p$ — "in the previous state we had p"
- $p \,\mathbf{U}^{-} q$ — "in the past at some pint q, and since then p"

Logic only with past-oriented operators: PLTLP; with both: PLTLB.

Semantics of PLTLB

Idea: We keep a $\mathbb{N}$-based time line x, but instead of x^i we now consider (x, i), so that we can refer also to states before i.

Then the old definitions become:

- $(x, i) \models (p\, \mathbf{U}\, q)$ iff there is a $j \geq i$ sich that $(x, j) \models q$ and for all k with $i \leq k < j$ we have $(x, k) \models p$
- $(x, i) \models \mathbf{X}p$ iff $(x, i + 1) \models p$
- $(x, i) \models \mathbf{F}q$ iff there is a $j \geq i$ such that $(x, j) \models q$
- $(x, i) \models \mathbf{G}q$ iff for all $j \geq i$ we have $(x, j) \models q$

For the new operators:

- $(x, i) \models (p\, \mathbf{U}^- q)$ iff there is a $j \leq i$ such that $(x, j) \models q$ and for all k with $j < k \leq i$ we have $(x, k) \models p$
- $(x, i) \models \mathbf{X}^- p$ iff $i > 0$ and $(x, i - 1) \models p$

Additional Expressivity of the Past Operators

Definition:

- Two formulae p and q are **globally equivalent**, $p \equiv_g q$, iff for all linear time structures x und for all times $i : \mathbb{N}$ we have $(x, i) \models p$ iff $(x, i) \models q$.
- Two formulae p and q are **initially equivalent**, $p \equiv_i q$, iff for all linear time structures x we have $(x, 0) \models p$ iff $(x, 0) \models q$.

Theorem: With respect to global equivalence, PLTLB is more expressive than PLTLF.

Proof: Consider two time structures x and y, that differ only in that $(x, 0) \models Q$ and $(y, 0) \models \neg\, Q$.

Then $(x, 1) \models \mathbf{F}^- Q$ and $(y, 1) \not\models \mathbf{F}^- Q$.

But since $x^1 = y^1$, PLTLF formulae cannot distinguish $(x, 1)$ and $(y, 1)$. □

Theorem: With respect to initial equivalence, PLTLB has the same expressivity as PLTLF.

Theorem: $p \equiv_g q$ iff $\mathbf{G}p \equiv_i \mathbf{G}q$.

Safety

- Safety properties: "nothing bad happens"
- Invariance properties: every finite prefix of the execution satisfies the invariance condition
- in PLTLB: initially equivalent to $\mathbf{G}p$ for a past formula p: "nothing bad has happened until now" must always be true.
- Every formula constructed from past operators, $\wedge$, $\vee$, $\mathbf{G}$ and $\mathbf{U}_w$ is a safety property, e.g.:

$$(p\,\mathbf{U}_w q) \equiv_i \mathbf{G}(\mathbf{G}^- p \vee \mathbf{F}^-(q \wedge \mathbf{X}^-\mathbf{G}^- p))$$ Exercise!

Safety Examples

- *Partial correctness* wrt. precondition ϕ ans postcondition ψ:

  If a program (with start label l_0 and halting label l_h) starts executing in a state satisfying the precondition ϕ and terminates, the the terminating state satisfies the postcondition ψ:

$$\mathrm{at}l_0 \wedge \phi \Rightarrow \mathbf{G}(\mathrm{at}l_h \Rightarrow \psi)$$

  This is initially equivalent to:

$$\mathbf{G}(\mathbf{F}^-(\neg\,(\mathrm{at}l_0 \wedge \phi) \wedge \mathbf{X}^-_w \mathit{false}) \vee \mathbf{G}(\mathrm{at}l_h \Rightarrow \psi))$$

  and therefore a safety property.

- *Mutual Exclusion*: $\mathbf{G}(\neg\,(\mathrm{atCS}_1 \wedge \mathrm{atCS}_2))$
- *Deadlock-freeness*: $\mathbf{G}(\mathrm{enabled}_1 \vee \ldots \vee \mathrm{enabled}_m)$

Liveness

- **Liveness**: "Something good will still happen (often enough)"
- p is an "*invincible*" past formula iff every finite sequence x has a finite extension x' such that p holds in the last state of x':
$$(x', length(x')) \models p$$
- A **pure liveness property** is a PLTLB formula that is initially equivalent to a formula $\mathbf{F}p$, $\mathbf{GF}p$ or $\mathbf{FG}p$, where p is an invincible past formula
- If p is a pure liveness property, then every finite sequence x can be extended to a finite or infinite sequence x' such that $(x', 0) \models p$
- **Temporal implication** $\mathbf{G}(p \Rightarrow \mathbf{F}q)$ (where p and q are past formulae) is a generic liveness property

ω-regular Languages — Deterministic Finite Automata

A **deterministic finite automaton** (S, s_0, F, δ) consists of

- state set S,
- starting state $s_0 \in S$,
- accepting state set $f \subseteq S$,
- transition **function** $\delta : S \times \Sigma \to S$.

An **infinite word** is an infinite sequence of letters $w : \mathbb{N} \to \Sigma$.

The automaton (S, s_0, f, δ) **accepts** the infinite word w iff the state sequence induced by w contains accepting states infinitely often.

"Büchi automaton"

Acceptance in PLTL: Define $M_w = (S, x, L)$ where x is the state sequence induced by w in the automaton, and $f \in L(s)$ iff $s \in f$.

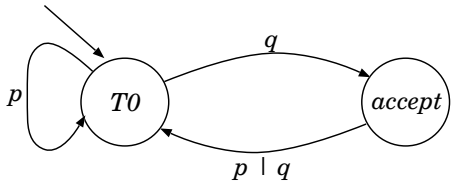
Then acceptance is the proposition $\mathbf{GF}f$ — a pure liveness property.

Büchi Automata and PLTL Formulae

Theorem 1 (Vardi, Wolper, 1983): Every PLTL formula can be translated into a Büchi automaton.

These automata accept exactly those infinite words that, when considered as time structures, satisfy the corresponding PLTL formula.

$\mathbf{G}(p\,\mathbf{U}\,q)$



PLTL Model Checking (e.g. in SPIN):

- Implementations are given as automata, too.
- Calculate intersection of implementation and negated specification as product automaton.