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COMP SCI 1FC3
[Gries & Schneider]
Theorem List 1
"Semi-Blank"

COMP SCI 1FC3 — Mathematics for Computing

6 February 2012

Substitution and Equality

(1.1) **Inference rule Substitution:**

(1.2) **Axiom, Reflexivity of =:**

(1.3) **Axiom, Symmetry of =:**

(1.4) **Inference rule Transitivity of =:**

(1.5) **Inference rule Leibniz:**

Equivalence

(3.1) **Axiom, Associativity of ≡:**

(3.2) **Axiom, Symmetry of ≡:**

(3.3) **Axiom, Identity of ≡:**

(3.4) *true*

(3.5) **Reflexivity of ≡:**

(3.7) **Metatheorem:** Any two theorems are equivalent.

Negation

(3.8) **Axiom, Definition of *false*:**

(3.9) **Axiom, Distributivity of ¬ over ≡:**

(3.10) **Axiom, Definition of ≢:**

(3.11) $\neg p \equiv q \equiv p \equiv \neg q$

(3.12) **Double negation:**

(3.13) **Negation of *false*:**

(3.14) $(p \not\equiv q) \ \equiv$

(3.15) $\neg p \equiv$

(3.16) **Symmetry of ≢:**

(3.17) **Associativity of ≢:**

(3.18) **Mutual associativity:**

(3.19) **Mutual interchangeability:**

Disjunction

(3.24) **Axiom, Symmetry of ∨:**

(3.25) **Axiom, Associativity of ∨:**

(3.26) **Axiom, Idempotency of ∨:**

(3.27) **Axiom, Distributivity of ∨ over ≡:**

(3.28) **Axiom, Excluded Middle:**

(3.29) **Zero of ∨:**

(3.30) **Identity of ∨:**

(3.31) **Distributivity of ∨ over ∨:**

(3.32) $p \vee q \ \equiv$

Conjunction

(3.35) **Axiom, Golden rule:**

(3.36) **Symmetry of ∧:**

(3.37) **Associativity of ∧:**

(3.38) **Idempotency of ∧:**

(3.39) **Identity of ∧:**

(3.40) **Zero of ∧:**

(3.41) **Distributivity of ∧ over ∧:**

(3.42) **Contradiction:**

(3.43) **Absorption:**

(3.44) **Absorption:**

(3.45) **Distributivity of ∨ over ∧:**

(3.46) **Distributivity of ∧ over ∨:**

(3.47) **De Morgan:**

(3.48) $p \wedge q \quad \equiv$

(3.49) $p \wedge (q \equiv r) \quad \equiv$

(3.50) $p \wedge (q \equiv p) \quad \equiv$

(3.51) **Replacement:** $(p \equiv q) \wedge (r \equiv p) \quad \equiv$

(3.52) **Definition of ≡:**

(3.53) **Definition of ≢ (Exclusive or):**

Implication

(3.57) **Axiom, Definition of Implication:**

(3.58) **Axiom, Consequence:**

(3.59) **(Alternative) Definition of Implication:**

(3.60) **(Dual) Definition of Implication:**

(3.61) **Contrapositive:**

Ex. 3.45 $p \Rightarrow q \quad \equiv$

Ex. 3.46 $p \Rightarrow q \quad \equiv$

(3.62) $p \Rightarrow (q \equiv r) \quad \equiv$

(3.63) **Distributivity of ⇒ over ≡:**

(3.64) **Self-distributivity of ⇒:**

(3.65) **Shunting:**

(3.66) $p \wedge (p \Rightarrow q) \quad \equiv$

(3.67) $p \wedge (q \Rightarrow p) \quad \equiv$

(3.68) $p \vee (p \Rightarrow q) \quad \equiv$

(3.69) $p \vee (q \Rightarrow p) \quad \equiv$

(3.70) $p \vee q \Rightarrow p \wedge q \quad \equiv$

(3.71) **Reflexivity of ⇒:**

(3.72) **Right zero of $\Rightarrow$:**

(3.73) **Left identity of $\Rightarrow$:**

(3.74) $p \Rightarrow false \quad \equiv$

(3.75) $false \Rightarrow$

(3.76a) **Weakening**/strengthening: $p \vee q$

(3.76b) **Weakening**/strengthening: $p \wedge q$

(3.76c) **Weakening**/strengthening: $p \wedge q \Rightarrow p \vee q$

(3.76d) **Weakening**/strengthening: $p \vee (q \wedge r)$

(3.76e) **Weakening**/strengthening: $p \wedge (q \vee r)$

(3.77) **Modus ponens:**

(3.78) **Case analysis:**

(3.79) **Case analysis:**

(3.80) **Mutual implication:**

(3.80b) **Reflexivity wrt. Equivalence:** — (not in the textbook)

(3.81) **Antisymmetry:**

(3.82a) **Transitivity:**

(3.82b) **Transitivity:**

(3.82c) **Transitivity:**

(4.1) $p \Rightarrow (q \Rightarrow p)$

(4.2) **Monotonicity of $\vee$:**

(4.3) **Monotonicity of $\wedge$:**

(4.4) **(Extended) Deduction Theorem:** Suppose adding $P_1, \ldots, P_n$ as axioms to propositional logic **E**, with the variables of the P_i considered to be constants, allows Q to be proved.
Then $P_1 \wedge \ldots \wedge P_n \Rightarrow Q$ is a theorem.

(4.6) $(p \vee q \vee r) \wedge (p \Rightarrow s) \wedge (q \Rightarrow s) \wedge (r \Rightarrow s) \;\Rightarrow\; s$

Integers

(15.1) **Axiom, Associativity:**

(15.2) **Axiom, Symmetry:**

(15.3) **Axiom, Additive identity:**

(15.4) **Axiom, Multiplicative identity:**

(15.5) **Axiom, Distributivity:**

(15.7) **Cancellation of $\cdot$:**

(15.8) **Cancellation of $+$:**

(15.13) **Axiom, Unary minus:**

(15.14) **Axiom, Subtraction:**

(15.9) **Zero of Multiplication:**

(15.10a) $a + z = a \quad \equiv \quad z = 0$

(15.10b) $a \neq 0 \quad \Rightarrow \quad (a \cdot z = a \quad \equiv \quad z = 1)$

(15.12) $x + a = 0 \wedge y + a = 0 \quad \Rightarrow \quad x = y$

(15.15) $x + a = 0 \quad \equiv \quad x = -a$

(15.16) $-a = -b \quad \equiv \quad a = b$

(15.17) $-(-a) = a$

(15.18) $-0 = 0$

(15.19) $-(a + b) = -a + -b$

(15.20) $-a = -1 \cdot a$

(15.21) $(-a) \cdot b = a \cdot (-b)$

(15.22) $a \cdot (-b) = -(a \cdot b)$

(15.23) $(-a) \cdot (-b) = a \cdot b$

(15.24) $a - 0 = a$

(15.25) $(a - b) + (c - d) = (a + c) - (b + d)$

(15.25a) **Mutual associativity of $+$ and $-$:**

(15.25b) $a - (b + c) = (a - b) - c$

(15.26) $(a - b) - (c - d) = (a + d) - (b + c)$

(15.27) $(a - b) \cdot (c - d) = (a \cdot c + b \cdot d) - (a \cdot d + b \cdot c)$

(15.28) $a - b = c - d \quad \equiv \quad a + d = b + c$

(15.29) $(a - b) \cdot c = a \cdot c - b \cdot c$

(15.30) **Axiom, Addition in pos:** $\mathsf{pos}.a \wedge \mathsf{pos}.b \;\Rightarrow\; \mathsf{pos}(a + b)$

(15.31) **Axiom, Multiplication in pos:** $\mathsf{pos}.a \wedge \mathsf{pos}.b \;\Rightarrow\; \mathsf{pos}(a \cdot b)$

(15.32) **Axiom:** $\neg\mathsf{pos}.0$

(15.33) **Axiom:** $b \neq 0 \quad \Rightarrow \quad (\mathsf{pos}.b \;\equiv\; \neg\mathsf{pos}(-b))$

(15.34) $b \neq 0 \quad \Rightarrow \quad \mathsf{pos}(b \cdot b)$

(15.35) $\mathsf{pos}.a \quad \Rightarrow \quad (\mathsf{pos}.b \quad \equiv \quad \mathsf{pos}(a \cdot b)$

(15.36) **Axiom, Less:** $a < b \quad \equiv \quad \mathsf{pos}(b - a)$

(15.37) **Axiom, Greater:** $a > b \quad \equiv \quad \mathsf{pos}(a - b)$

(15.38) **Axiom, At most:** $a \leq b \quad \equiv \quad a < b \;\vee\; a = b$

(15.39) **Axiom, At least:** $a \geq b \quad \equiv \quad a > b \;\vee\; a = b$

(15.40) **Positive elements:** $\mathsf{pos}.b \quad \equiv \quad 0 < b$

(15.41) **Transitivity:**
(a) $a < b \;\wedge\; b < c \;\Rightarrow\; a < c$
(b) $a \leq b \;\wedge\; b < c \;\Rightarrow\; a < c$
(c) $a < b \;\wedge\; b \leq c \;\Rightarrow\; a < c$
(d) $a \leq b \;\wedge\; b \leq c \;\Rightarrow\; a \leq c$

(15.42) **Monotonicity of $+$:** $a < b \;\equiv\; a + d < b + d$

(15.43) **Monotonicity of $\cdot$:** $0 < d \Rightarrow (a < b \;\equiv\; a \cdot d < b \cdot d)$

(15.44) **Trichotomy:** $(a < b \quad \equiv \quad a = b \quad \equiv \quad a > b) \wedge$
$\neg(a < b \quad \wedge \quad a = b \quad \wedge \quad a > b)$

(15.45) **Antisymmetry of $\leq$:** $a \leq b \quad \wedge \quad a \geq b \quad \equiv \quad a = b$

(15.46) **Reflexivity of $\leq$:** $a \leq a$

(15.47) $a = b \quad \equiv \quad (\forall z \;\bullet\; z \leq a \quad \equiv \quad z \leq b)$