Equivalence, Negation and Inequivalence

“Definition of ≡”: (p ≡ q) = (p = q)

(3.2) “Symmetry of ≡”: p ≡ q ≡ q ≡ p

(3.3) “Identity of ≡”: true ≡ q ≡ q

(3.5) “Reflexivity of ≡”: p ≡ p

(3.9) “Commutativity of ¬ with ≡” “Distributivity of ¬ over ≡”: ¬ (p ≡ q) ≡ (¬ p ≡ q)

(3.11) “¬ connection”: ¬ p ≡ q ≡ p ≡ ¬ q

(3.14): (p ≡ q) ≡ (¬ p ≡ q)

(3.15): ¬ p ≡ (p ≡ false)

Disjunction and Conjunction

(3.32): p ∨ q ≡ (p ∨ ¬ q) ≡ q

(3.35): “Golden rule”: p ∧ q ≡ p = q ≡ p ∨ q

(3.48): p ∧ q ≡ (p ∧ ¬ q) ≡ ¬ p

(3.49): “Semi-distributivity of ∧ over ≡”: p ∧ (q ≡ r) ≡ (p ∧ q) ≡ (p ∧ r) ≡ p

(3.50): “Strong Modus Ponens”: p ∧ (q ≡ p) ≡ p ∧ q

(3.51): “Replacement”: (p ≡ q) ∧ (r ≡ p) ≡ (p ≡ q) ∧ (r ≡ q)

(3.52): “Alternative definition of ≡”: p ≡ (q ≡ (p ∧ q) ∨ (¬ p ∧ ¬ q))

(3.53): “Exclusive or” “Alternative definition of ̂”: (p ≡ q) ≡ (¬ p ∧ q) ∨ (p ∧ ¬ q)

Impliation

(3.57): “Definition of ⇒”: p ⇒ q ≡ (p ∨ q) ≡ (p ∨ q) ≡ p

(3.58): “Definition of ≣” “Consequence” “⇐”: p = q ≡ q = p

(3.59): “Definition of ⇒”: p ⇒ q = ¬ p ⇒ q

(3.60): “Definition of ≣” “≡”: p = q = (p ∧ q) = (p ∧ q)

(3.61): “Contrapositive”: p ⇒ q = q ⇒ ¬ p

(3.62): p ⇒ (q ≡ r) = (p ∧ q ∨ ¬ (p ∧ ¬ p))

(3.63): “Distributivity of ⇒ over ≡”: p ⇒ (q ≡ r) = (p ⇒ q) ≡ (p ⇒ r)

(3.64): “Self-distributivity of ⇒”: p ⇒ (q ⇒ r) = (p ⇒ q) ⇒ (p ⇒ r)

(3.65): “Shunting”: p ∧ q ⇒ r ≡ p ⇒ (q ⇒ r)

(3.66): p ∧ (p ⇒ q) = p ∧ q

(3.67): p ∧ (q ⇒ p) = p

(3.68): p ∨ (p ⇒ q) = true

(3.69): p ∨ (q ⇒ p) = q ⇒ p

(3.70): p ∨ q ⇒ p ∧ q = (p ∨ q) = (p ∨ q)

(3.71): “Reflexivity of ⇒”: p ⇒ p

(3.72): “Right-zero of ⇒”: p = true

(3.73): “Left-identity of ⇒”: true = p ⇒ p

“Definition of ¬” (3.74): p = false ≡ ¬ p

(3.75): “ex falso quodlibet”: false ⇒ p

(3.76a): “Weakening”: p ⇒ p ⇒ q

(3.76b): “Weakening”: p ⇒ p ⇒ q

(3.76c): “Weakening”: p ∨ q = p

(3.76d): “Weakening”: p ∨ (q ∧ r) = p ∨ q

(3.76e): “Weakening”: p = q = (p ∧ q)

“Reflexivity of ⇒”: (p ≡ q) = (p = q)

(3.77): “Modus ponens”: p ∧ (p ⇒ q) = q

(3.78): “Case analysis”: (p =⇒ r) ∧ (q =⇒ r) = p ∨ q =⇒ r

(3.79): “Case analysis”: (p =⇒ r) ∧ (¬ p =⇒ r) = r

(3.80): “Mutual implication”: (p =⇒ q) ∧ (q =⇒ p) = (p ≡ q)

(3.81): “Antisymmetry of ⇒”: (p =⇒ q) ∧ (q =⇒ p) = (p ≡ q)

(3.82a): “Transitivity of ⇒”: (p =⇒ q) ∧ (q =⇒ r) = (p =⇒ r)

(3.82b): “Transitivity of ⇒”: (p ≡ q) ∧ (q =⇒ r) = (p =⇒ r)

(3.82c): “Transitivity of ⇒”: (p =⇒ q) ∧ (q =⇒ r) = (p =⇒ r)

“Implication strengthening”: p ⇒ q =⇒ p ⇒ p ∧ q

Leibniz as Axiom and Substitution/Replacement Laws

(3.83): “Leibniz” e = f =⇒ E[z := e] = E[z := f]

(3.84a): “Replacement” e = f =⇒ E[z := e] = E[z := f]

(3.84b): “Replacement” e = f =⇒ E[z := e] = e = f =⇒ E[z := f]

(3.84c): “Replacement” q ∧ e =⇒ E[z := e] = q ∧ e =⇒ E[z := f]

“Transitivity of =”: e = f ∧ f =⇒ E[z := e] = e = g


(3.85b): “Replace by ‘true’:” q ∧ p =⇒ E[z := p] = q ∧ p =⇒ E[z := true]

(3.85c): “Replace by ‘false’:” ¬ p =⇒ E[z := p] = ¬ p =⇒ E[z := false]

(3.85e): “Replace by ‘true’:” p =⇒ E[z := p] = E[z := true]


(3.87): “Replace by ‘true’:” p ∧ E[z := p] = p ∧ E[z := true]

(3.88): “Replace by ‘false’:” p ∨ E[z := p] = p ∨ E[z := false]

Monotonicity with Respect to Implication

(4.1) “Left-monotonicity of ⇀” “Monotonicity of ⇀”: (p =⇒ q) = (p ⇒ r =⇒ q ⇒ r)

“Monotonicity of ⇀”: (p =⇒ q) = (r =⇒ q) =⇒ (p =⇒ q)

(4.3): “Left-monotonicity of ∧” “Monotonicity of ∧”: (p =⇒ q) = (q ∧ r) =⇒ (p =⇒ q ∧ p)

“Antitonicity of ¬”: (p =⇒ q) = (¬ q =⇒ ¬ p)

“Monotonicity of ⇒” “Right-monotonicity of ⇒”: (p =⇒ q) = (q ⇒ r) =⇒ (r =⇒ q)

“Antitonicity of ⇒” “Left-antitonicity of ⇒”: (p ⇒ q) = (q ⇒ r) ⇒ (p ⇒ r)

Sum Quantification: General Quantifier Material Instantiated for Sum

“Leibniz for ∑ range”: (∀ x ∗ R1 ⊃ R2 x) = (∑ x | R1 ∗ E1 ) = (∑ x | R2 ∗ E2 )

“Leibniz for ∑ body”: (∀ x ∗ R =⇒ E11 ) = (∑ x | R ∗ E1 ) = (∑ x | R ∗ E2 )

(8.13): “Empty range for ∑”: (∑ x | false ∗ E ) = 0

(8.14): “One-point rule for ∑”: (∑ x | x = D ∗ E ) = E[x := D] — provided: ¬occurs(x, ‘D’)

(8.15): “Distributivity of ∑ over +”: (∑ x | R ∗ E1 + E2 ) = (∑ x | R ∗ E1 ) + (∑ x | R ∗ E2 )

(8.17): “Range split”: (∑ x | Q ∨ R ∗ E ) = (∑ x | Q ∨ R ∗ E ) + (∑ x | Q ∧ R ∗ E )

(8.16): “Disjoint range split for ∑”: (∀ x ∗ Q ∧ R =⇒ false ) = (∑ x | Q ∨ R ∗ E ) = (∑ x | Q ∨ R ∗ E ) + (∑ x | Q ∗ E )

(8.20): “Nesting for ∑”: (∑ x | O ∗ (Q ∗ y ∗ R ∗ E ) = (∑ x | y ∗ Q ∨ R ∗ E ) — provided: ¬occurs(y, ‘Q’)

“Replacement in ∑”: (∑ x | R ∨ E ) = E[y := f] = (∑ x | R ∨ E ) = E[y := f] = (∑ x | R ∨ E )

“Dummy list permutation for ∑”: (∑ x | y ∗ R ∗ E ) = (∑ x | y ∗ R ∗ E )

(8.19): “Interchange of dummies”: (∑ x | Q ∗ (Q ∗ y ∗ R ∗ P ) = (∑ x | Q ∗ (R ∗ y ∗ P ) = (∑ x | Q ∗ P ) = (∑ x | Q ∗ P ) = (∑ x | Q ∗ P ) — provided: ¬occurs(x, ‘R’), ¬occurs(y, ‘Q’)

Specific Material for Sum Quantification

“Distributivity of over \(\sum\)”: \(a \cdot (\sum x \in R \cdot E) = (\sum x \in R \cdot a \cdot E)\) — provided: \(\neg\text{occurs}(x', a)\)

“Zero \(\sum\) body”: \((\sum x \in R \cdot 0) = 0\)

“Definition of \(\leq\) in terms of \(<\)”: \(a \leq b \equiv a < b \lor a = b\)

“Definition of \(<\) in terms of ‘\(S\)’ and \(<\)”: \(a < b \equiv a < S b\)

“Split range at top”: \(m \leq n \Rightarrow (m \leq i < S n \equiv m \leq i < n \lor i = n)\)

“Split off term at top”: \((\sum i : N \mid i < S n \cdot E) = (\sum i : N \mid i < n \cdot E) + E[i := n]\) — provided: \(\neg\text{occurs}(i', n)\)

“Split off term at top”: \(m \leq n \Rightarrow (\sum i : m \leq i < S n \cdot E) = (\sum i : m \leq i < n \cdot E) + E[i := n]\) — provided: \(\neg\text{occurs}(i', m, n)\)

“Split off term at top using \(\leq\)”: \((\sum i : i \leq S n \cdot E) = (\sum i : i \leq n \cdot E) + E[i := n]\)

Universal Quantification

“Leibniz for \(\forall\) body”: \((\forall x \in R \cdot P) \equiv (\forall x \in R \cdot P)\)

(8.18) “Range split for \(\forall\)”: \((\forall x \in R \lor S \cdot P) \equiv (\forall x \in R \lor P) \lor (\forall x \in S \cdot P)\)

(9.5) “Distributivity of \(\forall\) over \(\lor\)”: \(\forall (\forall x \in R \cdot Q) \equiv (\forall x \in R \cdot P \lor Q)\) — provided: \(\neg\text{occurs}(x', P)\)

(9.6): \(P \lor (\forall x \in R \cdot \neg R) \equiv (\forall x \in R \cdot P)\)

“Distributivity of \(\forall\) over \(\forall\)”: \(\forall (\forall x \in R \cdot Q) \equiv (\forall x \in R \cdot P \lor Q)\) — provided: \(\neg\text{occurs}(x', P)\)

(9.7) “Distributivity of \(\forall\) over \(\land\)”: \(\forall (\forall x \in R \cdot P) \land (\forall x \in R \cdot Q) \equiv (\forall x \in R \cdot P \land Q)\)

(9.8) “True \(\forall\) body”: \((\forall x \in R \cdot \text{true})\)

“Introducing fresh \(\forall\)”: \(P \Rightarrow (\forall x \in R \cdot P)\) — provided: \(\neg\text{occurs}(x', P)\)

(9.9) “Sub-distributivity of \(\forall\) over \(\equiv\)”: \((\forall x \in R \cdot P \equiv Q) \Rightarrow (\forall x \in R \cdot P) \lor (\forall x \in R \cdot Q)\)

(10.10) “Range weakening for \(\forall\)”: \((\forall x \in R \cdot Q \lor R) \Rightarrow (\forall x \in R \cdot Q)\)

(11.9) “Body weakening for \(\forall\)”: \((\forall x \in R \cdot P \lor Q) \Rightarrow (\forall x \in R \cdot P)\)

(11.12) “Body monotonicity of \(\forall\)”: \((\forall x \in R \cdot Q \Rightarrow P) \Rightarrow (\forall x \in R \cdot Q) \Rightarrow (\forall x \in R \cdot P)\)

(11.12a) “Range monotonicity of \(\forall\)”: \((\forall x \in R \cdot Q \Rightarrow R) \Rightarrow (\forall x \in R \cdot Q) \Rightarrow (\forall x \in R \cdot P)\)

(11.13) “Instantiation”: \((\forall x \in R \cdot P) \Rightarrow P[x := E]\)

“Fresh \(\forall\)”: \(P \equiv (\forall x \in R \cdot P)\) — provided: \(\neg\text{occurs}(x', P)\)

Existential Quantification

(9.21) “Distributivity of \(\exists\) over \(\forall\)”: \(P \land (\exists x \in R \cdot Q) \equiv (\exists x \in R \cdot P \land Q)\)

(9.22): \(P \land (\exists x \in R \cdot Q) \equiv (\exists x \in R \cdot P)\)

“Distributivity of \(\exists\) over \(\forall\)”: \((\exists x \in R \cdot Q) \Rightarrow (P \land (\forall x \in R \cdot Q) \equiv (\forall x \in R \cdot P \land Q))\)

(9.23) “Distributivity of \(\forall\) over \(\exists\)”: \((\forall x \in R \cdot P \lor (\exists x \in R \cdot Q) \equiv (\exists x \in R \cdot P \lor Q)\)

(9.24) “False \(\exists\) body”: \((\exists x \in R \cdot P) \equiv \text{false} \equiv \text{false}\)

(9.25) “Range weakening for \(\exists\)”: \((\exists x \in R \cdot P) \Rightarrow (\exists x \in R \cdot Q \lor P \land R)\)

“Range weakening for \(\exists\)”: \((\exists x \in Q \land R \cdot P) \Rightarrow (\exists x \in R \cdot P)\)

(9.26) “Body weakening for \(\exists\)”: \((\exists x \in R \cdot P) \Rightarrow (\exists x \in R \cdot P \lor Q)\)

(9.26a) “Body weakening for \(\exists\)”: \((\exists x \in R \cdot P \lor Q) \Rightarrow (\exists x \in R \cdot P)\)

(9.27) “Body monotonicity of \(\exists\)”: \((\exists x \in R \cdot Q \Rightarrow R) \Rightarrow (\exists x \in Q \cdot R) \Rightarrow (\exists x \in R \cdot P)\)

“(Range monotonicity of \(\exists\)”: \((\forall x \in R \cdot P \Rightarrow R) \Rightarrow ((\exists x \in Q \cdot P) \Rightarrow (\exists x \in R \cdot P))\)

Introdction and Interchange for \(\exists\)

(9.28) “3-Introduction”: \(P[x := E] \Rightarrow (\exists x \in P)\)

(9.29a) “Interchange of quantifications”: \((\exists x \in (\forall y \in P) \Rightarrow (\forall y \in (\exists x \in P))\)

(9.30a) “Witness”: \((\exists x \in R \cdot P) \equiv Q \equiv (\forall x \in R \land P) \equiv Q\)

(9.30b) “Witness”: \((\exists x \in P) \equiv Q \equiv (\forall x \in P) \equiv Q\)

Set Theory

(11.3) “Set membership”: \(F \in \{ x \in R \cdot E \} \equiv (\exists x \in R \cdot F = E)\)

(11.7s) “Simple Membership”: \(e \in \{ x \in P \} \equiv P[x := e]\)

(11.7x) “Simple Membership”: \(x \in \{ x \in P \} \equiv P\)

(11.7y) “Simple Membership”: \((\forall x \in x \in \{ x \in P \}) \equiv P\)

“Membership in two-element set enumeration”:

“Membership in set enumeration”:

“Set Extensionality and Set Inclusion

(11.4) “Set extensionality” “Set equality” “Extensionality”: \(S = T \equiv (\forall e \in S \equiv e \in T)\)

(11.9) “Simple set comprehension equality”:

(11.13) “Subset” “Definition of \(\subseteq\)” “Set inclusion”: \(S \subseteq T \equiv (\forall e \in e \in S) \equiv e \in T)\)

“Subset” “Definition of \(\subseteq\)” “Set inclusion”:

“Subset membership” “Casting”:

(11.58) “Reflexivity of \(\subseteq\)”:

(11.59) “Transitivity of \(\subseteq\)”:

(11.57) “Antisymmetry of \(\subseteq\)”:

“Empty set”:

“Empty set”:

“Empty set is least” “Bottom set”:

“Universal set”:

“Universal set”:

“Universal set is greatest” “Top set”:

“Singleton Sets, Set Complement, Set Union and Intersection

“Singleton set membership”:

“Singleton set inclusion”:

“Singleton inclusion”:

“Complement”:

“Intersection”:

“Golden rule for \(n\) and \(\omega\)”:

“Set inclusion via \(n\)”:

“Set inclusion via \(\omega\)”:
Proper Subset

(11.14) “Proper subset” “Definition of c”:
S ⊂ T ≡ S ⊆ T ∧ S ≠ T

(11.61): S ⊆ T ≡ S ⊆ T ∧ ∼ (T ⊆ S)
(11.61): S ⊆ T ≡ S ⊆ T ∧ ∼ (T ⊆ S)
(11.63) “Inclusion in terms of c”:
S ⊂ T ≡ S ⊂ T ∨ S = T

(11.70) “Transitivity of c with c”:
X ⊆ Y ⇒ (Y ⊆ Z ⇒ X ⊆ Z)
(11.70) “Transitivity of c with c”:
X ⊆ Y ⇒ (Y ⊆ Z ⇒ X ⊆ Z)

Set Difference and Relative Pseudo-complement

(11.22) “Set difference”:
v ∈ S - T ≡ v ∈ S ∧ ∼ (v ∈ T)
(11.52): S ∩ (T - S) = {}
(11.54): S - (T ∪ U) = (S - T) ∩ (S - U)
“Characterisation of”:
S ⊆ A ⇒ B ⇒ S ∩ A ⊆ B

“Membership in”:
x ∈ A ⇒ B ⇒ x ∈ A ⇒ x ∈ B
“Definition of”:
A ⇒ B ⇒ A ∪ B

“Pseudocomplement of union”:
(A ∪ B) ⇒ C ⇒ (A ⇒ C) ∩ (B ⇒ C)

“Monotonicity of”:
B ⊆ C ⇒ A ⇒ B ⊆ A ⇒ C

Cartesian Products of Sets; Relationship

(14.2) “Pair equality”:
(b, c) = (b′, c′) ≡ b = b′ ∧ c = c′

“Definition of ‘fst’”:
fst(x, y) = x

“Definition of ‘snd’”:
snd(x, y) = y

“Membership in x”:
p ∈ S × T ≡ fst(p) ∈ S ∧ snd(p) ∈ T
(14.4) “Membership in x”:
(x, y) ∈ S × T ≡ x ∈ S ∧ y ∈ T
(14.5) “Membership in swapped x”:
(y, x) ∈ S × T ≡ (y, x) ∈ T × S
(14.6) “Empty factor in x”:
S = {} ⇒ S × T = {}

“Definition of ‘⇒’”:
A ⇒ B ≡ P (A × B)

“Infix relationship” “Definition of ‘(x, y)’”:
“Relation extensionality”:
R = S ≡ (∀x • (∀ y • x( (R) y) ≡ x( (S) y)))

(13.7) “Relation converse” “Relationship via ‘⇒’”:
R − y (R) x ≡ (x (R) y)

(13.8) “Relation composition”:
a( (R ∪ S) b) ≡ a( (R) b) ∨ a( (S) b)

(13.9) “Identity relation” “Relationship via ‘Id’”:

(13.10) “Relation inclusion”:
R ⊆ S ≡ (∀x • (∀ y • x( (R) y) ⇒ x( (S) y)))

(13.11) “Inclusion in terms of”:
R ⊆ S ⇒ (∀x • (∀ y • x( (R) y) x( (S) y)))

(13.12) “Membership in ‘∈’”:
x ∈ A ⇒ B ⇒ x ∈ A ⇒ x ∈ B

(13.13) “Definition of (x, y)” “Definition of” “Definition of ‘(x, y)’”:
(a ∩ s) = b ∩ (s ∩ b)

(13.14) “Snoc is not empty”:
Snoc is not empty:

(13.15) “Cancellation of ( )”:

(13.16) “Membership in ‘∈’”:

Sequences

(13.3) “Cons is not empty”:
Cons is not empty:

(13.4) “Cancellation of ( )”:
Cancellation of ( )

(13.5) “Cons decomposition”:
Cons decomposition:
xs = x ∨ (3 y • (3 z • yz = ys ∧ xz = zs))

(13.7) “Tail is different”:
Tail is different:

(13.8) “Membership in ‘∈’”:
Membership in ‘∈’:

(13.9) “Right-identity of ‘◦’”:
Right-identity of ‘◦’:

(13.10) “Associativity of ‘◦’”:
Associativity of ‘◦’:

(13.11) “Membership in ‘◦’”:
Membership in ‘◦’:

(13.12) “Subsequence of ‘◦’” “Definition of ‘◦’ for ‘◦’”:
Subsequence of ‘◦’ “Definition of ‘◦’ for ‘◦’”

(13.13) “Definition of ‘◦’” “Definition of ‘◦’ for ‘◦’”:
Definition of ‘◦’ “Definition of ‘◦’ for ‘◦’”

(13.14) “Snoc is not empty”:
Snoc is not empty:

(13.15) “Cancellation of ( )”:
Cancellation of ( )

(13.16) “Membership in ‘∈’”:
Membership in ‘∈’:

Concatenation

Left-identity of ‘◦’ “Definition of ‘◦’ for ‘◦’”

(13.18) “Mutual associativity of ‘◦’” “Definition of ‘◦’ for ‘◦’”:
Mutual associativity of ‘◦’ “Definition of ‘◦’ for ‘◦’”

(13.19) “Right-identity of ‘◦’”:
Right-identity of ‘◦’:

(13.20) “Associativity of ‘◦’”:
Associativity of ‘◦’:

(13.21) “Membership in ‘◦’”:
Membership in ‘◦’:

(13.22) “Mutual associativity of ‘◦’ with ‘◦’” “Definition of ‘◦’ for ‘◦’”:
Mutual associativity of ‘◦’ with ‘◦’ “Definition of ‘◦’ for ‘◦’”

(13.23) “Empty concatenation”:
Empty concatenation:
xs ⊆ ys = ε ≡ xs = ε ∧ ys = ε

Subsequences, Prefix, Segments

(13.24) “Empty subsequence”:
Empty subsequence:
ε ⊆ ys

(13.25) “Empty subsequence”:
Empty subsequence:
ε ⊆ ys

(13.26) “Subsequence” “Cons is not a subsequence of ‘◦’” “Definition of ‘◦’ for ‘◦’”:
Subsequence “Cons is not a subsequence of ‘◦’” “Definition of ‘◦’ for ‘◦’”

(13.27) “Subsequence anchored at head”:
Subsequence anchored at head:

(13.28) “Subsequence without head”:
Subsequence without head:

(13.29) “Proper subsequence” “Definition of ‘◦’”:
Proper subsequence “Definition of ‘◦’”

(13.30) “Reflexivity of ‘◦’”:
Reflexivity of ‘◦’:

(13.31) “Cons ‘ ‘-expands” “Cons ‘ ‘-expands”
Cons ‘ ‘-expands:

Subsequence of ‘◦’ “Definition of ‘◦’ for ‘◦’” “Definition of ‘◦’ for ‘◦’”

Subsequence of ‘◦’ “Definition of ‘◦’ for ‘◦’” “Definition of ‘◦’ for ‘◦’”

(13.34) “Membership of subsequence” “Definition of ‘◦’ for ‘◦’” “Definition of ‘◦’ for ‘◦’”:
Membership of subsequence “Definition of ‘◦’ for ‘◦’” “Definition of ‘◦’ for ‘◦’”

(13.35) “Empty prefix” “Cons is not prefix of ‘◦’” “Definition of ‘◦’ for ‘◦’” “Definition of ‘◦’ for ‘◦’”:
Empty prefix “Cons is not prefix of ‘◦’” “Definition of ‘◦’ for ‘◦’” “Definition of ‘◦’ for ‘◦’”

(13.36) “Empty prefix” “Cons is not prefix of ‘◦’” “Definition of ‘◦’ for ‘◦’” “Definition of ‘◦’ for ‘◦’”:
Empty prefix “Cons is not prefix of ‘◦’” “Definition of ‘◦’ for ‘◦’” “Definition of ‘◦’ for ‘◦’”

(13.37) “Not Prefix” “Cons is not prefix of ‘◦’” “Definition of ‘◦’ for ‘◦’” “Definition of ‘◦’ for ‘◦’”:
Not Prefix “Cons is not prefix of ‘◦’” “Definition of ‘◦’ for ‘◦’” “Definition of ‘◦’ for ‘◦’”

Prefix “Cons prefix” “Definition of ‘◦’ for ‘◦’” “Definition of ‘◦’ for ‘◦’”

Segment “Segment of ‘◦’” “Definition of ‘◦’ for ‘◦’” “Definition of ‘◦’ for ‘◦’”

Segment “Segment of ‘◦’” “Definition of ‘◦’ for ‘◦’” “Definition of ‘◦’ for ‘◦’”

Set Operations used as Relation Operations

“Relation union”:
“Relation intersection”:

“Relation difference”:
“Relation pseudocomplement”:
“Relation complement”:
“Empty relation”:
“Universal relation”:
“Singleton relation”:
“Singleton relation inclusion”:

Relation-specific Operations
Abstract Relation Algebra

"Reflexivity of \(\equiv\): \(R \subseteq R\)
"Transitivity of \(\equiv\): \(Q \subseteq R \Rightarrow R \subseteq S \Rightarrow Q \subseteq S\)
"Antisymmetry of \(\equiv\): \(R \nsubseteq S \Rightarrow S \nsubseteq R \Rightarrow R = S\)
"Transitivity of \(\equiv\): \(Q \subseteq R \Rightarrow R \subseteq S \Rightarrow Q \subseteq S\)
"Flipped Transitivity of \(\equiv\): \(R \subseteq S \Rightarrow Q \subseteq R \Rightarrow Q \subseteq S\)
"Reflexivity of \(\equiv\): \(R = S \Rightarrow R \subseteq S\)
"Mutual inclusion": \(R = S \iff R \subseteq S \land S \subseteq R\)
"Opposite inclusion": \(R \nsubseteq S \iff R \subseteq S\)
"Indirect Relation Equality from above": \(Q = R \iff (\forall S \cdot Q \subseteq S \iff R \subseteq S)\)
"Indirect Relation Equality from below": \(Q = R \iff (\forall S \cdot S \subseteq Q \iff S \subseteq R)\)
"Indirect Relation Inclusion from above": \(Q \subseteq R \iff (\forall S \cdot Q \subseteq S \Rightarrow R \subseteq S)\)
"Indirect Relation Inclusion from below": \(Q \subseteq R \iff (\forall S \cdot S \subseteq Q \Rightarrow R \subseteq S)\)

Composition

"Associativity of \(\circ\): \((Q \circ R) \circ S = Q \circ (R \circ S)\)
"Monotonicity of \(\circ\): \(P \subseteq Q \Rightarrow R \subseteq S \Rightarrow P \circ R \subseteq Q \circ S\)
"Monotonicity of \(\circ\): \(Q \subseteq R \Rightarrow R \subseteq S \Rightarrow Q \circ S \subseteq R \circ S\)
"Monotonicity of \(\circ\): \(R \subseteq S \Rightarrow Q \circ R \subseteq Q \circ S\)
"Identity of \(\circ\): \(Id \circ R = R\)
"Identity of \(\circ\): \(R \circ Id = R\)

Converse

"Self-inverse of \(\sim\): \((R \sim) \sim = R\)
"Cancelltion of \(\sim\): \(R \sim S \sim \iff R = S\)
"Monotonicity of \(\sim\): \(R \subseteq S \Rightarrow R \sim S \subseteq S\)
"Isotonicity of \(\sim\): \(R \subseteq S \iff R \sim S \subseteq S\)
"Converse of \(\circ\): \(Id \circ = Id\)
"Converse of \(\circ\): \(R \circ S \sim = S \sim R\)

Homogeneous Relation Properties

"Definition of reflexivity": \(R \equiv \text{id} \in R\)
"Definition of symmetry": \(R \equiv R^{-} \in R\)
"Definition of transitivity": \(R \equiv R^{-} \subseteq R\)
"Definition of idempotency": \(R \equiv R^{-} \subseteq R\)
"Definition of equivalence": \(R \equiv \text{id-equivalence} \in R \land R \equiv \text{symmetric} \land R \equiv \text{transitive} \)
"Definition of preorder": \(R \equiv R^{-} \subseteq \text{preorder} \land R \equiv \text{transitive} \)
"Definition of symmetry": \(R \equiv R^{-} \subseteq \text{symmetric} \land R \equiv \text{transitive} \)

Heterogeneous Relation Properties

"Definition of univalence": \(R \equiv \text{univalent} \iff R^{-} \subseteq \text{Id}\)
"Definition of totality": \(R \equiv \text{total} \iff \text{Id} \subseteq R \sim \)
"Definition of injectivity": \(R \equiv \text{injective} \iff R \sim \subseteq \text{Id}\)
"Definition of surjectivity": \(R \equiv \text{surjective} \iff \text{Id} \subseteq R^{-} \sim \)
"Definition of mappings": \(R \equiv \text{mapping} \iff R \equiv \text{univalent} \land R \equiv \text{total} \)
"Definition of bijectivity": \(R \equiv \text{bijective} \iff R \equiv \text{injective} \land R \equiv \text{surjective} \)
"Definition of mappings": \(R \equiv \text{mapping} \iff R \equiv \text{univalent} \land R \equiv \text{total} \)
"Definition of bijectivity": \(R \equiv \text{bijective} \iff R \equiv \text{injective} \land R \equiv \text{surjective} \)

Relation Algebra: Continuing with Intersection

"Characterisation of \(\cap\): \(Q \cap R \subseteq S \iff Q \subseteq S \land R \subseteq S\)
"Weakening for \(\cap\): \(Q \cap R \subseteq Q \land Q \cap R = R\)
"Symmetry of \(\cap\): \(Q \cap R = R \cap Q\)
"Associativity of \(\cap\): \((Q \cap R) \cap S = Q \cap (R \cap S)\)
"Idempotency of \(\cap\): \(R \cap R = R\)
"Monotonicity of \(\cap\): \(Q \subseteq R \Rightarrow Q \cap S \subseteq R \cap S\)
"Inclusion via \(\cap\): \(Q \cap R \subseteq Q \cap R = Q\)
"Sub-distributivity of \(\cap\) over \(\cap\): \((Q \cap R) \cap S = Q \cap (R \cap S)\)
"Sub-distributivity of \(\cap\) over \(\cap\): \((Q \cap R) \cap S = Q \cap (R \cap S)\)
"Converse of \(\cap\): \((R \cap S)^{-} = R^{-} \cap S^{-}\)
"Dedekind rule": \(Q \subseteq R \cap S \subseteq (Q \cap S \subseteq R^{-}) \iff (R \cap Q^{-} \subseteq S)\)
"Modal rule": \(Q \cap R \subseteq S \iff Q \cap (S \subseteq R^{-}) \iff R\)
"Modal rule": \(Q \subseteq S \iff (R \subseteq Q^{-} \subseteq S)\)
"Hesitation": \(R \subseteq R \cap R^{-} \subseteq R\)

Relation Algebra: Continuing with Union

"Characterisation of \(\cup\): \(Q \cup R \subseteq S \iff Q \subseteq S \lor R \subseteq S\)
"Weakening for \(\cup\): \(Q \cup R \subseteq Q \lor Q \cup R = Q\)
"Symmetry of \(\cup\): \(Q \cup R = R \cup Q\)
"Associativity of \(\cup\): \((Q \cup R) \cup S = Q \cup (R \cup S)\)
"Idempotency of \(\cup\): \(R \cup R = R\)
"Monotonicity of \(\cup\): \(Q \subseteq R \Rightarrow Q \cup S \subseteq R \cup S\)
"Inclusion via \(\cup\): \(Q \subseteq R \Rightarrow Q \cup R = R\)
"Distributivity of \(\cup\) over \(\cap\): \((Q \cup R) \cap S = Q \cap (R \cup S)\)
"Distributivity of \(\cup\) over \(\cap\): \((Q \cup R) \cap S = Q \cap (R \cup S)\)
"Union of converses": \(R^{-} \cap S^{-} \subseteq (R \cup S)\)
"Converse of \(\cup\): \((R \cup S)\sim = R^{-} \cup S^{-}\)
"Absorption of \(\cup\) by \(\cap\): \(Q \cap (Q \cup R) = Q\)
"Absorption of \(\cap\) by \(\cup\): \(Q \cap (Q \cup R) = Q\)
"Distributivity of \(\cap\) over \(\cup\): \((Q \cap R) \cup S = (Q \cap R) \cup (Q \cap S)\)
"Distributivity of \(\cup\) over \(\cap\): \((Q \cup R) \cap (Q \cap S) = (Q \cup R) \cap (Q \cap S)\)

Least Elements in the Inclusion Order

"Least relation": \(\bot \subseteq R\)
"Inclusion in \(\bot\): \(R \subseteq \bot \iff R = \bot\)
"Zero of \(\cap\): \(R \land \bot = \bot\)
"Identity of \(\cup\): \(\top \land R = R\)
"Converse of \(\bot\): \(\bot \sim = \bot\)
"Right-zero of \(\bot\): \(R \land \bot = \bot\)

Greatest Elements in the Inclusion Order

"Greatest relation": \(R \subseteq \top\)
"Inclusion of \(\top\): \(R \subseteq \top \iff R = \top\)
"Identity of \(\cap\): \(R \land \top = R\)
"Zero of \(\cup\): \(R \cup \top = R\)
"Converse of \(\top\): \(R \top = R\)
Relation Algebra: Complement

- Characterisation of \(~\): \( S \cap R = \perp \land S \cup R = \top \equiv S = \neg R \)
- Characterisation of \(~\): \( R \cap \neg R = \perp \land \neg R \cup R = \top \)
- Self-inverse of \(~\): \( \neg (\neg R) = R \)
- Antitonicity of \(~\): \( Q \subseteq R \equiv \neg R \subseteq \neg Q \)
- Anti-isotonicity of \(~\): \( Q \subseteq R \equiv \neg R \subseteq \neg Q \)
- \(~\) connection: \( Q \subseteq R \equiv \neg R \subseteq \neg Q \)
- Cancellation of \(~\): \( \neg Q = \neg R \)

Facts that can be shown by “Evaluation”

Assuming the Antecedent

\[
\begin{align*}
\text{Assuming } & p, q: \\
\text{... Assumption } & p \ldots \\
\text{Assuming } & p \text{ and using with } \ldots: \\
\text{... Assumption } & p \ldots
\end{align*}
\]

Case Analysis

\[
\begin{align*}
\text{By cases: } & p, q, r: \\
\text{Completeness: } & \\
\text{Case } & p: \\
\text{... Assumption } & p \ldots \\
\text{... }
\end{align*}
\]

Subproofs

\[
\begin{align*}
& ? \\
& \equiv \text{(Subproof for } \ldots): \\
& \{\text{proof indented as far as needed to avoid parse error!}\} \\
& ?
\end{align*}
\]

Nested subproofs currently may need to be indented even further than first-level subproofs!

CalculCheck Structured Proofs

Simple Induction

By induction on \( \text{`var : Ty}: \)

Base case: ?

Induction step: ?

... Induction hypothesis ...

Making base case, induction step, and induction hypothesis explicit:

By induction on \( \text{`var : Ty}: \)

Base case \( ?: \)

? Induction step \( ?: \)

? ... Induction hypothesis \( ? \ldots \)

(Remember that in nested inductions, induction hypotheses always need to be made explicit!)

These can also be used for proving theorems of shape

\[
\forall \text{ var : Ty } P
\]

by induction on precisely that universally-quantified variable, that is, \( \text{`on } \text{var : Ty}: \).

The induction hypothesis is then \( P \).

Example for sequences:

Theorem: \( \forall x : \text{Seq A } \cdot P \)

Proof:

By induction on \( \text{`xs : Seq A}: \)

Base case \( \text{`P[xs = \_ ]}: \)

? Induction step \( \forall x : A \cdot P[xs = x \cdot xs ]: \)

For any \( x: \)

?

Proving Universal Quantifications

For any \( \text{`var : Ty }\): ?

For any \( \text{`var : Ty} \) satisfying \( p\): ?

... Assumption \( p \ldots ?

Theorems Used as Proof Methods (Examples)

Using “Mutual implication”:

Subproof for \( \ldots \to \ldots \): ?

Subproof for \( \ldots \to \ldots \): ?

Using “Extensionality”:

Subproof for \( \forall x : \ldots \): ?

Side Proofs

Side proof for \( P\): ?

Continuing with goal \( ?: \)

? ... local property \( P \ldots ?

Disabling Hints Producing Time-outs

Add “?,” “ at the beginning of the hint: ?

≡ ( ?, “Golden rule”)
Selected CalcCheckWeb Key Bindings
(See Getting Started with CalcCheckWeb for the complete listing.)
The following key bindings work the same in both edit and command modes:

- **Ctrl-Enter** performs a syntax check on the contents of all code cells before and up to the current cell.
- **Ctrl-Alt-Enter** performs proof checks (if enabled) on the contents of all code cells before and up to the current cell.
- **Shift-Alt-RightArrow** enlarges the width of the current code cell entry area by a small amount.
- **Ctrl-Shift-Alt-RightArrow** enlarges the width of the current code cell entry area by a large amount.
- **Shift-Alt-LeftArrow** reduces the width of the current code cell entry area by a small amount.
- **Ctrl-Shift-Alt-LeftArrow** reduces the width of the current code cell entry area by a large amount.
- **Ctrl-Shift-v** (for visible spaces) toggles display of initial spaces on each line as \( \equiv \) characters.

**ONLY if you are logged in via Avenue:**

**Ctrl-s** saves the notebook on the server.

To be safest, use in command mode, e.g. after clicking on the area of a code box where the line number would be displayed.

Check the pop-up whether it is the CalcCheckWeb pop-up saying “...Notebook saved to ...

(Links for reloading the last three saved versions are displayed when you view the notebook again.)

In **edit mode**, you have the following **key bindings**:
- **Esc** enters command mode.
- **Alt-i** or **Alt-SPACE** inserts one space in the current line and in all non-empty lines below it, until a line is encountered that is not indented more than to the cursor position.
- **Alt-BACKSPACE** deletes only a space character to the left of the current cursor position, and also from lines below it, until a line is encountered that is not indented at least to the cursor position.
- **Alt-DELETE** deletes only a space character to the right of the current cursor position, and also from lines below it, until a line is encountered that is not indented more than to the cursor position.

The last three bindings also work with Shift.

<table>
<thead>
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<th>Symbol</th>
<th>Key sequence(s)</th>
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<td>\equiv, ==</td>
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\( \in \)
\( \\PP, \powerset \)
\( \\union \)
\( \\intersection \)
\( \\bigunion \)
\( \\bigintersection \)
\( \\top \)
\( \\bot \)
\( \\pseudocompl \)
\( \\subsetneq, \subseteq \)
\( \\supseteq, \supset \)
\( \\subset \)
\( \\supset \)
\( \\universe \)

### Some important symbols:

<table>
<thead>
<tr>
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