McMaster University
Department of Computing and Software
Dr. Wolfram Kahl

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Handout 1

Design and Selection of Programming Languages

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Some Notation

- For a **set** S, its **cardinality** is written $\#S$ and its **power set** $\mathbb{P}\,S$.
- The **empty set** may be written $\{\}$ or $\varnothing$.
- **Set comprehension** is frequently written using the pattern

$$\{\mathit{declaration} \mid \mathit{predicate} \bullet \mathit{term}\}$$

  and denotes the set containing those values of *term* that arise from binding variables in the *declaration* to elements satisfying the *predicate*. For example:

$$\{n : \mathbb{N} \mid n < 5 \bullet n^2\} = \{0, 1, 4, 9, 16\}$$

  If the *predicate* is omitted, it is understood to be *true*. If the *term* is omitted, it is understood to be the tuple of all variables introduced in the *declaration*, in the same order.

- **Quantification** also follows this pattern:

$$\begin{array}{lcl} \forall\, \mathit{declaration} \mid \mathit{predicate} \bullet \mathit{formula} & \Leftrightarrow & \forall\, \mathit{declaration} \bullet (\mathit{predicate} \Rightarrow \mathit{formula}) \\ \exists\, \mathit{declaration} \mid \mathit{predicate} \bullet \mathit{formula} & \Leftrightarrow & \exists\, \mathit{declaration} \bullet (\mathit{predicate} \wedge \mathit{formula}) \end{array}$$

- The **cartesian product** of two sets A and B is written $A \times B$; it can be considered as defined in the following way:

$$A \times B = \{a : A;\ b : B \bullet (a, b)\}$$

- Given two sets A and B, a **relation** from A to B is a subset of $A \times B$; the set of all relations from A to B is denoted by $A \leftrightarrow B$; we therefore have:

$$A \leftrightarrow B = \mathbb{P}\,(A \times B)$$

- The **domain** (of definition) of a relation $R : A \leftrightarrow B$ is the following subset of A:

$$\mathsf{dom}\, R = \{a : A \mid (\exists\, b : B \bullet (a, b) \in R)\}$$

  The **range** of a relation $R : A \leftrightarrow B$ is the following subset of B:

$$\mathsf{ran}\, R = \{b : B \mid (\exists\, a : A \bullet (a, b) \in R)\}$$

- For every set A, the **identical relation** may be written $\mathrm{id}\, A$ or ${}_A$; we have:

$$\mathrm{id}\, A = {}_A = \{a : A \bullet (a, a)\}$$

- For two relations $R : A \leftrightarrow B$ and $S : B \leftrightarrow C$, their **composition** $R{;}S$ is an element of $A \leftrightarrow C$, and is defined as follows:

$$R{;}S = \{a : A;\ c : C \mid (\exists\, b : B \bullet (a, b) \in R \wedge (b, c) \in S)\}$$

- Every relation $R : A \leftrightarrow B$ has a **converse** (transposed) relation $R^{\smile} : B \leftrightarrow A$ with:

$$R^{\smile} = \{a : A;\ b : B \mid (a, b) \in R \bullet (b, a)\}$$

- A relation $R : A \leftrightarrow B$ is called
  - **univalent** (a **function**) iff $R^{\smile};R \subseteq {}_B$, or, equivalently, iff

    $$\forall\, x : A;\ y_1, y_2 : B \bullet (x, y_1) \in R \land (x, y_2) \in R \Rightarrow y_1 = y_2 \ \ ;$$

  - **total** iff ${}_A \subseteq R;R^{\smile}$, or, equivalently, iff $\mathsf{dom}\, R = A$, or equivalently, iff

    $$\forall\, x : A \bullet \exists\, y : B \bullet (x, y) \in R \ \ ;$$

  - **injective** iff $R;R^{\smile} \subseteq {}_A$, or equivalently, iff

    $$\forall\, x_1, x_2 : A;\ y : B \bullet (x_1, y) \in R \land (x_2, y) \in R \Rightarrow x_1 = x_2 \ \ ;$$

  - **surjective** iff ${}_B \subseteq R^{\smile};R$, or, equivalently, iff $\mathsf{ran}\, R = B$, or equivalently, iff

    $$\forall\, y : B \bullet \exists\, x : A \bullet (x, y) \in R \ \ ;$$

  - a **mapping** iff R is a total function;
  - **bijective** iff R is injective and surjective.

  The following notations are used:

  - $A \nrightarrow B$ is the set of all partial functions (i.e., univalent relations) from A to B;
  - $A \rightarrow B$ is the set of all mappings (i.e., total functions) from A to B;
  - $A \rightarrowtail\!\!\!\!\!+\ B$ is the set of all univalent and injective relations from A to B;
  - $A \rightarrowtail B$ is the set of all injective mappings from A to B;
  - $A \twoheadrightarrow\!\!\!\!\!+\ B$ is the set of all univalent and surjective relations from A to B;
  - $A \twoheadrightarrow B$ is the set of all surjective mappings from A to B;
  - $A \rightarrowtail\!\!\!\!\twoheadrightarrow B$ is the set of all bijective mappings from A to B.

- A relation $R : A \leftrightarrow A$, i.e., where source and target are identical, is called *homogenous*. A homogeneous relation $R : A \leftrightarrow A$ is called:

  - **reflexive** iff ${}_A \subseteq R$;
  - **irreflexive** iff ${}_A \cap R = \varnothing$;
  - **symmetric** iff $R = R^{\smile}$;
  - **asymmetric** iff $R \cap R^{\smile} = \varnothing$;
  - **antisymmetric** iff $R \cap R^{\smile} \subseteq {}_A$;
  - **transitive** iff $R;R \subseteq R$;
  - a **preorder** iff R is reflexive and transitive;
  - an **order** iff R is an antisymmetric preorder;
  - an **equivalence** iff R is reflexive, symmetric, and transitive;
  - a **partial equivalence relation (PER)** iff R is symmetric and transitive.