**2DM3 2018 — Final Theorem List**

Equivalence, Negation and Inequivalence

“Definition of ≡”: (p ≡ q) = (p = q)

(3.2) “Symmetry of ≡”: ?

(3.3) “Identity of ≡”: ?

(3.5) “Reflexivity of ≡”: p ≡ p

(3.9) “Commutativity of ~ with ≡” “Distributivity of ~ over ≡”: ?

(3.11) “~ connection”: ~ p ≡ q ≡ p ~ q

(3.14): (p ≠ q) ≡ (~ p ≡ q)

(3.15): ~ p ≡ (p ≡ false)

Disjunction and Conjunction

(3.32): ?

(3.35) “Golden rule”: ?

(3.48): ?

(3.49) “Semi-distributivity of ∧ over ≡”: ?

(3.50) “Strong Modus Ponens”: ?

(3.51) “Replacement”: (p ≡ q) ∧ (r ≡ p) ≡ (p ≡ q) ∧ (r ≡ q)

(3.52) “Alternative definition of ≡”: q ∨ (~ p ∧ ~ q)

(3.53) “Exclusive or” “Alternative definition of #”: ?

Implication

(3.57) “Definition of ⇒”: ?

(3.58) “Definition of ⇔” “Consequence”: ?

(3.59) “Definition of ≡⇒”: ?

(3.60) “Definition of ⇒≡”: ?

(3.61) “Contrapositive”: ?

(3.62): p ⇒ (q ≡ r) ≡ (p ∧ q) ≡ p ∧ r

(3.63) “Distributivity of ⇒ over ≡”: ?

(3.64) “Self-distributivity of ⇒”: ?

(3.65) “Shunting”: ?

(3.66): p ∧ (p ⇒ q) ≡ p ∧ q

(3.67): p ∧ (q ⇒ p) ≡ p

(3.68): p ∨ (p ⇒ q) ≡ true

(3.69): p ∨ (q ⇒ p) ≡ q ⇒ p

(3.70): p ∨ q ⇒ p ∧ q ≡ (p ≡ q)

(3.71) “Reflexivity of ⇒”: ?

(3.72) “Right-zero of ⇒”: ?

(3.73) “Left-identity of ⇒”: ?

“Definition of ¬” (3.74): ?

(3.75) ”ex falso quodlibet” *?

(3.76a) “Weakening”: ?

(3.76b) “Weakening”: ?

(3.76c) “Weakening”: ?

(3.76d) “Weakening”: ?

(3.76e) “Weakening”: ?

“Reflexivity of ⇒”: ?

(3.77) “Modus ponens”: ?

(3.78) “Case analysis”: ?

(3.79) “Case analysis”: ?

(3.80) “Mutual implication”: ?

(3.81) “Antisymmetry of ⇒”: ?

(3.82a) “Transitivity of ⇒”: ?

(3.82b) “Transitivity of ⇒”: ?

(3.82c) “Transitivity of ⇒”: ?

“Implication strengthening”: p ⇒ q ∨ p ⇒ p ∧ q

Leibniz as Axiom and Substitution/Replacement Laws

(3.83) “Leibniz”: e = f ⇒ E[z := e] = E[z := f]

(3.84a) “Replacement”: e = f ∧ E[z := e] ≡ e = f ∧ E[z := f]

(3.84b) “Replacement”: e = f ⇒ E[z := e] ≡ e = f ⇒ E[z := f]

(3.84c) “Replacement”: q ∧ e = f ⇒ E[z := e] ∧ q ∧ e = f ⇒ E[z := f]

“Transitivity of ≡”: e = f ∧ f = g ⇒ e = g

(3.85a) “Replace by ’true’”: p ⇒ E[z := p] ⇒ p ⇒ E[z := true]

(3.85b) “Replace by ’true’”: q ∧ p ⇒ E[z := p] ∧ q ∧ p ⇒ E[z := true]

(3.85c) “Replace by ’false’”: ¬ p ⇒ E[z := p] ⇒ ¬ p ⇒ E[z := false]

(3.85e) “Replace by ’true’”: p ⇒ E[z := p] = E[z := true]

(3.86a) “Replace by ’false’”: E[z := p] ⇒ p ⇒ E[z := false] ⇒ p

(3.86b) “Replace by ’false’”: E[z := p] ⇒ p ∨ q ⇒ E[z := true] ⇒ p ∨ q

(3.87) “Replace by ’true’”: p ∧ E[z := p] ⇒ p ∧ E[z := true]

(3.88) “Replace by ’false’”: p ∨ E[z := p] ⇒ p ∨ E[z := false]

Monotonicity with Respect to Implication

(4.2) “Left-monotonicity of ∨” “Monotonicity of ∨”: ?

“Monotonicity of ∨”: (p ⇒ q) ⇒ ((r ⇒ s) ⇒ (p ∨ r ⇒ q ∨ s))

(4.3) “Left-monotonicity of ∧” “Monotonicity of ∧”: ?

“Monotonicity of ∧”: (p ⇒ p′) ⇒ ((q ⇒ q′) ⇒ (p ∧ q ⇒ p′ ∧ q′))

“Antitonicity of ∨”:

“Antitonicity of ⇒” “Right-antitonicity of ⇒”:

“Antitonicity of ⇒” “Left-antitonicity of ⇒”:

Sum Quantification: General Quantifier Material Instantiated for Sum

“Leibniz for Σ range”:

(3.89) “Leibniz for Σ body”:

(3.90) “Empty range for Σ”:

(8.13) “Empty range for Σ”:

(8.14) “One-point rule for Σ”:

(8.15) “Distributivity of Σ over ∨”:

(8.17) “Range split”:

(8.18) “Disjoint range split for Σ”:

(8.20) “Nesting for Σ”:

(8.21) “Dummy renaming for Σ” “α-conversion”:

“Replacement in Σ”:

“Dummy list permutation for Σ”:

(8.19) “Interchange of dummies”:

(8.21) “Dummy renaming for Σ” “α-conversion”:

— provided: ¬occurs(x, ’R’), ¬occurs(y, ’Q’)

— provided: ¬occurs(y, ’E, R’)

— provided: ¬occurs(y, ’R, E’), ¬occurs(y, ’Q’)

— provided: ¬occurs(y, ’E, R’), ¬occurs(y, ’Q’)

(2DM3 2018 — Final Theorem List)
Specific Material for Sum Quantification

“Distributivity of ∑ over ∑”: \( a \cdot (\sum x | R \cdot E) = (\sum x | R \cdot a \cdot E) \) — provided: \( \neg \text{occurs}(x', a') \)

“Zero ∑ body”: \( (\sum x | R \cdot 0) = 0 \)

“Definition of ≤ in terms of <”: \( a \leq b \equiv a < b \vee a = b \)

“Definition of ≤ in terms of ‘S’ and <”: \( a \leq b \equiv a < S b \)

“Split range at top”: \( m \leq n \Rightarrow (m \leq i < S n \equiv m \leq i < n \vee i = n) \)

“Split off term at top”: \( \{ i : N | i \leq S n \cdot E \} = \{ i : N | i \leq n \cdot E \} + E[i = n] \)

— provided: \( \neg \text{occurs}(i, 'n') \)

“Split off term at top using ≤”: \( \{ i | i \leq S n \cdot E \} = \{ i | i \leq n \cdot E \} + E[i = n] \)

— provided: \( \neg \text{occurs}(i, 'n') \)

Universal Quantification

“Leibniz for ∀ body”: \( (\forall x | R \cdot P_1 \equiv P_2) \Rightarrow ((\forall x | R \cdot P_1) \equiv (\forall x | R \cdot P_2)) \)

(8.18) “Range split for ∀ v”: \( (\forall x | R v S \cdot P) \equiv (\forall x | R \cdot P) \wedge (\forall x | S \cdot P) \)

(9.5) “Distributivity of v over ∀ v”: \( v \wedge (\forall x | R \cdot Q) \equiv (\forall x | R \cdot v \wedge Q) \)

— provided: \( \neg \text{occurs}(x', 'P) \)

(9.6): \( P \vee (\forall x | R \cdot (\neg R)) \equiv (\forall x | R \cdot P) \)

“Distributivity of \Rightarrow over ∀ v”: \( P \Rightarrow (\forall x | R \cdot Q) \equiv (\forall x | R \cdot P \Rightarrow Q) \) — prov. \( \neg \text{occurs}(x', 'P) \)

(9.7) “Distributivity of \lor over ∀ v": \( \lor (\forall x | R \cdot \neg R) \Rightarrow (\forall x | R \cdot P) \wedge (\forall x | R \cdot Q) \) — provided: \( \neg \text{occurs}(x', 'P) \)

(9.8) “True ∀ body”: \( (\forall x | R \cdot \text{true}) \)

“Introducing fresh ∀ v": \( P \equiv (\forall y | R \cdot P) \)

— provided: \( \neg \text{occurs}(x', 'P) \)

(9.9) “Sub-distributivity of ∧ over ∀ v"”: \( (\forall x | R \cdot P \equiv Q) \Rightarrow ((\forall x | R \cdot P) \equiv (\forall x | R \cdot Q)) \)

(9.10) “Range weakening for ∀ v": \( (\forall x | R v Q \cdot P) \Rightarrow (\forall x | Q \cdot P) \)

(9.11) “Body weakening for ∀ v": \( (\forall x | R \cdot P \wedge Q) \Rightarrow (\forall x | Q \cdot P) \)

(9.12) “Body monotonicity for ∀ v": \( (\forall x | R \cdot Q \Rightarrow P) \Rightarrow (\forall x | R \cdot Q) \Rightarrow (\forall x | R \cdot P) \)

— provided: \( \neg \text{occurs}(x', 'P) \)

(9.12a) “Range antimonotonicity of ∀ v": \( (\forall x | R \cdot Q \Rightarrow P) \Rightarrow (\forall x | R \cdot Q) \Rightarrow (\forall x | R \cdot P) \)

(9.13) “Instantiation”: \( \neg \)

“Fresh ∀ v": \( P \equiv (\forall x | P) \)

— provided: \( \neg \text{occurs}(x', 'P) \)

Existential Quantification

(9.21) “Distributivity of \lor over ∃ v": \( P \lor (\exists x | R \cdot Q) \equiv (\exists x | R \cdot P \lor Q) \)

— provided: \( \neg \text{occurs}(x', 'P) \)

(9.22): \( P \lor (\exists x \cdot R) \equiv (\exists x \cdot R \lor Q) \)

“Distributivity of \lor over ∃ v": \( (\exists x \cdot R) \Rightarrow (P \lor (\forall x | R \cdot Q)) \equiv (\forall x | R \cdot P \lor Q) \)

— provided: \( \neg \text{occurs}(x', 'P) \)

(9.23) “Distributivity of v over ∃ v": \( (\exists x \cdot R) \Rightarrow (P v (\exists x | R \cdot Q)) \equiv (\exists x | R \cdot P \lor Q) \)

— provided: \( \neg \text{occurs}(x', 'P) \)

(9.24) “False ∃ body”: \( (\exists x | R \cdot \text{false}) \equiv \text{false} \)

(9.25) “Range weakening for ∃ v": \( (\exists x | R \cdot P) \Rightarrow (\exists x | Q v R \cdot P) \)

“Range weakening for ∃ v": \( (\exists x | Q \wedge R \cdot P) \Rightarrow (\exists x | R \cdot P) \)

(9.26) “Body weakening for ∃ v": \( (\exists x | R \cdot P) \Rightarrow (\exists x | R \cdot Q) \)

(9.26a) “Body weakening for ∃ v": \( (\exists x | R \cdot P \wedge Q) \Rightarrow (\exists x | R \cdot P) \)

(9.27) “Body monotonicity of ∃ v": \( (\forall x | R \cdot Q \Rightarrow P) \Rightarrow ((\exists x | R \cdot Q) \Rightarrow (\exists x | R \cdot P)) \)

“Range monotonicity of ∃ v": \( (\forall x \cdot Q \Rightarrow R) \Rightarrow ((\exists x | R \cdot P) \Rightarrow (\exists x | R \cdot P)) \)

“Range monotonicity of ∃ v": \( (\forall x | P \cdot Q \Rightarrow R) \Rightarrow ((\exists x | Q \cdot P) \Rightarrow (\exists x | R \cdot P)) \)

Introduction and Interchange for ∃

(9.28) “3-Introduction”: \( \exists \)

(9.29a) “Interchange of quantifications”: \( (\exists x | (\forall y \cdot P)) \Rightarrow (\forall y \cdot (\exists x \cdot P)) \)

(9.30a) “Witness”: \( (\exists x | R \cdot P) \Rightarrow Q \equiv (\forall x | R \wedge P \Rightarrow Q) \) — provided: \( \neg \text{occurs}(x', 'Q) \)

(9.30b) “Witness”: \( (\exists x \cdot P) \Rightarrow Q \equiv (\forall x \cdot P \Rightarrow Q) \) — provided: \( \neg \text{occurs}(x', 'Q) \)

Set Theory

(11.3) “Set membership”: \( \exists \)
(11.7a) “Simple Membership”: \( \exists \)
(11.7b) “Simple Membership”: \( \exists \)

“Membership in two-element set enumeration”: \( \exists \)

“Membership in set enumeration”: \( \exists \)

Set Extensionality and Set Inclusion

(11.4) “Set extensionality” “Set equality” “Extensionality”: \( \exists \)
(11.9) “Simple set comprehension extensionality”: \( \exists \)
(11.13) “Subset” “Definition of ≤” “Set inclusion”: \( \exists \)

“Subset” “Definition of ≤” “Set inclusion”: \( \exists \)

“Subset membership” “Casting”: \( \exists \)

(11.58) “Reflexivity of ≤”: \( \exists \)

“Reflexivity of ≤”: \( \exists \)

(11.59) “Transitivity of ≤”: \( \exists \)

“Flipped transitivity of ≤”: \( \exists \)

(11.57) “Antisymmetry of ≤”: \( \exists \)

“Empty set”: \( \{ \} \equiv \{ x | \text{false} \} \)

“Empty set”: \( x \in \{ \} \equiv \text{false} \)

“Empty set is least” “Bottom set”: \( \{ \} \subseteq X \)

“Universal set”: \( U = \{ x | \text{true} \} \)

“Universal set”: \( x \in U \)

“Universal set is greatest” “Top set”: \( X \subseteq U \)

(11.56) “Simple set comprehension inclusion”: \( \{ x | P \} \subseteq \{ x | Q \} \equiv (\forall x | P \Rightarrow Q) \)

Singleton Sets, Set Complement, Set Union and Intersection

“Singleton set membership”: \( \exists \)

“Singleton set inclusion”: \( \exists \)

“Complement”: \( \exists \)

(11.19) “Self-inverse of complement”: \( \exists \)

“Lower ~ connection for ≤”: \( X \subseteq Y \equiv \neg Y \subseteq X \)

“Upper ~ connection for ≤”: \( X \subseteq Y \equiv \neg Y \subseteq X \)

“Union”: \( \exists \)

“Intersection”: \( \exists \)

“Golden rule for n and u”: \( \exists \)

“Set inclusion via n”: \( \exists \)

“Set inclusion via u”: \( \exists \)
Proper Subset

(11.14) “Proper subset” “Definition of c”: ?
(11.61): S ⊆ T ⇔ S ⊆ T ∧ (T ⊈ S)
(11.61): S ⊆ T ⇔ S ⊆ T ∧ (T ⊈ S)
(11.63) “Inclusion in terms of c”: ?
(11.70) “Transitivity of with c”: ?
(11.70) “Transitivity of with c”: ?

Set Difference and Relative Pseudo-complement

(11.22) “Set difference”: ?
(11.52): S ∩ (T - S) = {}
(11.54): S - (T ∪ U) = (S - T) ∩ (S - U)
“Characterisation of →”: S ∈ A ⇒ B ⇒ S ∈ A ∧ B
“Membership in →”: x ∈ A ⇒ B ⇒ x ∈ A ⇒ x ∈ B
“Definition of →”: A ⇒ B ⇒ = A ∪ B
“Pseudocomplement of union”: (A ∪ B) ⇒ C = (A ⇒ C) ∩ (B ⇒ C)
“Monotonicity of →”: B ⊆ A ⇒ A ⇒ B ⊆ A ⇒ C

Cartesian Products of Sets; Relationship

(14.2) “Pair equality”: {b, c} = {b’, c’} ⇔ b = b’ ∧ c = c’
“Definition of ‘fst’”: fst (x, y) = x
“Definition of ‘snd’”: snd (x, y) = y
“Membership in ×”: p ∈ S × T ⇔ fst p ∈ S ∧ snd p ∈ T
(14.4) “Membership in ×”: (x, y) ∈ S × T ⇔ x ∈ S ∧ y ∈ T
(14.5) “Membership in swapped ×”: (x, y) ∈ S × T ⇔ (y, x) ∈ T × S
(14.6) “Empty factor in ×”: S = {} ⇒ S × T = {}
“Definition of →”:
“Infix relationship” “Definition of ‘A’”:
“Relation extensionality”:
“Relation inclusion”:
“Relation inclusion”:
“Relation inclusion”:

Set Operations used as Relation Operations

“Relation union”:
“Relation intersection”:
“Relation difference”:
“Relation pseudocomplement”:
“Relation complement”:
“Empty relation”:
“Universal relation”:
(∀ A : Type • (∀ B : Type • a (A × B) b))
“Singleton relation”:
“Singleton relation inclusion”:

Relation-specific Operations

“Relation converse” “Relationship via ‘’”:
“Relation composition”:
“Identity relation” “Relationship via ‘Id’”:

Sequences

(13.3) “Cons is not empty”:
“Cons is not empty”:
(13.4) “Cancellation of <”:
(13.6) “Cons decomposition”:
xs = ℰ (y • (x ys • xs ≠ y ys))
(13.7) “Tail is different”:

Sequence Membership , Snoc ▷

“Membership in”:
“Membership in”:
(13.12) “Definition of ▷” “Definition of ▷ for”:
(13.13) “Definition of ▷” “Definition of ▷ for”:
(13.14) “Snoc is not empty”:
“Snoc is not empty”:
(13.15) “Cancellation of ▷”:
(13.16) “Membership in ▷”:

Concatenation

(13.17) “Left-identity of ” “Definition of for ”:
(13.18) “Mutual associativity of with ” “Definition of for ”:
(13.19) “Right-identity of ”:
(13.20) “Associativity of ”:
(13.21) “Membership in ”:
(13.22) “Mutual associativity of with ▷”:
(13.23) “Empty concatenation”:
xs ▷ ys = ℰ ⊆ xs = ℰ ∧ ys = ℰ

Subsequences, Prefix, Segments

(13.25) “Empty subsequence”:
(13.26) “Subsequence” “Cons is not a subsequence of ℰ”:
(13.27) “Subsequence anchored at head”:
(13.28) “Subsequence without head”:
(13.29) “Proper subsequence” “Definition of ℰ”:
(13.30) “Reflexivity of ℰ”:
(13.31) “Cons -expands”:
(13.33) “Subsequence of ℰ”:
(13.34) “Membership of subsequence”:
(13.35) “Membership of subsequence”:
(13.36) “Empty prefix”:
(13.37) “Not Prefix” “Cons is not prefix of ℰ”:
(13.38) “Prefix” “Cons prefix”:
(13.39) “Segment” “Segment of ℰ”:
(13.40) “Segment” “Segment of ▷”:
isseg xs ℰ ⊆ xs = ℰ

Abstract Relation Algebra

“Reflexivity of ≦$: $\forall R : \text{Id} \subseteq R$
“Transitivity of $\leq$: $\forall R : R \subseteq S \Rightarrow R \leq S$
“Antisymmetry of $\leq$: $?\equiv R \subseteq S$
“Transitivity of $\leq$: $\forall R : R \subseteq S$
“Flipped Transitivity of $\leq$: $\forall R : R \subseteq S$
“Reflexivity of $\leq$: $R = S \Rightarrow R \subseteq S$
“Mutual inclusion”: $\forall R : R \subseteq S \Leftrightarrow R \subseteq S$
“Opposite inclusion”: $R \subseteq S \Leftrightarrow S \subseteq R$
“Indirect Relation Equality from above”: $\forall R : R \subseteq S$
“Indirect Relation Equality from below”: $\forall R : R \subseteq S$
“Indirect Relation Inclusion from above”: $\forall R : R \subseteq S$
“Indirect Relation Inclusion from below”: $\forall R : R \subseteq S$

Composition

“Associativity of $\circ$: $\forall R : (R \circ S) \circ T = R \circ (S \circ T)$
“Monotonicity of $\circ$: $\forall R : (R \subseteq S) \Rightarrow (R \circ T \subseteq S \circ T)$
“Monotonicity of $\circ$: $\forall R : (R \subseteq S) \Rightarrow (R \circ T \subseteq S \circ T)$
“Identity of $\circ$: $\forall R : R \circ \text{Id} = R$
“Identity of $\circ$: $\forall R : R \circ \text{Id} = R$

Converse

“Self-inverse of $^{-1}$: $\forall R : (R \circ R)^{-1} = R$
“Cancellation of $^{-1}$: $\forall R : (R \subseteq S) \Rightarrow R \circ (S \circ R) = R \circ S$
“Monotonicity of $^{-1}$: $\forall R : (R \subseteq S) \Rightarrow (R \circ S)^{-1} \subseteq S^{-1}$
“Isotonicity of $^{-1}$: $\forall R : (R \subseteq S) \Rightarrow (R \circ S)^{-1} \subseteq S^{-1}$
“Converse of $\text{Id}^{-1}$: $\forall R : R \circ \text{Id}^{-1} = R$
“Converse of $\circ$: $\forall R : R \circ (R^{-1} \circ R) = R$

Homogeneous Relation Properties

“Definition of reflexivity”: $\text{is-reflexive } R \equiv \text{Id} \subseteq R$
“Definition of symmetry”: $\text{is-symmetric } R \equiv R^{-1} \subseteq R$
“Definition of transitivity”: $\text{is-transitive } R \equiv R \circ S \subseteq R$
“Definition of idempotency”: $\forall R : R \circ R = R$
“Definition of equivalence”: $\forall R : R \circ S = S \circ R$
“Definition of preorder”: $\forall R : R \circ S = S \circ R$
“Definition of symmetry”: $\forall R : R \subseteq S$

Heterogeneous Relation Properties

“Definition of univalence”: $\text{is-univalent } R \equiv R^{-1} \circ R \subseteq \text{Id}$
“Definition of totality”: $\text{is-total } R \equiv \text{Id} \subseteq R^{-1} \circ R$
“Definition of injectivity”: $\text{is-injective } R \equiv R \circ R^{-1} \subseteq \text{Id}$
“Definition of surjectivity”: $\text{is-surjective } R \equiv \text{Id} \subseteq R^{-1} \circ R$
“Definition of mappings”: $\forall R : R \subseteq S$
“Definition of bijection”: $\forall R : R \subseteq S$
“Definition of bijection”: $\forall R : R \subseteq S$

Relation Algebra: Continuing with Union

“Characterisation of $\cup$: $Q \cup R \subseteq S \equiv R$
“Weakening for $\cup$: $\forall R : Q \cup R \subseteq S$
“Symmetry of $\cup$: $\forall R : R \cup S = S \cup R$
“Associativity of $\cup$: $\forall R : (R \cup S) \cup T = R \cup (S \cup T)$
“Idempotency of $\cup$: $\forall R : R \cup R = R$
“Monotonicity of $\cup$: $\forall R : (R \subseteq S) \Rightarrow (R \cup T \subseteq S \cup T)$
“Inclusion via $\cup$: $\forall R : R \subseteq S \Rightarrow R \cup T \subseteq S \cup T$
“Dedekind rule”: $Q \cup R \subseteq S \subseteq (Q \cup S \cup R \cup T \cup S) \cup (Q \cup S \cup R \cup T \cup S)$
“Modal rule”: $Q \cup R \subseteq S \subseteq (Q \cup S \cup R \cup T \cup S) \cup (Q \cup S \cup R \cup T \cup S)$
“Hesitation”: $R \subseteq S \Rightarrow R \subseteq S$

Relation Algebra: Continuing with Intersection

“Characterisation of $\cap$: $Q \cap R \subseteq S \equiv R$
“Weakening for $\cap$: $\forall R : Q \cap R \subseteq S$
“Symmetry of $\cap$: $\forall R : R \cap S = S \cap R$
“Associativity of $\cap$: $\forall R : (R \cap S) \cap T = R \cap (S \cap T)$
“Idempotency of $\cap$: $\forall R : R \cap R = R$
“Monotonicity of $\cap$: $\forall R : (R \subseteq S) \Rightarrow (R \cap T \subseteq S \cap T)$
“Inclusion via $\cap$: $\forall R : R \subseteq S \Rightarrow R \cap T \subseteq S \cap T$
“Distributivity of $\cap$ over $\cup$: $\forall R : (R \cap S) \cap T = R \cap (S \cap T)$
“Distributivity of $\cap$ over $\cup$: $\forall R : (R \cap S) \cap T = R \cap (S \cap T)$
“Union of converses”: $\forall R : R \circ S \subseteq S \circ R$
“Converse of $\cup$: $\forall R : R \subseteq S \Rightarrow R \cap T \subseteq S \cap T$
“Absorption of $\cup$ by $\cap$: $\forall R : (R \cup S) \cap T = R \cap T$
“Absorption of $\cap$ by $\cup$: $\forall R : (R \cap S) \cup T = R \cup T$
“Distributivity of $\cap$ over $\cup$: $\forall R : (R \cap S) \cap T = R \cap (S \cap T)$
“Distributivity of $\cup$ over $\cap$: $\forall R : (R \cup S) \cap T = R \cap (S \cap T)$

Least Elements in the Inclusion Order

“Least relation”: $\bot \subseteq R$
“Inclusion in $\bot$: $R \subseteq \bot \equiv R = \bot$
“Zero of $\cap$: $\bot \cap R = \bot$
“Identity of $\cup$: $\bot \cup R = R$
“Converse of $\bot$: $\bot^{-1} = \bot$
“Left-zero of $\circ$: $\bot \circ R = \bot$
“Right-zero of $\circ$: $R \circ \bot = \bot$

Greatest Elements in the Inclusion Order

“Greatest relation”: $\top \subseteq R$
“Inclusion of $\top$: $\top \subseteq R \equiv R = \top$
“Identity of $\top$: $\top \cap R = R$
“Zero of $\cup$: $\top \cup R = \top$
“Converse of $\top$: $\top^{-1} = \top$
Relation Algebra: Complement

- Characterisation of ~: $S \cap R = \perp \iff S = \lnot R$
- Characterisation of ~: $R \cap \lnot R = \perp \iff R = S$
- Self-inverse of ~: $\lnot (\lnot R) = R$
- Antitonicity of ~: $Q \subseteq R \Rightarrow \lnot R \subseteq \lnot Q$
- Anti-isotonicity of ~: $Q \subseteq R \iff \lnot R \subseteq \lnot Q$
- ~ connection: $Q \subseteq R \iff -R \subseteq -Q$
- ~ connection: $Q \subseteq R \iff -R \subseteq -Q$
- Cancellation of ~: $\lnot (\lnot R) = R$
- Equality with ~: $Q = \lnot R$
- Anti-disjunction on ~: $Q \lor R = \bot$
- Inclusion via intersection with complement: $\lnot Q \subseteq \lnot (Q \cap R)$
- Inclusion via intersection with complement: $\lnot Q \subseteq \lnot (Q \cup R)$
- Contrapositive of ~ with \&: $Q \cap R \subseteq S \subseteq \lnot R$
- Anti-Schröder: $Q \cap R \subseteq S \iff \lnot R \subseteq R$
- Schröder: $Q \cap R \subseteq S \iff \lnot R \subseteq R$
- Complement of \&: $R \cap \perp = \perp$
- Complement of converse inclusion: $\lnot (R \cup S) \subseteq \lnot R$
- Complement of converse: $\lnot (R \cup S) \subseteq \lnot R$
- De Morgan for \&: $\lnot (Q \cap R) = -Q \lor -R$
- De Morgan for \&: $\lnot (Q \cup R) = -Q \land -R$
- De Morgan for \&: $\lnot (Q \cup R) = -Q \land -R$

CalcCheck Structured Proofs

Simple Induction

<table>
<thead>
<tr>
<th>By induction on <code>var : Ty</code>:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base case: <code>?</code>:</td>
</tr>
<tr>
<td>Induction step <code>?</code>:</td>
</tr>
<tr>
<td>... Induction hypothesis <code>?</code></td>
</tr>
</tbody>
</table>

Making base case, induction step, and induction hypothesis explicit:

<table>
<thead>
<tr>
<th>By induction on <code>var : Ty</code>:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base case <code>?</code>:</td>
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<tr>
<td>Induction step <code>?</code>:</td>
</tr>
<tr>
<td>... Induction hypothesis <code>?</code></td>
</tr>
</tbody>
</table>

(remember that in nested inductions, induction hypotheses always need to be made explicit!)

These can also be used for proving theorems of shape

$\forall \: \text{var} : Ty \cdot P$

by induction on precisely that universally-quantified variable, that is, "on `var : Ty`".

The induction hypothesis is then $P$.

Example for sequences:

```
Theorem: $\forall \: x:\text{Seq} A \cdot P$
Proof:
   By induction on `x : Seq A`:
   Base case `P[x = x]`:
     ?
   Induction step `$\forall \: x : A \cdot P[x = x \cdot x]`:
     For any `x`:
       ?
```

Proving Universal Quantifications

```
For any `\: \text{var} : Ty`: `?`
```

Facts that can be shown by "Evaluation"

Only where Evaluation is enabled:

```
Fact `6 ⋅ 7 = 42`
Assuming the Antecedent
```

```
Assuming `p`, `q`:
  ?
  ... Assumption `p` ...
```

Assuming `p` and using with ...

```
  ?
  ... Assumption `p` ...
```

Case Analysis

```
By cases: `p`, `\neg q`, `\neg r`
Completeness:
  ?
  Case `p`:
    ?
    ... Assumption `p` ...
    ...
```

Subproofs

```
  ?
  $(\text{Subproof for `...`})$
  \{ proof indented as far as needed to avoid parse error! \}
)
  
```

Nested subproofs currently may need to be indented even further than first-level subproofs!

Proving Universal Quantifications

```
For any `\: \text{var} : Ty` satisfying `p`:
  ?
  ... Assumption `p` ...
```

Theorems Used as Proof Methods (Examples)

```
Using "Mutual implication":
Subproof for `\: \rightarrow \: `:
  ?
Subproof for `\: \rightarrow \: `:
  ?
```

Using "Extensionality":
Subproof for `\forall \: x \cdot \: `:
  ?

Side Proofs

```
Side proof for `\text{P}`:
  ?
Continuing with goal `\: ?`:
  ?
  ... local property `\text{P}` ...
  ?
```

Disabling Hints Producing Time-outs

Add `?`, "at the beginning of the hint:

```
≡( ?, "Golden rule")
```

```
Selected CalcCheckWeb Key Bindings

(See Getting Started with CalcCheckWeb for the complete listing.)

The following key bindings work the same in both edit and command modes:

- **Ctrl-Enter** performs a syntax check on the contents of all code cells before and up to the current cell.
- **Ctrl-Alt-Enter** performs proof checks (if enabled) on the contents of all code cells before and up to the current cell.
- **Shift-Alt-RightArrow** enlarges the width of the current code cell entry area by a small amount.
- **Ctrl-Shift-Alt-RightArrow** enlarges the width of the current code cell entry area by a large amount.
- **Shift-Alt-LeftArrow** reduces the width of the current code cell entry area by a small amount.
- **Ctrl-Shift-Alt-LeftArrow** reduces the width of the current code cell entry area by a large amount.
- **Ctrl-Shift-v** (for visible spaces) toggles display of initial spaces on each line as \`\` characters.

**ONLY** if you are logged in via Avenue:

- **Ctrl-s** saves the notebook on the server.
  To be safest, use in command mode, e.g. after clicking on the area of a code box where the line number would be displayed.
  Check the pop-up whether it is the CalcCheckWeb pop-up saying “...Notebook saved to...”.

In edit mode, you have the following key bindings:

- **Esc** enters command mode.
- **Alt-i** or **Alt-SPACE** inserts one space in the current line and in all non-empty lines below it, until a line is encountered that is not indented more than to the cursor position.
- **Alt-BACKSPACE** deletes only a space character to the left of the current cursor position, and also from lines below it, until a line is encountered that is not indented at least to the cursor position.
- **Alt-DELETE** deletes only a space character to the right of the current cursor position, and also from lines below it, until a line is encountered that is not indented more than to the cursor position.

The last three bindings also work with **Shift**.

### Some important symbols:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Key sequence(s)</th>
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<tbody>
<tr>
<td>\equiv</td>
<td>\equiv, ==</td>
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<tr>
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