Exercise 11.1 — Correctness Proof for Gödel Numbering — 20% of Final 2004

Let int variables \( a, b, p, \) and \( i, \) and the following program fragment \( P \) in a Pascal-like programming language be given:

\[
(a, i) := (p, 0);

\text{while } a > i \text{ do}
\]
\[
i := i + 1;
\]
\[
a := a - i
\]
\[\text{od;}
\]
\[
b := i - a;
\]

Prove partial correctness of \( P \) with respect to the precondition \( \{ p \geq 0 \} \) and the postcondition

\[
p = \frac{(a + b)(a + b + 1)}{2} + a \land a \geq 0 \land b \geq 0
\]
documenting all intermediate steps, and showing also the implications used.

**Hint:** For producing this proof, you need no creativity at all, but a high degree of diligence.

**Background:** \( P \) decodes the natural number stored in \( p \) as a pair \((a, b)\) of two natural numbers; this encoding is a simple kind of Gödel numbering.

Exercise 11.2 — Axiomatic Semantics: Partial Correctness Proof — 24% of Final, 2003

Consider the following program fragment in a language providing a Java-like printing statement, given an \( n \)-element Java-like array \( a \):

\[
(i, m) := (0, 0);

\text{while } i \neq n \text{ do}
\]
\[
(i, m) := (i + 1, (m * i + a[i]) / (i + 1));
\]
\[
\text{println}(i + " + m)
\]
\[\text{od}
\]

(a) What is the output of this program for \( n = 5 \) and \( a \) containing the sequence \( 4, 2, 9, 1, 4 \)?

(b) What does this program do? (Short verbal description.)

(c) For this program without the println statement, prove partial correctness with respect to the precondition \( \{ n \geq 0 \} \) and the postcondition \( \{ m \cdot n = \sum_{j=0}^{n-1} a[j] \} \).

**Important:** Justify implications you use, and pay attention to definedness of operations!