Name: ....
Student Number ....

Software Engineering 2F04

DAY CLASS — Dr. Mark Lawford
DURATION OF EXAMINATION: 2 Hours
McMaster University Midterm Examination — October 23, 2001

THIS EXAMINATION PAPER INCLUDES 4 PAGES AND 3 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

Special Instructions: The use of calculators, notes, and text books is not permitted during this exam. Answer all questions in the provided answer booklets. Fill in your name and student number and sign each booklet you use.

1. **Propositional Semantics: Building the World with NOR** (35 marks)

   A common type of logic gate that is compact and easy to build is the NOR gate. Just as P NAND Q was the negation of P and Q (i.e. $\neg(P \wedge Q)$), P NOR Q is the negation of $P \vee Q$ (i.e. $\neg(P \vee Q)$).

   a) (2 marks) Write down the truth table for P NOR Q.

   b) Check if NOR is:

   i) (2 marks) commutative

   ii) (4 marks) associative

   c) (2 marks) Write down the simplest propositional formula ϕ that *only uses* the NOR operator such that $\phi \equiv \neg p$.

   d) (3 marks) Write down the simplest contradiction that uses only the propositional variable p and the binary operator NOR (i.e. find the simplest formula ψ using p and NOR such that $\psi \equiv \bot$).

   e) (3 marks) Create a tautology using only NOR and the propositional variable p.

   f) (10 marks) Show that you can "build the world" with NOR gates, that is, show that any propositional formula can be written in term of the NOR operator by showing how the operators $\neg, \vee, \wedge, \rightarrow$ and $\leftrightarrow$ can be written in terms of NOR.

   g) (4 marks) Write the simplest formula possible for $(\neg p \wedge \neg q)$ only using the propositional variables p, q, and the NOR operator.

   h) (5 marks) Consider the formula:

   $$(p \vee q) \wedge (r \vee s)$$

   It is in Conjunctive Normal Form so it has only two "levels" of gates, i.e., travelling from a propositional variable (atom) in the parse tree to the root of the parse tree, we encounter 2 operator nodes first $\vee$, then $\wedge$.

   NOR gates are particularly well suited to directly implement formulas that are in Conjunctive Normal Form (CNF). Write down a formula with two levels of NOR gates that is equivalent to $(p \vee q) \wedge (r \vee s)$.

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2. Propositional Syntax (and Semantics) (35 marks total)

a) (10 marks) Formally prove using only the proof rules attached on page 4 that

$$\vdash (s \wedge m \rightarrow \neg l) \rightarrow (s \rightarrow \neg(m \wedge l))$$

b) (5 marks) Show that

$$(s \rightarrow \neg(m \wedge l)) \vdash (s \wedge m \rightarrow \neg l)$$

by the means of your choice. Provide proper justification for your answer.

c) (5 marks) Is $(s \wedge m \rightarrow \neg l) \equiv (s \rightarrow \neg(m \wedge l))$? Justify your answer.

d) (5 marks) In PVS when when you are asked to find a proof of the form $\Gamma, \neg\phi \vdash \psi$, the `(FLATTEN)` command changes this into a proof obligation of the form $\Gamma \vdash \phi \vee \psi$
Use the proof rules on page to show why if $\Gamma \vdash \phi \vee \psi$, then $\Gamma, \neg\phi \vdash \psi$.

e) (10 marks) We would like to determine if the following is a valid argument:

$$a \rightarrow (b \leftrightarrow c), b \rightarrow d, b \vee c \rightarrow \neg d \overset{?}{\models} \neg d \rightarrow \neg a$$

i) Write down the PVS input (definitions and theorem) that you would type into PVS to determine if the above is a valid argument.

ii) Applying the command `(BDDSIMP)` to this theorem results in the following sequent:

```
Question2d :

{-1}  a
  |-------
{1}   b
{2}   c
{3}   d
```

Use this to find a counter example that shows that the argument is not valid. Check that it is indeed a counter example.

3. Predicate Logic (30 Marks Total)

The formal language of Predicate Logic makes use of the connectives

$$\wedge, \rightarrow, \neg, \leftrightarrow, \vee$$

and quantifiers $\forall, \exists$.

To reduce the number of parentheses required when writing down formulas, one typically assigns an order of precedence to the connectives and quantifiers from those with the highest precedence (do first!) to those with the lowest precedence (do last!).

a) (2 marks) What is the order of precedence employed by:

i) The textbook by Huth+Ryan?

ii) The automated proof assistant PVS?

b) (10 marks) For the following formula which we will refer to as ϕ:

$$\exists x P(f(x), y) \wedge Q(x) \rightarrow \neg Q(x) \vee P(y, f(y))$$

Continued on page 3

i) Assuming Huth+Ryan's order of precedence, draw the parse tree for the formula ϕ.

ii) What are the free variable of this formula?

iii) What variables occur bound in this formula?

iv) For ϕ as defined above, write down the formulas resulting from each of the following substitutions. Indicate which of the substitutions are vaild.

(i) $\phi[u|y]$ (ii) $\phi[f(y)|y]$ (iii) $\phi[u|x]$ (iv) $\phi[y|x]$

c) (5 marks) In the following let:

$$\begin{aligned} m(x,y) &:= x\text{'s mark in } y. \\ S(x) &:= x \text{ is a student.} \\ C(y) &:= y \text{ is a class.} \end{aligned}$$

Using the standard interpretations for $=$ and $>$, translate the following sentence into predicate logic using the above symbols and meanings:

"There is a student such that in every class she has the highest mark."

d) Consider the following sequence of formulas:

$$\begin{aligned} \Gamma &:= \phi_1,\ \phi_2 \\ &= \forall x Q(f(f(x)),x),\ \exists x(Q(f(x),x) \land \forall y(\neg Q(y,x) \to \neg Q(y,f(y)))) \end{aligned}$$

The intended interpretation for these formulas is given by the model $\mathcal{M}$ with:

$$\begin{aligned} A &:= \mathbb{Z} = \{\ldots -2,-1,0,1,2,\ldots\} \text{ is the universe,} \\ Q^{\mathcal{M}} &:= \{(x,y) \in \mathbb{Z}^2 | x = y\} \text{ is the interpretation of symbol } Q, \end{aligned}$$

and the interpretation of f, denoted $f^{\mathcal{M}}$, is yet to be determined.

In your summer job as a software developer[1] you are given Γ as a formal specification and told to create two different versions of $f^{\mathcal{M}}$.

i) (5 marks) First, find an $f^{\mathcal{M}}$ such that for this interpretation of f we have $\mathcal{M} \not\models \phi_1$ and $\mathcal{M} \models \phi_2$.

ii) (8 marks) Next, find a different $f^{\mathcal{M}}$ such that for this interpretation of f we have $\mathcal{M} \models \Gamma$.

"You know when you start doing all those funny symbols? Well I don't understand any of that." - 2F04 student

[1] Now that this program is accredited, you aren't a "real" software engineer until you graduate, get the required experience, and the pass the Law & Ethics exams.

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Proof Rules

	introduction	elimination
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi} \wedge i$	$\dfrac{\phi \wedge \psi}{\phi} \wedge e_1 \quad \dfrac{\phi \wedge \psi}{\psi} \wedge e_2$
$\vee$	$\dfrac{\phi}{\phi \vee \psi} \vee i_1 \quad \dfrac{\psi}{\phi \vee \psi} \vee i_2$	$\dfrac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi \\ \vdots \\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi \\ \vdots \\ \chi\end{array}}}{\chi} \vee e$
$\rightarrow$	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi} \rightarrow e$	$\dfrac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \rightarrow \psi} \rightarrow i$
$\leftrightarrow$	$\dfrac{\phi \rightarrow \psi \quad \psi \rightarrow \phi}{\phi \leftrightarrow \psi} \leftrightarrow i$	$\dfrac{\phi \leftrightarrow \psi}{\phi \rightarrow \psi} \leftrightarrow e_1 \quad \dfrac{\phi \leftrightarrow \psi}{\psi \rightarrow \phi} \leftrightarrow e_2$
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \bot\end{array}}}{\neg\phi} \neg i$	$\dfrac{\phi \quad \neg\phi}{\bot} \neg e$
$\neg\neg$	$\dfrac{\phi}{\neg\neg\phi} \neg\neg i$	$\dfrac{\neg\neg\phi}{\phi} \neg\neg e$
$\bot$	see $\neg e$	$\dfrac{\bot}{\phi} \bot e$

Additional Proof Rules

$$\dfrac{\phi \rightarrow \psi}{\neg\phi \vee \psi} \rightarrow 2\vee \qquad \dfrac{\neg\phi \vee \psi}{\phi \rightarrow \psi} \vee 2 \rightarrow$$

$$\dfrac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi} \mathrm{MT} \qquad \dfrac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi} \mathrm{RAA}$$

$$\dfrac{}{\phi \vee \neg\phi} \mathrm{LEM}$$

The End