2DM3 2018 — Final Theorem List

Equivalence, Negation and Inequivalence

“Definition of ≡”: $(p \equiv q) = (p = q)$
(3.2) “Symmetry of ≡”: ?
(3.3) “Identity of ≡”: ?
(3.5) “Reflexivity of ≡”: $p \equiv p$
(3.9) “Commutativity of ¬ with ≡” “Distributivity of ¬ over ≡”: ?
(3.11) “¬ connection”: $\neg p \equiv q \equiv p \equiv \neg q$
(3.14): $(p \not\equiv q) \equiv (\neg p \equiv q)$
(3.15): $\neg p \equiv (p \equiv \mathrm{false})$

Disjunction and Conjunction

(3.32): ?
(3.35) “Golden rule”: ?
(3.48): ?
(3.49) “Semi-distributivity of ∧ over ≡”: ?
(3.50) “Strong Modus Ponens”: ?
(3.51) “Replacement”: $(p \equiv q) \wedge (r \equiv p) \equiv (p \equiv q) \wedge (r \equiv q)$
(3.52) “Alternative definition of ≡”: ? $q) \vee (\neg p \wedge \neg q))$
(3.53) “Exclusive or” “Alternative definition of ≢”: ?

Implication

(3.57) “Definition of ⇒”: ?
(3.58) “Definition of ⇐” “Consequence”: ?
(3.59) “Definition of ⇒”: ?
(3.60) “Definition of ⇒”: ?
(3.61) “Contrapositive”: ?
(3.62): $p \Rightarrow (q \equiv r) \equiv (p \wedge q \equiv p \wedge r)$
(3.63) “Distributivity of ⇒ over ≡”: ?
(3.64) “Self-distributivity of ⇒”: ?
(3.65) “Shunting”: ?
(3.66): $p \wedge (p \Rightarrow q) \equiv p \wedge q$
(3.67): $p \wedge (q \Rightarrow p) \equiv p$
(3.68): $p \vee (p \Rightarrow q) \equiv \mathrm{true}$
(3.69): $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$
(3.70): $p \vee q \Rightarrow p \wedge q \equiv (p \equiv q)$
(3.71) “Reflexivity of ⇒”: ?
(3.72) “Right-zero of ⇒”: ?
(3.73) “Left-identity of ⇒”: ?
“Definition of ¬” (3.74): ?
(3.75) “ex falso quodlibet”: ?
(3.76a) “Weakening”: ?
(3.76a) “Weakening”: ?
(3.76b) “Weakening”: ?
(3.76c) “Weakening”: ?
(3.76d) “Weakening”: ?
(3.76e) “Weakening”: ?
“Reflexivity of ⇒”: ?
(3.77) “Modus ponens”: ?
(3.78) “Case analysis”: ?
(3.79) “Case analysis”: ?
(3.80) “Mutual implication”: ?
(3.81) “Antisymmetry of ⇒”: ?
(3.82a) “Transitivity of ⇒”: ?
(3.82b) “Transitivity of ⇒”: ?
(3.82c) “Transitivity of ⇒”: ?
“Implication strengthening”: $p \Rightarrow q \equiv p \Rightarrow p \wedge q$

Leibniz as Axiom and Substitution/Replacement Laws

(3.83) “Leibniz”: $e = f \Rightarrow E[z := e] = E[z := f]$
(3.84a) “Replacement”: $e = f \wedge E[z := e] \equiv e = f \wedge E[z := f]$
(3.84b) “Replacement”: $e = f \Rightarrow E[z := e] \equiv e = f \Rightarrow E[z := f]$
(3.84c) “Replacement”: $q \wedge e = f \Rightarrow E[z := e] \equiv q \wedge e = f \Rightarrow E[z := f]$
“Transitivity of =”: $e = f \wedge f = g \Rightarrow e = g$
(3.85a) “Replace by ‘true’”: $p \Rightarrow E[z := p] \equiv p \Rightarrow E[z := \mathrm{true}]$
(3.85b) “Replace by ‘true’”: $q \wedge p \Rightarrow E[z := p] \equiv q \wedge p \Rightarrow E[z := \mathrm{true}]$
(3.85c) “Replace by ‘false’”: $\neg p \Rightarrow E[z := p] \equiv \neg p \Rightarrow E[z := \mathrm{false}]$
(3.85e) “Replace by ‘true’”: $p \Rightarrow E[z := p] = E[z := \mathrm{true}]$
(3.86a) “Replace by ‘false’”: $E[z := p] \Rightarrow p \equiv E[z := \mathrm{false}] \Rightarrow p$
(3.86b) “Replace by ‘false’”: $E[z := p] \Rightarrow p \vee q \equiv E[z := \mathrm{false}] \Rightarrow p \vee q$
(3.87) “Replace by ‘true’”: $p \wedge E[z := p] \equiv p \wedge E[z := \mathrm{true}]$
(3.88) “Replace by ‘false’”: $p \vee E[z := p] \equiv p \vee E[z := \mathrm{false}]$

Monotonicity with Respect to Implication

(4.2) “Left-monotonicity of ∨” “Monotonicity of ∨”: ?
“Monotonicity of ∨”: $(p \Rightarrow q) \Rightarrow ((r \Rightarrow s) \Rightarrow (p \vee r \Rightarrow q \vee s))$
(4.3) “Left-monotonicity of ∧” “Monotonicity of ∧”: ?
“Monotonicity of ∧”: $(p \Rightarrow p') \Rightarrow ((q \Rightarrow q') \Rightarrow (p \wedge q \Rightarrow p' \wedge q'))$
“Antitonicity of ¬”: ?
“Monotonicity of ⇒” “Right-monotonicity of ⇒”: ?
“Antitonicity of ⇒” “Left-antitonicity of ⇒”: ?

Sum Quantification: General Quantifier Material Instantiated for Sum

“Leibniz for ∑ range”: $(\forall x \bullet R_1 \;\equiv\; R_2) \Rightarrow (\sum x \mid R_1 \bullet E) = (\sum x \mid R_2 \bullet E)$
“Leibniz for ∑ body”: $(\forall x \bullet R \Rightarrow E_1 = E_2) \Rightarrow (\sum x \mid R \bullet E_1) = (\sum x \mid R \bullet E_2)$
(8.13) “Empty range for ∑”: $(\sum x \mid \mathrm{false} \bullet E) = 0$
(8.14) “One-point rule for ∑”: $(\sum x \mid x = D \bullet E) = E[x := D]$ — provided: $\neg$*occurs*(‘x‘, ‘D‘)
(8.15) “Distributivity of ∑ over +”: $(\sum x \mid R \bullet E_1 + E_2) = (\sum x \mid R \bullet E_1) + (\sum x \mid R \bullet E_2)$
(8.17) “Range split”: $(\sum x \mid Q \vee R \bullet E) + (\sum x \mid Q \wedge R \bullet E) = (\sum x \mid Q \bullet E) + (\sum x \mid R \bullet E)$
(8.16) “Disjoint range split for ∑”: $(\forall x \bullet Q \wedge R \;\equiv\; \mathrm{false}) \Rightarrow (\sum x \mid Q \vee R \bullet E) = (\sum x \mid Q \bullet E) + (\sum x \mid R \bullet E)$
(8.20) “Nesting for ∑”: $(\sum x \mid Q \bullet (\sum y \mid R \bullet E)) = (\sum x, y \mid Q \wedge R \bullet E)$
— provided: $\neg$*occurs*(‘y‘, ‘Q‘)
“Replacement in ∑”: $(\sum x \mid R \wedge e = f \bullet E[y := e]) = (\sum x \mid R \wedge e = f \bullet E[y := f])$
“Dummy list permutation for ∑”: $(\sum x, y \mid R \bullet E) = (\sum y, x \mid R \bullet E)$
(8.19) “Interchange of dummies”: $(\sum x \mid Q \bullet (\sum y \mid R \bullet P)) = (\sum y \mid R \bullet (\sum x \mid Q \bullet P))$
— provided: $\neg$*occurs*(‘x‘, ‘R‘), $\neg$*occurs*(‘y‘, ‘Q‘)
(8.21) “Dummy renaming for ∑” “α-conversion”: $(\sum x \mid R \bullet E) = (\sum y \mid R[x := y] \bullet E[x := y])$
— provided: $\neg$*occurs*(‘y‘, ‘E, R‘)

Specific Material for Sum Quantification

“Distributivity of · over $\sum$”: $a \cdot (\sum x \mid R \bullet E) = (\sum x \mid R \bullet a \cdot E)$ — provided: $\neg occurs$(‘x’, ‘a’)
“Zero $\sum$ body”: $(\sum x \mid R \bullet 0) = 0$

“Definition of $\leq$ in terms of $<$”: $a \leq b \;\equiv\; a < b \vee a = b$
“Definition of $\leq$ in terms of ‘S’ and $<$”: $a \leq b \;\equiv\; a < S\,b$
“Split range at top”: $m \leq n \Rightarrow (m \leq i < S\,n \;\equiv\; m \leq i < n \vee i = n)$
“Split off term at top”: $(\sum i : \mathbb{N} \mid i < S\,n \bullet E) = (\sum i : \mathbb{N} \mid i < n \bullet E) + E[i := n]$
— provided: $\neg occurs$(‘i’, ‘n’)
“Split off term at top”: $m \leq n \Rightarrow (\sum i \mid m \leq i < S\,n \bullet E) = (\sum i \mid m \leq i < n \bullet E) + E[i := n]$
— provided: $\neg occurs$(‘i’, ‘m, n’)
“Split off term at top using $\leq$”: $(\sum i \mid i \leq S\,n \bullet E) = (\sum i \mid i \leq n \bullet E) + E[i := S\,n]$
— provided: $\neg occurs$(‘i’, ‘n’)

Universal Quantification

“Leibniz for $\forall$ body”: $(\forall x \mid R \bullet P_1 \;\equiv\; P_2) \Rightarrow ((\forall x \mid R \bullet P_1) \;\equiv\; (\forall x \mid R \bullet P_2))$
(8.18) “Range split for $\forall$”: $(\forall x \mid R \vee S \bullet P) \;\equiv\; (\forall x \mid R \bullet P) \wedge (\forall x \mid S \bullet P)$
(9.5) “Distributivity of $\vee$ over $\forall$”: $P \vee (\forall x \mid R \bullet Q) \;\equiv\; (\forall x \mid R \bullet P \vee Q)$
— provided: $\neg occurs$(‘x’, ‘P’)
(9.6): $P \vee (\forall x \bullet \neg R) \;\equiv\; (\forall x \mid R \bullet P)$ — provided: $\neg occurs$(‘x’, ‘P’)
“Distributivity of $\Rightarrow$ over $\forall$”: $P \Rightarrow (\forall x \mid R \bullet Q) \;\equiv\; (\forall x \mid R \bullet P \Rightarrow Q)$ — prov. $\neg occurs$(‘x’, ‘P’)
(9.7) “Distributivity of $\wedge$ over $\forall$”: $\neg(\forall x \bullet \neg R) \Rightarrow (P \wedge (\forall x \mid R \bullet Q) \;\equiv\; (\forall x \mid R \bullet P \wedge Q))$
— provided: $\neg occurs$(‘x’, ‘P’)

(9.8) “True $\forall$ body”: $(\forall x \mid R \bullet true)$
“Introducing fresh $\forall$”: $P \Rightarrow (\forall x \mid R \bullet P)$ — provided: $\neg occurs$(‘x’, ‘P’)
(9.9) “Sub-distributivity of $\forall$ over $\equiv$”: $(\forall x \mid R \bullet P \;\equiv\; Q) \Rightarrow ((\forall x \mid R \bullet P) \;\equiv\; (\forall x \mid R \bullet Q))$
(9.10) “Range weakening for $\forall$”: $(\forall x \mid Q \vee R \bullet P) \Rightarrow (\forall x \mid Q \bullet P)$
(9.11) “Body weakening for $\forall$”: $(\forall x \mid R \bullet P \wedge Q) \Rightarrow (\forall x \mid R \bullet P)$
(9.12) “Body monotonicity of $\forall$”: $(\forall x \mid R \bullet Q \Rightarrow P) \Rightarrow ((\forall x \mid R \bullet Q) \Rightarrow (\forall x \mid R \bullet P))$
(9.12a) “Range antitonicity of $\forall$”: $(\forall x \bullet Q \Rightarrow R) \Rightarrow ((\forall x \mid R \bullet P) \Rightarrow (\forall x \mid Q \bullet P))$
(9.12a) “Range antitonicity of $\forall$”: $(\forall x \mid \neg P \bullet Q \Rightarrow R) \Rightarrow ((\forall x \mid R \bullet P) \Rightarrow (\forall x \mid Q \bullet P))$
(9.13) “Instantiation”: ?
“Fresh $\forall$”: $P \;\equiv\; (\forall x \bullet P)$ — provided: $\neg occurs$(‘x’, ‘P’)

Existential Quantification

(9.21) “Distributivity of $\wedge$ over $\exists$”: $P \wedge (\exists x \mid R \bullet Q) \;\equiv\; (\exists x \mid R \bullet P \wedge Q)$
— provided: $\neg occurs$(‘x’, ‘P’)
(9.22): $P \wedge (\exists x \bullet R) \;\equiv\; (\exists x \mid R \bullet P)$ — provided: $\neg occurs$(‘x’, ‘P’)
“Distributivity of $\wedge$ over $\forall$”: $(\exists x \bullet R) \Rightarrow (P \wedge (\forall x \mid R \bullet Q) \;\equiv\; (\forall x \mid R \bullet P \wedge Q))$
— provided: $\neg occurs$(‘x’, ‘P’)
(9.23) “Distributivity of $\vee$ over $\exists$”: $(\exists x \bullet R) \Rightarrow (P \vee (\exists x \mid R \bullet Q) \;\equiv\; (\exists x \mid R \bullet P \vee Q))$
— provided: $\neg occurs$(‘x’, ‘P’)
(9.24) “False $\exists$ body”: $(\exists x \mid R \bullet false) \;\equiv\; false$
(9.25) “Range weakening for $\exists$”: $(\exists x \mid R \bullet P) \Rightarrow (\exists x \mid Q \vee R \bullet P)$
“Range weakening for $\exists$”: $(\exists x \mid Q \wedge R \bullet P) \Rightarrow (\exists x \mid R \bullet P)$
(9.26) “Body weakening for $\exists$”: $(\exists x \mid R \bullet P) \Rightarrow (\exists x \mid R \bullet P \vee Q)$
(9.26a) “Body weakening for $\exists$”: $(\exists x \mid R \bullet P \wedge Q) \Rightarrow (\exists x \mid R \bullet P)$
(9.27) “Body monotonicity of $\exists$”: $(\forall x \mid R \bullet Q \Rightarrow P) \Rightarrow ((\exists x \mid R \bullet Q) \Rightarrow (\exists x \mid R \bullet P))$
“Range monotonicity of $\exists$”: $(\forall x \bullet Q \Rightarrow R) \Rightarrow ((\exists x \mid Q \bullet P) \Rightarrow (\exists x \mid R \bullet P))$
“Range monotonicity of $\exists$”: $(\forall x \mid P \bullet Q \Rightarrow R) \Rightarrow ((\exists x \mid Q \bullet P) \Rightarrow (\exists x \mid R \bullet P))$

Introduction and Interchange for $\exists$

(9.28) “$\exists$-Introduction”: ?
(9.29a) “Interchange of quantifications”: $(\exists x \bullet (\forall y \bullet P)) \Rightarrow (\forall y \bullet (\exists x \bullet P))$
(9.30a) “Witness”: $(\exists x \mid R \bullet P) \Rightarrow Q \;\equiv\; (\forall x \bullet R \wedge P \Rightarrow Q)$ — provided: $\neg occurs$(‘x’, ‘Q’)
(9.30b) “Witness”: $(\exists x \bullet P) \Rightarrow Q \;\equiv\; (\forall x \bullet P \Rightarrow Q)$ — provided: $\neg occurs$(‘x’, ‘Q’)

Set Theory

(11.3) “Set membership”: ?
(11.7s) “Simple Membership”: ?
(11.7x) “Simple Membership”: ?
(11.7$\forall$) “Simple Membership”: ?
“Membership in two-element set enumeration”: ?
“Membership in set enumeration”: ?

Set Extensionality and Set Inclusion

(11.4) “Set extensionality” “Set equality” “Extensionality”: ?
(11.9) “Simple set comprehension equality”: ?
(11.13) “Subset” “Definition of $\subseteq$” “Set inclusion”: ?
“Subset” “Definition of $\subseteq$” “Set inclusion”: ?
“Subset membership” “Casting”: ?
(11.58) “Reflexivity of $\subseteq$”: ?
“Reflexivity of $\subseteq$”: ?
(11.59) “Transitivity of $\subseteq$”: ?
“Flipped transitivity of $\subseteq$”: ?
(11.57) “Antisymmetry of $\subseteq$”: ?
“Empty set”: $\{\} = \{ x \mid false \}$
“Empty set”: $x \in \{\} \;\equiv\; false$
“Empty set is least” “Bottom set”: $\{\} \subseteq X$
“Universal set”: $U = \{ x \mid true \}$
“Universal set”: $x \in U$
“Universal set is greatest” “Top set”: $X \subseteq U$
(11.56) “Simple set comprehension inclusion”: $\{ x \mid P \} \subseteq \{ x \mid Q \} \;\equiv\; (\forall x \bullet P \Rightarrow Q)$

Singleton Sets, Set Complement, Set Union and Intersection

“Singleton set membership”: ?
“Singleton set inclusion”: ?
“Complement”: ?
(11.19) “Self-inverse of complement”: ?
“Lower $\sim$ connection for $\subseteq$”: $\sim X \subseteq Y \;\equiv\; \sim Y \subseteq X$
“Upper $\sim$ connection for $\subseteq$”: $X \subseteq \sim Y \;\equiv\; Y \subseteq \sim X$
“Union”: ?
“Intersection”: ?
“Golden rule for $\cap$ and $\cup$”: ?
“Set inclusion via $\cap$”: ?
“Set inclusion via $\cup$”: ?

Proper Subset

(11.14) “Proper subset” “Definition of ⊂”: ?
(11.61): $S \subset T \equiv S \subseteq T \wedge \neg (T \subseteq S)$
(11.61): $S \subset T \equiv S \subseteq T \wedge \neg (T \subseteq S)$
(11.63) “Inclusion in terms of ⊂”: ?
(11.70) “Transitivity of ⊆ with ⊂”: ?
(11.70) “Transitivity of ⊆ with ⊂”: ?

Set Difference and Relative Pseudo-complement

(11.22) “Set difference”: ?
(11.52): $S \cap (T - S) = \{\}$
(11.54): $S - (T \cup U) = (S - T) \cap (S - U)$
“Characterisation of ⇨”: $S \subseteq A \Rightarrow B \equiv S \cap A \subseteq B$
“Membership in ⇨”: $x \in A \Rightarrow B \equiv x \in A \Rightarrow x \in B$
“Definition of ⇨”: $A \Rightarrow B = {\sim} A \cup B$
“Pseudocomplement of union”: $(A \cup B) \Rightarrow C = (A \Rightarrow C) \cap (B \Rightarrow C)$
“Monotonicity of ⇨”: $B \subseteq C \Rightarrow A \Rightarrow B \subseteq A \Rightarrow C$

Cartesian Products of Sets; Relationship

(14.2) “Pair equality”: $\langle b, c \rangle = \langle b', c' \rangle \equiv b = b' \wedge c = c'$
“Definition of ‘fst’”: $\mathrm{fst}\ \langle x, y \rangle = x$
“Definition of ‘snd’”: $\mathrm{snd}\ \langle x, y \rangle = y$
“Membership in ×”: $p \in S \times T \equiv \mathrm{fst}\ p \in S \wedge \mathrm{snd}\ p \in T$
(14.4) “Membership in ×”: $\langle x, y \rangle \in S \times T \equiv x \in S \wedge y \in T$
(14.5) “Membership in swapped ×”: $\langle x, y \rangle \in S \times T \equiv \langle y, x \rangle \in T \times S$
(14.6) “Empty factor in ×”: $S = \{\} \Rightarrow S \times T = \{\}$

“Definition of ↔”: ?
“Infix relationship” “Definition of ‘_⦇_⦈_’”: ?
“Relation extensionality”: ?
“Relation inclusion”: ?
“Relation inclusion”: ?
“Relation inclusion”: ?

Set Operations used as Relation Operations

“Relation union”: ?
“Relation intersection”: ?
“Relation difference”: ?
“Relation pseudocomplement”: $a ⦇ R \Rightarrow S ⦈ b \equiv a ⦇ R ⦈ b \Rightarrow a ⦇ S ⦈ b$
“Relation complement”: ?
“Empty relation”: ?
“Universal relation”: $(\forall A : \mathrm{Type} \bullet (\forall B : \mathrm{Type} \bullet a ⦇ A \times B ⦈ b))$
“Singleton relation”: ?
“Singleton relation inclusion”: ?

Relation-specific Operations

“Relation converse” “Relationship via ˘”: ?
“Relation composition”: ?
“Identity relation” “Relationship via ‘Id’”: ?

Sequences

(13.3) “Cons is not empty”: ?
“Cons is not empty”: ?
(13.4) “Cancellation of ◁”: ?
(13.6) “Cons decomposition”: $xs = \epsilon \vee (\exists\ y \bullet (\exists\ ys \bullet xs = y \lhd ys))$
(13.7) “Tail is different”: $x \lhd xs \neq xs$

Sequence Membership ∈, Snoc ▷

“Membership in ϵ”: ?
“Membership in ◁”: ?
(13.12) “Definition of ▷” “Definition of ▷ for ϵ”: ?
(13.13) “Definition of ▷” “Definition of ▷ for ◁”: ?
(13.14) “Snoc is not empty”: ?
“Snoc is not empty”: ?
(13.15) “Cancellation of ▷”: ?
(13.16) “Membership in ▷”: ?

Concatenation

(13.17) “Left-identity of ⌢” “Definition of ⌢ for ϵ”: ?
(13.18) “Mutual associativity of ◁ with ⌢” “Definition of ⌢ for ◁”: ?

(13.19) “Right-identity of ⌢”: ?
(13.20) “Associativity of ⌢”: ?
(13.21) “Membership in ⌢”: ?
(13.22) “Mutual associativity of ⌢ with ▷”: ?
(13.23) “Empty concatenation”: $xs \frown ys = \epsilon \equiv xs = \epsilon \wedge ys = \epsilon$

Subsequences, Prefix, Segments

(13.25) “Empty subsequence”: $\epsilon \subseteq ys$
(13.26) “Subsequence” “Cons is not a subsequence of ϵ”: $\neg (x \lhd xs \subseteq \epsilon)$
(13.27) “Subsequence anchored at head”: $x \lhd ys \subseteq x \lhd zs \equiv ys \subseteq zs$
(13.28) “Subsequence without head”: $x \neq y \Rightarrow (x \lhd xs \subseteq y \lhd ys \equiv x \lhd xs \subseteq ys)$
(13.29) “Proper subsequence” “Definition of ⊂”: $xs \subset ys \equiv xs \subseteq ys \wedge xs \neq ys$
(13.30) “Reflexivity of ⊆”: $xs \subseteq xs$
(13.31) “Cons ⊆-expands”: $ys \subseteq x \lhd ys$
(13.33) “Subsequence of ϵ”: $xs \subseteq \epsilon \equiv xs = \epsilon$
(13.34) “Membership of subsequence”: $ys \subseteq zs \Rightarrow x \in ys \Rightarrow x \in zs$

(13.36) “Empty prefix”: $\mathrm{isprefix}\ \epsilon\ xs$
(13.37) “Not Prefix” “Cons is not prefix of ϵ”: $\mathrm{isprefix}\ (x \lhd xs)\ \epsilon \equiv \mathrm{false}$
(13.38) “Prefix” “Cons prefix”: $\mathrm{isprefix}\ (x \lhd xs)\ (y \lhd ys) \equiv x = y \wedge \mathrm{isprefix}\ xs\ ys$

(13.39) “Segment” “Segment of ϵ”: $\mathrm{isseg}\ xs\ \epsilon \equiv xs = \epsilon$
(13.40) “Segment” “Segment of ◁”: $\mathrm{isseg}\ xs\ (y \lhd ys) \equiv \mathrm{isprefix}\ xs\ (y \lhd ys) \vee \mathrm{isseg}\ xs\ ys$

Abstract Relation Algebra

“Reflexivity of $\subseteq$”: ?
“Transitivity of $\subseteq$”: ?
“Antisymmetry of $\subseteq$”: ?
“Transitivity of $\subseteq$”: ?
“Flipped Transitivity of $\subseteq$”: ?
“Reflexivity of $\subseteq$”: $R = S \Rightarrow R \subseteq S$
“Mutual inclusion”: ?
“Opposite inclusion”: $R \supseteq S \;\equiv\; S \subseteq R$
“Indirect Relation Equality from above”: ?
“Indirect Relation Equality from below”: ?
“Indirect Relation Inclusion from above”: ?
“Indirect Relation Inclusion from below”: ?

Composition

“Associativity of $\fatsemi$”: ?
“Monotonicity of $\fatsemi$”: ?
“Monotonicity of $\fatsemi$”: ?
“Monotonicity of $\fatsemi$”: ?
“Identity of $\fatsemi$”: ?
“Identity of $\fatsemi$”: ?

Converse

“Self-inverse of $\smile$”: ?
“Cancellation of $\smile$”: ?
“Monotonicity of $\smile$”: ?
“Isotonicity of $\smile$”: ?
“Converse of ‘Id’”: ?
“Converse of $\fatsemi$”: ?

Homogeneous Relation Properties

“Definition of reflexivity”: is-reflexive $R \;\equiv\; Id \subseteq R$
“Definition of symmetry”: is-symmetric $R \;\equiv\; R^{\smile} \subseteq R$
“Definition of transitivity”: is-transitive $R \;\equiv\; R \fatsemi R \subseteq R$
“Definition of idempotency”: ?
“Definition of equivalence”:
?
“Definition of preorder”: ?
“Definition of symmetry”: is-symmetric $R \;\equiv\; R^{\smile} = R$

Heterogeneous Relation Properties

“Definition of univalence”: is-univalent $R \;\equiv\; R^{\smile} \fatsemi R \subseteq Id$
“Definition of totality”: is-total $R \;\equiv\; Id \subseteq R \fatsemi R^{\smile}$
“Definition of injectivity”: is-injective $R \;\equiv\; R \fatsemi R^{\smile} \subseteq Id$
“Definition of surjectivity”: is-surjective $R \;\equiv\; Id \subseteq R^{\smile} \fatsemi R$
“Definition of mappings”: ?
“Definition of bijectivity”: ?
“Definition of mappings”: ?
“Definition of bijectivity”: ?

Relation Algebra: Continuing with Intersection

“Characterisation of $\cap$”: $Q \subseteq R \cap S \;\equiv$?
“Weakening for $\cap$”: ? $\wedge$?
“Symmetry of $\cap$”: ?
“Associativity of $\cap$”: ?
“Idempotency of $\cap$”: ?
“Monotonicity of $\cap$”: ?
“Inclusion via $\cap$”: ?
“Sub-distributivity of $\fatsemi$ over $\cap$”: ?
“Sub-distributivity of $\fatsemi$ over $\cap$”: ?
“Converse of $\cap$”: ?
“Dedekind rule”: $Q \fatsemi R \cap S \subseteq (Q \cap S \fatsemi R^{\smile}) \fatsemi (R \cap Q^{\smile} \fatsemi S)$
“Modal rule”: $Q \fatsemi R \cap S \subseteq (Q \cap S \fatsemi R^{\smile}) \fatsemi R$
“Modal rule”: $Q \fatsemi R \cap S \subseteq Q \fatsemi (R \cap Q^{\smile} \fatsemi S)$
“Hesitation”: $R \subseteq R \fatsemi R^{\smile} \fatsemi R$

Relation Algebra: Continuing with Union

“Characterisation of $\cup$”: $Q \cup R \subseteq S \;\equiv$?
“Weakening for $\cup$”: ? $\wedge$?
“Symmetry of $\cup$”: ?
“Associativity of $\cup$”: ?
“Idempotency of $\cup$”: ?
“Monotonicity of $\cup$”: ?
“Inclusion via $\cup$”: ?
“Distributivity of $\fatsemi$ over $\cup$”: ?
“Distributivity of $\fatsemi$ over $\cup$”: ?
“Union of converses”: ?
“Converse of $\cup$”: ?
“Absorption of $\cup$ by $\cap$”: ?
“Absorption of $\cap$ by $\cup$”: ?
“Distributivity of $\cap$ over $\cup$”: ?
“Distributivity of $\cup$ over $\cap$”: ?

Least Elements in the Inclusion Order

“Least relation”: $\bot \subseteq R$
“Inclusion in $\bot$”: $R \subseteq \bot \;\equiv\; R = \bot$
“Zero of $\cap$”: $\bot \cap R = \bot$
“Identity of $\cup$”: $\bot \cup R = R$
“Converse of $\bot$”: $\bot^{\smile} = \bot$
“Left-zero of $\fatsemi$”: $\bot \fatsemi R = \bot$
“Right-zero of $\fatsemi$”: $R \fatsemi \bot = \bot$

Greatest Elements in the Inclusion Order

“Greatest relation”: $R \subseteq \top$
“Inclusion of $\top$”: $\top \subseteq R \;\equiv\; R = \top$
“Identity of $\cap$”: $\top \cap R = R$
“Zero of $\cup$”: $\top \cup R = \top$
“Converse of $\top$”: $\top^{\smile} = \top$

Relation Algebra: Complement

"Characterisation of ∼": $S \cap R = \bot \wedge S \cup R = \top \;\equiv\; S = {\sim} R$
"Characterisation of ∼": ${\sim} R \cap R = \bot \wedge {\sim} R \cup R = \top$
"Self-inverse of ∼": ${\sim} ({\sim} R) = R$
"Antitonicity of ∼": $Q \subseteq R \Rightarrow {\sim} R \subseteq {\sim} Q$
"Anti-isotonicity of ∼": $Q \subseteq R \;\equiv\; {\sim} R \subseteq {\sim} Q$
"∼ connection": ${\sim} Q \subseteq R \;\equiv\; {\sim} R \subseteq Q$
"∼ connection": $Q \subseteq {\sim} R \;\equiv\; R \subseteq {\sim} Q$
"Cancellation of ∼": ${\sim} Q = {\sim} R \;\equiv\; Q = R$
"Equality with ∼": $Q = {\sim} R \;\equiv\; R = {\sim} Q$

"Complement of ⊤": ${\sim} \top = \bot$
"Converse of complement inclusion": $({\sim} R)\breve{} \subseteq {\sim} R\breve{}$
"Converse of ∼" "Complement of converse" "Complement of ˘": $({\sim} R)\breve{} = {\sim} R\breve{}$
"De Morgan for ∩": ${\sim} (Q \cap R) = {\sim} Q \cup {\sim} R$
"De Morgan for ∪": ${\sim} (Q \cup R) = {\sim} Q \cap {\sim} R$
"Inclusion via intersection with complement" "Inclusion via ∩ ∼": $R \subseteq S \;\equiv\; R \cap {\sim} S \subseteq \bot$
"Inclusion via intersection with complement" "Inclusion via ∩ ∼": $R \subseteq S \;\equiv\; R \cap {\sim} S = \bot$
"Contrapositive of ⊆ with ∩": $Q \cap R \subseteq S \;\equiv\; Q \cap {\sim} S \subseteq {\sim} R$
"Schröder": $Q \,;\, R \subseteq S \;\equiv\; Q\breve{} \,;\, {\sim} S \subseteq {\sim} R$
"Schröder": $Q \,;\, R \subseteq S \;\equiv\; {\sim} S \,;\, R\breve{} \subseteq {\sim} Q$

CalcCheck Structured Proofs

Simple Induction

```
By induction on `var : Ty`:
  Base case:
    ?
  Induction step:
      ?
    ... Induction hypothesis ...
      ?
```

Making base case, induction step, and induction hypothesis explicit:

```
By induction on `var : Ty`:
  Base case `?`:
    ?
  Induction step `?`:
      ?
    ... Induction hypothesis `?` ...
      ?
```

(Remember that in nested inductions, induction hypotheses always need to be made explicit!)

These can also be used for proving theorems of shape

∀ `var : Ty` • P

by induction on precisely that universally-quantified variable, that is, "on `` `var : Ty` ``:".

The induction hypothesis is then P.

Example for sequences:

```
Theorem: ∀ xs : Seq A • P
Proof:
  By induction on `xs : Seq A`:
    Base case `P[xs ≔ ϵ]`:
        ?
    Induction step `∀ x : A • P[xs ≔ x ◃ xs]`:
      For any `x`:
        ?
```

Facts that can be shown by "Evaluation"

Only where Evaluation is enabled:
  Fact `` `6 · 7 = 42` ``

Assuming the Antecedent

```
Assuming `p`, `q`:
    ?
  ... Assumption `p` ...
    ?
```

```
Assuming `p` and using with ...:
    ?
  ... Assumption `p` ...
    ?
```

Case Analysis

```
By cases: `p`, `q`, `r`
  Completeness:
    ?
  Case `p`:
      ?
    ... Assumption `p` ...
      ?
  ...
```

Subproofs

```
   ?
  ≡⟨ Subproof for `...`:
       ⟪ proof indented as far as needed
         to avoid parse error! ⟫
   ⟩
   ?
```

Nested subproofs currently may need to be indented even further than first-level subproofs!

Proving Universal Quantifications

```
For any `var : Ty`:
    ?
```

```
For any `var : Ty` satisfying `p`:
      ?
    ... Assumption `p` ...
      ?
```

Theorems Used as Proof Methods (Examples)

```
Using "Mutual implication":
  Subproof for `... ⇒ ...`:
      ?
  Subproof for `... ⇒ ...`:
      ?
```

```
Using "Extensionality":
  Subproof for `∀ x • ...`:
    For any `x`:
        ?
```

Side Proofs

```
Side proof for `P`:
  ?
Continuing with goal `?`:
    ?
  ... local property `P` ...
    ?
```

Disabling Hints Producing Time-outs

Add "?, " at the beginning of the hint:

```
  ≡⟨ ?, "Golden rule" ⟩
```

Selected CalcCheck$_{\text{Web}}$ Key Bindings

(See Getting Started with CalcCheck$_{\text{Web}}$ for the complete listing.)

The following key bindings work the same in **both edit and command modes**:

`Ctrl-Enter` performs a syntax check on the contents of all code cells before and up to the current cell.

`Ctrl-Alt-Enter` performs proof checks (if enabled) on the contents of all code cells before and up to the current cell.

`Shift-Alt-RightArrow` enlarges the width of the current code cell entry area by a small amount

`Ctrl-Shift-Alt-RightArrow` enlarges the width of the current code cell entry area by a large amount

`Shift-Alt-LeftArrow` reduces the width of the current code cell entry area by a small amount

`Ctrl-Shift-Alt-LeftArrow` reduces the width of the current code cell entry area by a large amount

`Ctrl-Shift-v` (for visible spaces) toggles display of initial spaces on each line as "␣" characters.

> ONLY if you are logged in via Avenue:
> `Ctrl-s` saves the notebook on the server.
> **To be safest, use in command mode, e.g. after clicking on the area of a code box where the line number would be displayed.**
> **Check the pop-up whether it is the CalcCheck-Web pop-up saying "...Notebook saved to ...".**
> (Links for reloading the last three saved versions are displayed when you view the notebook again.)

In **edit mode**, you have the following **key bindings**:

`Esc` enters command mode

`Alt-i` ***or*** `Alt-SPACE` inserts one space in the current line and in all non-empty lines below it, until a line is encountered that is not indented more than to the cursor position.

`Alt-BACKSPACE` deletes **only a space character** to the left of the current cursor position, and also from lines below it, until a line is encountered that is not indented at least to the cursor position.

`Alt-DELETE` deletes **only a space character** to the right of the current cursor position, and also from lines below it, until a line is encountered that is not indented more than to the cursor position.

The last three bindings also work with `Shift`.

Some important symbols:

Symbol	Key sequence(s)
$\equiv$	`\equiv`, `\==`
$\not\equiv$	`\nequiv`
$\neg$	`\lnot`
$\wedge$	`\land`
$\vee$	`\lor`
$\Rightarrow$	`\implies`, `\=>`
$\Leftarrow$	`\follows`
$\neq$	`\neq`
$\forall$	`\forall`
$\exists$	`\exists`
$\sum$	`\sum`
$\prod$	`\product`
$\mid$	`\with`
$\bullet$	`\spot`
$\downarrow$	`\min`
$\uparrow$	`\max`
$\mathbb{B}$	`\BB`, `\bool`
$\mathbb{N}$	`\NN`, `\nat`
$\mathbb{Z}$	`\ZZ`, `\int`
$\in$	`\in`
$\mathbb{P}$	`\PP`, `\powerset`
$\cup$	`\union`
$\cap$	`\intersection`
$\bigcup$	`\bigunion`
$\bigcap$	`\bigintersection`
$\bot$	`\bot`
$\top$	`\top`
⇨	`\pseudocompl`
$\subseteq$	`\subseteq`, `\(=`
$\supseteq$	`\supseteq`, `\)=`
$\subset$	`\subset`
$\supset$	`\supset`
U	`\universe`

Symbol	Key sequence(s)
$\times$	`\times`
$\leftrightarrow$	`\rel`
⦇	`\lrel`, `\((`, `\([`
⦈	`\rrel`, `\))`, `\])`
⨾	`\rcomp`, `\fcomp`, `\;;`
$\breve{}$	`\converse`, `\u{}`
$^{+}$	`\^+`
*	`*`
╱	`\lres`
╲	`\rres`
ϵ	`\eps`, `\emptyseq`
$\triangleleft$	`\cons`
$\triangleright$	`\snoc`
$\frown$	`\catenate`

Contents