Higher Order Logic

Outline

- Motivation
- Identifying Sets with Predicates
- Higher Order Logic and Higher Order WFF
- The Order of a
  1. Predicate
  2. Quantifier
  3. WFF
  4. Logic
- Applications

Motivation

How could you try to express reachability on an arbitrary graph using predicate logic?

Let $R(x, y)$ be the transition relation of the graph, then state u could reach v if:

$$\begin{array}{lcr} \text{0 steps} & u = v & \vee \\ \text{1 steps} & R(u, v)) & \vee \\ \text{2 steps} & \exists x_1 (R(u, x_1) \wedge R(x, v)) & \vee \\ \text{3 steps} & \exists x_1 \exists x_2 (R(u, x_1) \wedge R(x_1, x_2) \wedge R(x_2, v)) & \vee \\ \vdots & \vdots & \vdots \end{array}$$

But this is an infinitely long formula and hence not a WFF of predicate logic.

Motivation

Predicate Logic is also often referred to as *1st order logic*. We'll see why shortly.

Currently there are some important things regarding software that we cannot say in 1st order logic.

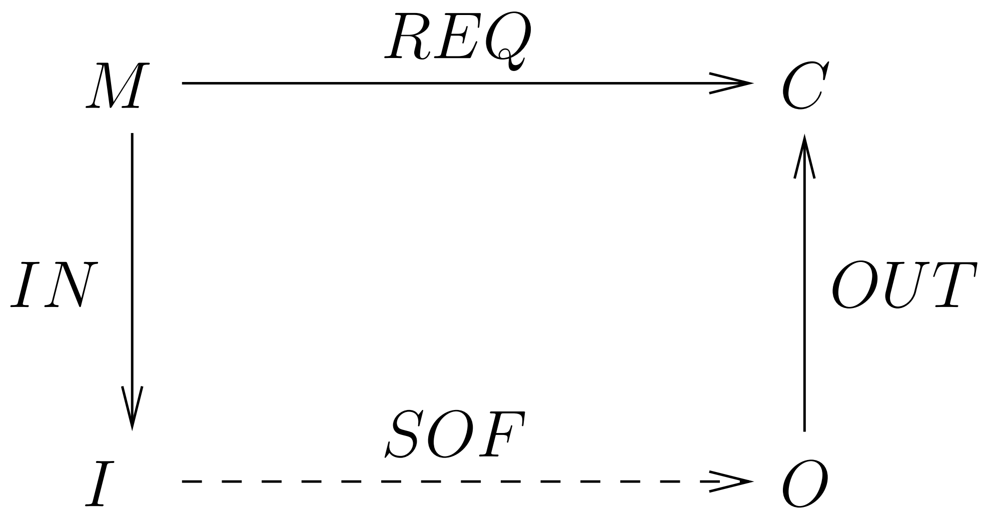
For example: Formally stating the rule for mathematical induction (MI).

$$\forall P(P(0) \wedge (\forall m(P(m) \rightarrow P(s(m)))) \rightarrow (\forall n P(n)))$$

where the intended interpretation $s^{\mathcal{M}}(m) := m + 1$.

Here the first $\forall$ seems to be binding a predicate symbol rather than a variable so this is not a WFF in predicate logic.

Motivation: Program Specification & Verification



If *REQ* is a function specifying the abstract requirements (desired behaviour) of the software and *SOF* is to be the concrete implementation (the "program"), we want to know if:

$$\exists SOF\, \forall m \in M(REQ(m) = OUT(SOF(IN(m))))$$

i.e. does a program exist that meets the requirements?

Identifying Sets with Predicates (and Functions)

Recall that if $P(x)$ occurs in a predicate logic formula ϕ, when we create a model for ϕ we choose universe A and an interpretation $P^{\mathcal{M}} \subseteq A$ or alternatively we have to provide a characteristic function $P^{\mathcal{M}} : A \to \{True, False\}$ and say $P^{\mathcal{M}}(x)$ is True iff $x \in P^{\mathcal{M}}$.

Thus $\forall x(P(x) \to Q(x))$ can be interpreted as $P^{\mathcal{M}} \subseteq Q^{\mathcal{M}}$.

Dropping the model superscript we can identify sets and predicates so, e.g. $S(x, y, z)$ is True iff $(x, y, z) \in S$.

Recall that functions a just specialized relations or equivalently sets

e.g. the function $f(x) = 3x$ defines $\{(x, y)|y = 3x\}$

Higher Order Logic and Higher Order WFF

(Informal) Definition of Higher Order Logic:

A logic is called *higher order* if it allows sets to be quantified or if it allows sets to be elements of other sets.

A WFF (well formed formula) that quantifies a set or has a set as an argument to a predicate is called a higher-order WFF.

Example:

$\exists S S(x)$	Set S is quantified.
$S(x) \land T(S)$	The set S is an element of T.
$\exists g \forall x (f(x) = h(g(x))$	Function g is a set that is quantified.
$P(f(x)) \lor Q(f)$	Function f is a set that is an element of set Q

Order

So far we have really only used propositional logic (*zero order logic*) and predicate logic (*first order logic*).

How do we obtain second, third, . . . , n^{th} order logic?

- A predicate has order 1 if all of its arguments are terms. Otherwise the predicate has order $n + 1$ where n is the max order of its arguments.
- A quantifier has order 1 if it quantifies an individual variable. Otherwise the quantifier has order $n + 1$ where n is the order of the predicate (or function) being quantified.
- The *order of a WFF* is the max of the order of its quantifiers and predicates.
- The *order of a logic* is the max of the order of its WFFs (i.e. an n^{th} order logic is a logic who's WFFs have order n or less.)

Order (cont)

You can think of order as telling you how many levels you are from directly "eating" individuals.

1st Order Examples	
$S(x)$	S has order 1
$\forall x S(x)$	Both S and $\forall x$ have order 1
2nd Order Examples	
$\forall P \forall x \forall y (R(x,y) \rightarrow P(x,y))$	P has order 1 and $\forall P$ has order 2
$\forall y (P(f(x)) \wedge Q(y,f))$	$\forall y$, P, f have order 1, Q has order 2

Applications: Specifying reachability

Suppose $R(x, y)$ denotes $x \to y$ in a directed graph. We now have a formula for reachability that is independent of any particular model. State v is reachable from u if the following formula holds:

$$
\begin{aligned}
&\forall P(\forall x \forall y \forall z \\
&\qquad (P(x, x) \wedge \\
&\qquad (P(x, y) \wedge P(y, z) \to P(x, z)) \wedge \\
&\qquad (R(x, y) \to P(x, y))) \\
&\to P(u, v))
\end{aligned}
$$