2DM3 2018 — Midterm 2 Theorem List — (filled, **not** provided in Midterm 2)

Equivalence, Negation and Inequivalence

"Definition of ≡": $(p \equiv q) = (p = q)$
(3.2) "Symmetry of ≡": $p \equiv q \equiv q \equiv p$
(3.3) "Identity of ≡": $true \equiv q \equiv q$
(3.5) "Reflexivity of ≡": $p \equiv p$
(3.9) "Commutativity of ¬ with ≡" "Distributivity of ¬ over ≡": $\neg (p \equiv q) \equiv (\neg p \equiv q)$
(3.11) "¬ connection": $\neg p \equiv q \equiv p \equiv \neg q$
(3.14): $(p \not\equiv q) \equiv (\neg p \equiv q)$
(3.15): $\neg p \equiv (p \equiv false)$

Disjunction and Conjunction

(3.32): $p \lor q \equiv (p \lor \neg q \equiv p)$
(3.35) "Golden rule": $p \land q \equiv p \equiv q \equiv p \lor q$
(3.48): $p \land q \equiv (p \land \neg q \equiv \neg p)$
(3.49) "Semi-distributivity of ∧ over ≡": $p \land (q \equiv r) \equiv (p \land q \equiv (p \land r \equiv p))$
(3.50) "Strong Modus Ponens": $p \land (q \equiv p) \equiv p \land q$
(3.51) "Replacement": $(p \equiv q) \land (r \equiv p) \equiv (p \equiv q) \land (r \equiv q)$
(3.52) "Alternative definition of ≡": $p \equiv (q \equiv (p \land q) \lor (\neg p \land \neg q))$
(3.53) "Exclusive or" "Alternative definition of ≢": $(p \not\equiv q) \equiv (\neg p \land q) \lor (p \land \neg q)$

Implication

(3.57) "Definition of ⇒": $p \Rightarrow q \equiv (p \lor q \equiv q)$
(3.58) "Definition of ⇐" "Consequence": $p \Leftarrow q \equiv q \Rightarrow p$
(3.59) "Definition of ⇒": $p \Rightarrow q \equiv \neg p \lor q$
(3.60) "Definition of ⇒": $p \Rightarrow q \equiv (p \land q \equiv p)$
(3.61) "Contrapositive": $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$
(3.62): $p \Rightarrow (q \equiv r) \equiv (p \land q \equiv p \land r)$
(3.63) "Distributivity of ⇒ over ≡": $p \Rightarrow (q \equiv r) \equiv (p \Rightarrow q \equiv p \Rightarrow r)$
(3.64) "Self-distributivity of ⇒": $p \Rightarrow (q \Rightarrow r) \equiv (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$
(3.65) "Shunting": $p \land q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$
(3.66): $p \land (p \Rightarrow q) \equiv p \land q$
(3.67): $p \land (q \Rightarrow p) \equiv p$
(3.68): $p \lor (p \Rightarrow q) \equiv true$
(3.69): $p \lor (q \Rightarrow p) \equiv q \Rightarrow p$
(3.70): $p \lor q \Rightarrow p \land q \equiv (p \equiv q)$
(3.71) "Reflexivity of ⇒": $p \Rightarrow p$
(3.72) "Right-zero of ⇒": $p \Rightarrow true$
(3.73) "Left-identity of ⇒": $true \Rightarrow p \equiv p$
"Definition of ¬" (3.74): $p \Rightarrow false \equiv \neg p$
(3.75) "ex falso quodlibet": $false \Rightarrow p$
(3.76a) "Weakening": $p \Rightarrow p \lor q$
(3.76a) "Weakening": $p \Rightarrow p \lor q$
(3.76b) "Weakening": $p \land q \Rightarrow p$
(3.76c) "Weakening": $p \land q \Rightarrow p \lor q$
(3.76d) "Weakening": $p \lor (q \land r) \Rightarrow p \lor q$
(3.76e) "Weakening": $p \land q \Rightarrow p \land (q \lor r)$
"Reflexivity of ⇒": $(p \equiv q) \Rightarrow (p \Rightarrow q)$
(3.77) "Modus ponens": $p \land (p \Rightarrow q) \Rightarrow q$
(3.78) "Case analysis": $(p \Rightarrow r) \land (q \Rightarrow r) \equiv p \lor q \Rightarrow r$
(3.79) "Case analysis": $(p \Rightarrow r) \land (\neg p \Rightarrow r) \equiv r$
(3.80) "Mutual implication": $(p \Rightarrow q) \land (q \Rightarrow p) \equiv (p \equiv q)$
(3.81) "Antisymmetry of ⇒": $(p \Rightarrow q) \land (q \Rightarrow p) \Rightarrow (p \equiv q)$
(3.82a) "Transitivity of ⇒": $(p \Rightarrow q) \land (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
(3.82b) "Transitivity of ⇒": $(p \equiv q) \land (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
(3.82c) "Transitivity of ⇒": $(p \Rightarrow q) \land (q \equiv r) \Rightarrow (p \Rightarrow r)$
"Implication strengthening": $p \Rightarrow q \equiv p \Rightarrow p \land q$

Leibniz as Axiom and Substitution/Replacement Laws

(3.83) "Leibniz": $e = f \Rightarrow E[z := e] = E[z := f]$
(3.84a) "Replacement": $e = f \land E[z := e] \equiv e = f \land E[z := f]$
(3.84b) "Replacement": $e = f \Rightarrow E[z := e] \equiv e = f \Rightarrow E[z := f]$
(3.84c) "Replacement": $q \land e = f \Rightarrow E[z := e] \equiv q \land e = f \Rightarrow E[z := f]$
"Transitivity of =": $e = f \land f = g \Rightarrow e = g$
(3.85a) "Replace by 'true'": $p \Rightarrow E[z := p] \equiv p \Rightarrow E[z := true]$
(3.85b) "Replace by 'true'": $q \land p \Rightarrow E[z := p] \equiv q \land p \Rightarrow E[z := true]$
(3.85c) "Replace by 'false'": $\neg p \Rightarrow E[z := p] \equiv \neg p \Rightarrow E[z := false]$
(3.85e) "Replace by 'true'": $p \Rightarrow E[z := p] = E[z := true]$
(3.86a) "Replace by 'false'": $E[z := p] \Rightarrow p \equiv E[z := false] \Rightarrow p$
(3.86b) "Replace by 'false'": $E[z := p] \Rightarrow p \lor q \equiv E[z := false] \Rightarrow p \lor q$
(3.87) "Replace by 'true'": $p \land E[z := p] \equiv p \land E[z := true]$
(3.88) "Replace by 'false'": $p \lor E[z := p] \equiv p \lor E[z := false]$

Monotonicity with Respect to Implication

(4.2) "Left-monotonicity of ∨" "Monotonicity of ∨": $(p \Rightarrow q) \Rightarrow (p \lor r \Rightarrow q \lor r)$
"Monotonicity of ∨": $(p \Rightarrow q) \Rightarrow ((r \Rightarrow s) \Rightarrow (p \lor r \Rightarrow q \lor s))$
(4.3) "Left-monotonicity of ∧" "Monotonicity of ∧": $(p \Rightarrow q) \Rightarrow (p \land r \Rightarrow q \land r)$
"Monotonicity of ∧": $(p \Rightarrow p') \Rightarrow ((q \Rightarrow q') \Rightarrow (p \land q \Rightarrow p' \land q'))$
"Antitonicity of ¬": $(p \Rightarrow q) \Rightarrow (\neg q \Rightarrow \neg p)$
"Monotonicity of ⇒" "Right-monotonicity of ⇒": $(p \Rightarrow q) \Rightarrow ((r \Rightarrow p) \Rightarrow (r \Rightarrow q))$
"Antitonicity of ⇒" "Left-antitonicity of ⇒": $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$

Sum Quantification: General Quantifier Material Instantiated for Sum

"Leibniz for ∑ range": $(\forall x \bullet R_1 \equiv R_2) \Rightarrow (\sum x \mid R_1 \bullet E) = (\sum x \mid R_2 \bullet E)$
"Leibniz for ∑ body": $(\forall x \bullet R \Rightarrow E_1 = E_2) \Rightarrow (\sum x \mid R \bullet E_1) = (\sum x \mid R \bullet E_2)$
(8.13) "Empty range for ∑": $(\sum x \mid false \bullet E) = 0$
(8.14) "One-point rule for ∑": $(\sum x \mid x = D \bullet E) = E[x := D]$ — provided: ¬*occurs*('x', 'D')
(8.15) "Distributivity of ∑ over +": $(\sum x \mid R \bullet E_1 + E_2) = (\sum x \mid R \bullet E_1) + (\sum x \mid R \bullet E_2)$
(8.17) "Range split": $(\sum x \mid Q \lor R \bullet E) + (\sum x \mid Q \land R \bullet E) = (\sum x \mid Q \bullet E) + (\sum x \mid R \bullet E)$
(8.16) "Disjoint range split for ∑": $(\forall x \bullet Q \land R \equiv false) \Rightarrow (\sum x \mid Q \lor R \bullet E) = (\sum x \mid Q \bullet E) + (\sum x \mid R \bullet E)$
(8.20) "Nesting for ∑": $(\sum x \mid Q \bullet (\sum y \mid R \bullet E)) = (\sum x, y \mid Q \land R \bullet E)$
— provided: ¬*occurs*('y', 'Q')
"Replacement in ∑": $(\sum x \mid R \land e = f \bullet E[y := e]) = (\sum x \mid R \land e = f \bullet E[y := f])$
"Dummy list permutation for ∑": $(\sum x, y \mid R \bullet E) = (\sum y, x \mid R \bullet E)$
(8.19) "Interchange of dummies": $(\sum x \mid Q \bullet (\sum y \mid R \bullet P)) = (\sum y \mid R \bullet (\sum x \mid Q \bullet P))$
— provided: ¬*occurs*('x', 'R'), ¬*occurs*('y', 'Q')
(8.21) "Dummy renaming for ∑" "α-conversion": $(\sum x \mid R \bullet E) = (\sum y \mid R[x := y] \bullet E[x := y])$
— provided: ¬*occurs*('y', 'E, R')

Specific Material for Sum Quantification

"Distributivity of $\cdot$ over $\sum$": $a \cdot (\sum x \mid R \bullet E) = (\sum x \mid R \bullet a \cdot E)$ — provided: $\neg$*occurs*('x', 'a')
"Zero $\sum$ body": $(\sum x \mid R \bullet 0) = 0$

Manipulating Ranges over $\mathbb{N}$

"Definition of $\leq$ in terms of $<$": $a \leq b \equiv a < b \vee a = b$
"Definition of $\leq$ in terms of 'S' and $<$": $a \leq b \equiv a < S\ b$
"Split range at top": $m \leq n \Rightarrow (m \leq i < S\ n \equiv m \leq i < n \vee i = n)$
"Split off term" "Split off term at top": $(\sum i : \mathbb{N} \mid i < S\ n \bullet E) = (\sum i : \mathbb{N} \mid i < n \bullet E) + E[i := n]$ — provided: $\neg$*occurs*('i', 'n')
"Split off term" "Split off term at top": $m \leq n \Rightarrow (\sum i \mid m \leq i < S\ n \bullet E) = (\sum i \mid m \leq i < n \bullet E) + E[i := n]$ — provided: $\neg$*occurs*('i', 'm, n')
"Split off term at top using $\leq$": $(\sum i \mid i \leq S\ n \bullet E) = (\sum i \mid i \leq n \bullet E) + E[i := S\ n]$ — provided: $\neg$*occurs*('i', 'n')

Universal Quantification

"Leibniz for $\forall$ body": $(\forall x \mid R \bullet P_1 \equiv P_2) \Rightarrow ((\forall x \mid R \bullet P_1) \equiv (\forall x \mid R \bullet P_2))$
(8.18) "Range split for $\forall$": $(\forall x \mid R \vee S \bullet P) \equiv (\forall x \mid R \bullet P) \wedge (\forall x \mid S \bullet P)$
(9.5) "Distributivity of $\vee$ over $\forall$": $P \vee (\forall x \mid R \bullet Q) \equiv (\forall x \mid R \bullet P \vee Q)$ — provided: $\neg$*occurs*('x', 'P')
(9.6): $P \vee (\forall x \bullet \neg R) \equiv (\forall x \mid R \bullet P)$ — provided: $\neg$*occurs*('x', 'P')
"Distributivity of $\Rightarrow$ over $\forall$": $P \Rightarrow (\forall x \mid R \bullet Q) \equiv (\forall x \mid R \bullet P \Rightarrow Q)$ — provided: $\neg$*occurs*('x', 'P')
(9.7) "Distributivity of $\wedge$ over $\forall$": $\neg (\forall x \bullet \neg R) \Rightarrow (P \wedge (\forall x \mid R \bullet Q) \equiv (\forall x \mid R \bullet P \wedge Q))$ — provided: $\neg$*occurs*('x', 'P')
(9.8) "True $\forall$ body": $(\forall x \mid R \bullet true)$
"Introducing fresh $\forall$": $P \Rightarrow (\forall x \mid R \bullet P)$ — provided: $\neg$*occurs*('x', 'P')
(9.9) "Sub-distributivity of $\forall$ over $\equiv$": $(\forall x \mid R \bullet P \equiv Q) \Rightarrow ((\forall x \mid R \bullet P) \equiv (\forall x \mid R \bullet Q))$
(9.10) "Range weakening for $\forall$": $(\forall x \mid Q \vee R \bullet P) \Rightarrow (\forall x \mid Q \bullet P)$
(9.11) "Body weakening for $\forall$": $(\forall x \mid R \bullet P \wedge Q) \Rightarrow (\forall x \mid R \bullet P)$
(9.12) "Body monotonicity of $\forall$": $(\forall x \mid R \bullet Q \Rightarrow P) \Rightarrow ((\forall x \mid R \bullet Q) \Rightarrow (\forall x \mid R \bullet P))$
(9.12a) "Range antitonicity of $\forall$": $(\forall x \bullet Q \Rightarrow R) \Rightarrow ((\forall x \mid R \bullet P) \Rightarrow (\forall x \mid Q \bullet P))$
(9.12a) "Range antitonicity of $\forall$": $(\forall x \mid \neg P \bullet Q \Rightarrow R) \Rightarrow ((\forall x \mid R \bullet P) \Rightarrow (\forall x \mid Q \bullet P))$
(9.13) "Instantiation": $(\forall x \bullet P) \Rightarrow P[x := E]$
"Fresh $\forall$": $P \equiv (\forall x \bullet P)$ — provided: $\neg$*occurs*('x', 'P')

Existential Quantification

(9.21) "Distributivity of $\wedge$ over $\exists$": $P \wedge (\exists x \mid R \bullet Q) \equiv (\exists x \mid R \bullet P \wedge Q)$ — provided: $\neg$*occurs*('x', 'P')
(9.22): $P \wedge (\exists x \bullet R) \equiv (\exists x \mid R \bullet P)$ — provided: $\neg$*occurs*('x', 'P')
"Distributivity of $\wedge$ over $\forall$": $(\exists x \bullet R) \Rightarrow (P \wedge (\forall x \mid R \bullet Q) \equiv (\forall x \mid R \bullet P \wedge Q))$ — provided: $\neg$*occurs*('x', 'P')
(9.23) "Distributivity of $\vee$ over $\exists$": $(\exists x \bullet R) \Rightarrow (P \vee (\exists x \mid R \bullet Q) \equiv (\exists x \mid R \bullet P \vee Q))$ — provided: $\neg$*occurs*('x', 'P')
(9.24) "False $\exists$ body": $(\exists x \mid R \bullet false) \equiv false$
(9.25) "Range weakening for $\exists$": $(\exists x \mid R \bullet P) \Rightarrow (\exists x \mid Q \vee R \bullet P)$
"Range weakening for $\exists$": $(\exists x \mid Q \wedge R \bullet P) \Rightarrow (\exists x \mid R \bullet P)$
(9.26) "Body weakening for $\exists$": $(\exists x \mid R \bullet P) \Rightarrow (\exists x \mid R \bullet P \vee Q)$
(9.26a) "Body weakening for $\exists$": $(\exists x \mid R \bullet P \wedge Q) \Rightarrow (\exists x \mid R \bullet P)$
(9.27) "Body monotonicity of $\exists$": $(\forall x \mid R \bullet Q \Rightarrow P) \Rightarrow ((\exists x \mid R \bullet Q) \Rightarrow (\exists x \mid R \bullet P))$
"Range monotonicity of $\exists$": $(\forall x \bullet Q \Rightarrow R) \Rightarrow ((\exists x \mid Q \bullet P) \Rightarrow (\exists x \mid R \bullet P))$
"Range monotonicity of $\exists$": $(\forall x \mid P \bullet Q \Rightarrow R) \Rightarrow ((\exists x \mid Q \bullet P) \Rightarrow (\exists x \mid R \bullet P))$

Introduction and Interchange for $\exists$

(9.28) "$\exists$-Introduction": $P[x := E] \Rightarrow (\exists x \bullet P)$
(9.29a) "Interchange of quantifications": $(\exists x \bullet (\forall y \bullet P)) \Rightarrow (\forall y \bullet (\exists x \bullet P))$
(9.30a) "Witness": $(\exists x \mid R \bullet P) \Rightarrow Q \equiv (\forall x \bullet R \wedge P \Rightarrow Q)$ — provided: $\neg$*occurs*('x', 'Q')
(9.30b) "Witness": $(\exists x \bullet P) \Rightarrow Q \equiv (\forall x \bullet P \Rightarrow Q)$ — provided: $\neg$*occurs*('x', 'Q')

Set Membership Properties

(11.3) "Set membership": $F \in \{ x \mid R \bullet E \} \equiv (\exists x \mid R \bullet F = E)$ — provided: $\neg$*occurs*('x', 'F')
(11.7s) "Simple Membership": $e \in \{ x \mid P \} \equiv P[x := e]$
(11.7x) "Simple Membership": $x \in \{ x \mid P \} \equiv P$
(11.7$\forall$) "Simple Membership": $(\forall x \bullet x \in \{ x \mid P \} \equiv P)$
"Membership in two-element set enumeration": $x \in \{x, y\}$
"Membership in set enumeration": $x \in \{ u \mid u = x \vee R \}$ — provided: $\neg$*occurs*('u', 'x')

Set Extensionality and Set Inclusion

(11.4) "Set extensionality" "Set equality" "Extensionality": $S = T \equiv (\forall e \bullet e \in S \equiv e \in T)$ — provided: $\neg$*occurs*('e', 'S, T')
(11.9) "Simple set comprehension equality": $\{ x \mid Q \} = \{ x \mid R \} \equiv (\forall x \bullet Q \equiv R)$
(11.13) "Subset" "Definition of $\subseteq$" "Set inclusion": $S \subseteq T \equiv (\forall e \mid e \in S \bullet e \in T)$ — provided: $\neg$*occurs*('e', 'S, T')
"Subset" "Definition of $\subseteq$" "Set inclusion": $S \subseteq T \equiv (\forall e \bullet e \in S \Rightarrow e \in T)$ — provided: $\neg$*occurs*('e', 'S, T')
"Subset membership" "Casting": $X \subseteq Y \Rightarrow (x \in X \Rightarrow x \in Y)$
(11.58) "Reflexivity of $\subseteq$": $X \subseteq X$
"Reflexivity of $\subseteq$": $S = T \Rightarrow S \subseteq T$
(11.59) "Transitivity of $\subseteq$": $X \subseteq Y \Rightarrow (Y \subseteq Z \Rightarrow X \subseteq Z)$
"Flipped transitivity of $\subseteq$": $Y \subseteq Z \Rightarrow (X \subseteq Y \Rightarrow X \subseteq Z)$
(11.57) "Antisymmetry of $\subseteq$": $X \subseteq Y \Rightarrow (Y \subseteq X \Rightarrow X = Y)$
"Empty set": $\{\} = \{ x \mid false \}$
"Empty set": $x \in \{\} \equiv false$
"Empty set is least" "Bottom set": $\{\} \subseteq X$
"Universal set": $U = \{ x \mid true \}$
"Universal set": $x \in U$
"Universal set is greatest" "Top set": $X \subseteq U$
(11.56) "Simple set comprehension inclusion": $\{ x \mid P \} \subseteq \{ x \mid Q \} \equiv (\forall x \bullet P \Rightarrow Q)$

Singleton Sets, Set Complement, Set Union and Intersection

"Singleton set membership": $x \in \{y\} \equiv x = y$
"Singleton set inclusion": $\{x\} \subseteq S \equiv x \in S$
"Complement": $e \in \sim S \equiv \neg (e \in S)$
(11.19) "Self-inverse of complement": $\sim (\sim S) = S$
"Lower $\sim$ connection for $\subseteq$": $\sim X \subseteq Y \equiv \sim Y \subseteq X$
"Upper $\sim$ connection for $\subseteq$": $X \subseteq \sim Y \equiv Y \subseteq \sim X$
"Union": $e \in S \cup T \equiv e \in S \vee e \in T$
"Intersection": $e \in S \cap T \equiv e \in S \wedge e \in T$
"Golden rule for $\cap$ and $\cup$": $S \cap T = S \equiv T = S \cup T$

"Set inclusion via ∩": $S \subseteq T \equiv S \cap T = S$
"Set inclusion via ∪": $S \subseteq T \equiv S \cup T = T$

Proper Subset

(11.14) "Proper subset" "Definition of ⊂": $S \subset T \equiv S \subseteq T \land S \neq T$
(11.61): $S \subset T \equiv S \subseteq T \land \neg (T \subseteq S)$
(11.61): $S \subset T \equiv S \subseteq T \land \neg (T \subseteq S)$
(11.63) "Inclusion in terms of ⊂": $S \subseteq T \equiv S \subset T \lor S = T$
(11.70) "Transitivity of ⊆ with ⊂": $X \subseteq Y \Rightarrow (Y \subset Z \Rightarrow X \subset Z)$
(11.70) "Transitivity of ⊆ with ⊂": $X \subseteq Y \Rightarrow (Y \subset Z \Rightarrow X \subset Z)$

Set Difference and Relative Pseudo-complement

(11.22) "Set difference": $v \in S - T \equiv v \in S \land \neg (v \in T)$
(11.52): $S \cap (T - S) = \{\}$
(11.54): $S - (T \cup U) = (S - T) \cap (S - U)$
"Characterisation of ⇨": $S \subseteq A ⇨ B \equiv S \cap A \subseteq B$
"Membership in ⇨": $x \in A ⇨ B \equiv x \in A \Rightarrow x \in B$
"Definition of ⇨": $A ⇨ B = {\sim} A \cup B$
"Pseudocomplement of union": $(A \cup B) ⇨ C = (A ⇨ C) \cap (B ⇨ C)$
"Monotonicity of ⇨": $B \subseteq C \Rightarrow A ⇨ B \subseteq A ⇨ C$

Relations via Set Theory

"Definition of ↔": $A \leftrightarrow B = \mathbb{P}(A \times B)$

"Infix relationship" "Definition of '_⦗_⦘_'": $a ⦗ R ⦘ b \equiv \langle a, b \rangle \in R$
"Relation extensionality": $R = S \equiv (\forall x \bullet (\forall y \bullet x ⦗ R ⦘ y \equiv x ⦗ S ⦘ y))$
— provided: $\neg occurs$('x, y', 'R, S')
"Relation inclusion": $R \subseteq S \equiv (\forall x \bullet (\forall y \bullet x ⦗ R ⦘ y \Rightarrow x ⦗ S ⦘ y))$
— provided: $\neg occurs$('x, y', 'R, S')
"Relation inclusion": $R \subseteq S \equiv (\forall x \bullet (\forall y \mid x ⦗ R ⦘ y \bullet x ⦗ S ⦘ y))$
— provided: $\neg occurs$('x, y', 'R, S')
"Relation inclusion": $R \subseteq S \equiv (\forall x, y \mid x ⦗ R ⦘ y \bullet x ⦗ S ⦘ y)$ — provided: $\neg occurs$('x, y', 'R, S')

Set Operations used as Relation Operations

"Relation union": $a ⦗ R \cup S ⦘ b \equiv a ⦗ R ⦘ b \lor a ⦗ S ⦘ b$
"Relation intersection": $a ⦗ R \cap S ⦘ b \equiv a ⦗ R ⦘ b \land a ⦗ S ⦘ b$
"Relation difference": $a ⦗ R - S ⦘ b \equiv a ⦗ R ⦘ b \land \neg (a ⦗ S ⦘ b)$
"Relation pseudocomplement": $a ⦗ R ⇨ S ⦘ b \equiv a ⦗ R ⦘ b \Rightarrow a ⦗ S ⦘ b$
"Relation complement": $a ⦗ {\sim} R ⦘ b \equiv \neg (a ⦗ R ⦘ b)$
"Empty relation": $a ⦗ \{\} ⦘ b \equiv \mathrm{false}$
"Universal relation": $(\forall A : \mathrm{Type} \bullet (\forall B : \mathrm{Type} \bullet a ⦗ A \times B ⦘ b))$
"Singleton relation": $a_1 ⦗ \{\langle a_2, b_2 \rangle\} ⦘ b_1 \equiv a_1 = a_2 \land b_1 = b_2$
"Singleton relation inclusion": $\{\langle a, b \rangle\} \subseteq R \equiv a ⦗ R ⦘ b$

Relation-specific Operations

"Relation converse" "Relationship via ˘": $y ⦗ R˘ ⦘ x \equiv x ⦗ R ⦘ y$
"Relation composition": $a ⦗ R ⨾ S ⦘ c \equiv (\exists b \bullet a ⦗ R ⦘ b \land b ⦗ S ⦘ c)$
"Identity relation" "Relationship via 'Id'": $x ⦗ \mathrm{Id} ⦘ y \equiv x = y$

CalcCheck Structured Proofs

Simple Induction

```
By induction on `var : Ty`:
  Base case:
    ?
  Induction step:
      ?
    ... Induction hypothesis ...
      ?
```

Making base case, induction step, and induction hypothesis explicit:

```
By induction on `var : Ty`:
  Base case `?`:
    ?
  Induction step `?`:
      ?
    ... Induction hypothesis `?` ...
      ?
```

(Remember that in nested inductions, induction hypotheses always need to be made explicit!)

These can also be used for proving theorems of shape

$\forall$ `var :  Ty` $\bullet$ P

by induction on precisely that universally-quantified variable, that is, "on `` `var : Ty` ``:".
The induction hypothesis is then P.
Example for sequences:

```
Theorem: ∀ xs : Seq A • P
Proof:
  By induction on `xs : Seq A`:
    Base case `P[xs ≔ ϵ]`:
        ?
    Induction step `∀ x : A • P[xs ≔ x ◃ xs]`:
      For any `x`:
          ?
```

Assuming the Antecedent

```
Assuming `p`, `q`:
    ?
  ... Assumption `p` ...
    ?
```

```
Assuming `p` and using with ...:
    ?
  ... Assumption `p` ...
    ?
```

Case Analysis

```
By cases: `p`, `q`, `r`
  Completeness:
    ?
  Case `p`:
      ?
    ... Assumption `p` ...
      ?
  ...
```

Subproofs

```
   ?
 ≡⟨ Subproof for `...`:
        ⟪ proof indented as far as needed
          to avoid parse error! ⟫
   ⟩
   ?
```

Nested subproofs currently may need to be indented even further than first-level subproofs!

Proving Universal Quantifications

```
For any `var : Ty`:
    ?
```

```
For any `var : Ty` satisfying `p`:
      ?
    ... Assumption `p` ...
      ?
```

Theorems Used as Proof Methods (Examples)

```
Using “Mutual implication”:
  Subproof for `... ⇒ ...`:
      ?
  Subproof for `... ⇒ ...`:
      ?
```

```
Using “Extensionality”:
  Subproof for `∀ x • ...`:
    For any `x`:
        ?
```

Side Proofs

```
Side proof for `P`:
  ?
Continuing with goal `?`:
    ?
  ... local property `P` ...
    ?
```

Disabling Hints Producing Time-outs

Add “`?, `” at the beginning of the hint:

```
    ≡⟨ ?, “Golden rule” ⟩
```

Selected CalcCheck$_{\text{Web}}$ Key Bindings

(See Getting Started with CalcCheck$_{\text{Web}}$ for the complete listing.)

The following key bindings work the same in **both edit and command modes**:

`Ctrl-Enter` performs a syntax check on the contents of all code cells before and up to the current cell.

`Ctrl-Alt-Enter` performs proof checks (if enabled) on the contents of all code cells before and up to the current cell.

`Shift-Alt-RightArrow` enlarges the width of the current code cell entry area by a small amount

`Ctrl-Shift-Alt-RightArrow` enlarges the width of the current code cell entry area by a large amount

`Shift-Alt-LeftArrow` reduces the width of the current code cell entry area by a small amount

`Ctrl-Shift-Alt-LeftArrow` reduces the width of the current code cell entry area by a large amount

`Ctrl-Shift-v` (for visible spaces) toggles display of initial spaces on each line as “␣” characters.

ONLY if you are logged in via Avenue:

`Ctrl-s` saves the notebook on the server.
(Links for reloading the last three saved versions are displayed when you the notebook again later.)

In **edit mode**, you have the following **key bindings**:

`Esc` enters command mode

`Alt-SPACE` ***or*** `Alt-i` inserts one space in the current line and in all non-empty lines below it, until a line is encountered that is not indented more than to the cursor position.

`Alt-BACKSPACE` deletes **only a space character** to the left of the current cursor position, and also from lines below it, until a line is encountered that is not indented at least to the cursor position.

`Alt-DELETE` deletes **only a space character** to the right of the current cursor position, and also from lines below it, until a line is encountered that is not indented more than to the cursor position.

The last three bindings also work with the `Shift` *key pressed.*

Some important symbols:

Symbol	Key sequence(s)
⇒	`\implies`, `\=>`
⇐	`\follows`
≢	`\nequiv`
≠	`\neq`
∀	`\forall`
∃	`\exists`
∑	`\sum`
∏	`\product`
$\mid$	`\with`
•	`\spot`
↓	`\min`
↑	`\max`
𝔹	`\BB`, `\bool`
ℕ	`\NN`, `\nat`
ℤ	`\ZZ`, `\int`
∈	`\in`
ℙ	`\PP`, `\powerset`
∪	`\union`
∩	`\intersection`
⇨	`\pseudocompl`
⊆	`\subseteq`
⊂	`\subset`
U	`\universe`
×	`\times`
↔	`\rel`
⦇	`\lrel`, `\((`, `\([`
⦈	`\rrel`, `\))`, `\])`
⨾	`\rcomp`, `\fcomp`, `\;;`
˘	`\converse`, `\u{}`
╱	`\lres`
╲	`\rres`
ϵ	`\eps`, `\emptyseq`
◁	`\cons`
▷	`\snoc`
⌢	`\catenate`