Propositional Logic:

Part II - Syntax & Proofs

Outline

- Syntax of Propositional Formulas
- Motivating Proofs
- Syntactic Entailment $\vdash$ and Proofs
- Proof Rules for Natural Deduction
- Axioms, theories and theorems
- Consistency & completeness

Language of Propositional Calculus

Def: A *propositional formula* is constructed inductively from the symbols for

- propositional variables: $p, q, r, \ldots$ or $p_1, p_2, \ldots$
- connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- parentheses: $(,)$
- constants: $\top, \bot$

by the following rules:

1. A propositional variable or constant symbol $(\top, \bot)$ is a formula.
2. If ϕ and ψ are formulas, then so are:

$$(\neg\phi), (\phi \wedge \psi), (\phi \vee \psi), (\phi \rightarrow \psi), (\phi \leftrightarrow \psi)$$

Note: Removing $\top, \bot$ from the above provides the def. of *sentential formulas* in Rubin. Semantically $\top = T$, $\bot = F$ though often T, F are used directly in formulas when engineers abuse notation.

WFF Smackdown

Propositional formulas are also called *propositional sentences* or *well formed formulas* (WFF).

When we use precedence of logical connectives and associativity of $\wedge, \vee, \leftrightarrow$ to drop (smackdown!) parentheses it is understood that this is shorthand for the fully parenthesized expressions.

Note: To further reduce use of $(,)$ some def's of formula use order of precedence:

$$\neg, \wedge, \vee, \begin{array}{c}\rightarrow\\ \leftrightarrow\end{array} \text{ instead of } \neg, \begin{array}{c}\wedge\\ \vee\end{array}, \begin{array}{c}\rightarrow\\ \leftrightarrow\end{array}$$

As we will see, PVS uses the 1st order of precedence.

BNF and Parse Trees

The Backus Naur form (BNF) for the definition of a propositional formula is:

$$\phi ::= p|\bot|\top|(\neg\phi)|(\phi \wedge \phi)|(\phi \vee \phi)|(\phi \rightarrow \phi)$$

Here p denotes any propositional variable and each occurrence of ϕ to the right of ::= represents any formula constructed thus far.

We can apply this inductive definition in reverse to construct a formula's *parse tree*. A parse tree represents a WFF ϕ if

i) the root is an atomic formula and nothing else (i.e. ϕ is p), or

ii) the root is $\neg$ and there is only one well formed subtree, or

iii) the root is $\wedge, \vee, \rightarrow$ or $\leftrightarrow$ and there are two well formed subtrees.

Note: All leaf nodes will be atomic (e.g. $p, \bot$ or $\top$)

Subformulas and Subtrees

Def: If ϕ is a propositional formula, the *subformulas* of ϕ are give as follows:

- ϕ is a subformula of ϕ,
- if ϕ is $\neg\psi$ then ψ is a subformula of ϕ,
- if ϕ is $(\phi_1 \wedge \phi_2), (\phi_1 \vee \phi_2), (\phi_1 \rightarrow \phi_2)$, or $(\phi_1 \leftrightarrow \phi_2)$, then both ϕ_1 and ϕ_2 are subformulas of ϕ
- if ψ is a subformula of ϕ, then all subformulas of ψ are subformulas of ϕ.

The subformulas of ϕ correspond to all of the subtrees of ϕ's parse tree.

Example: Consider $p \leftrightarrow (q \wedge \neg p \rightarrow q \vee r)$. The fully parenthesized formula is:

$$(p \leftrightarrow ((q \wedge (\neg p)) \rightarrow (q \vee r)))$$

Motivating Proofs

Limitations of Truth Tables

of rows in truth table = 2^n where n = # of propositional variables in formula

Formula with 10 propositional variables has truth table with $2^{10} = 1024$ rows - too big to do by hand!

Safety critical shutdown system with 3 redundant controllers each with > 20 boolean inputs & majority vote logic on shutdown could have specification of a propositional formula using $> 3 \cdot 20 = 60$ boolean variables. Truth table would have $2^{60} > 1.15 \times 10^{18}$ rows!

Would require $\frac{2^{60}}{8 \times 10^9} = 144{,}115{,}188 > 144$ million GB/column!!

Motivating Proofs (cont)

Formal proof systems provide a way of examining the structure or *syntax* of formulas to determine the validity of an argument without resorting to truth tables.

E.g. Know $p, p \rightarrow q \models q$ (modus ponens - a.k.a $\rightarrow_e$). Therefore

$$c \vee \neg d, c \vee \neg d \rightarrow (a \wedge b \leftrightarrow c) \quad \models \quad a \wedge b \leftrightarrow c$$

Formal proof systems can decompose a problem into sub-problems (sub-proofs) that are of a manageable size.

E.g. If $\phi_1 \ldots, \phi_n \models \psi_1$ and $\phi_1, \ldots \phi_n \models \psi_2$ then $\phi_1, \ldots, \phi_n \models \psi_1 \wedge \psi_2$.

Proof Rules and Proofs

These are examples of valid *rules of inference* or *proof rules*. E.g. Knowing that formulas ϕ and $\phi \rightarrow \psi$ are true allows us to *infer* or *deduce* that ψ is true.

An example of an invalid rule of inference would be knowing that ψ is true and $\phi \rightarrow \psi$ is true, we conclude ϕ. Why?

Def: A *proof of* ψ *from premises* $\phi_1, \ldots, \phi_n$ is a finite sequence of propositions ending with ψ, such that each member of the sequence is either a premise (ϕ_i), or is derived from previous members of the sequence by a valid rule of inference.

In this case we say that $\phi_1, \ldots, \phi_n$ *syntactically entails* ψ and write

$$\phi_1, \ldots, \phi_n \vdash \psi$$

and say that $\phi_1, \ldots, \phi_n \vdash \psi$ is a *valid sequent*.

Proof Rules of Natural Deduction

While there exists more than one proof system for propositional (and predicate) logic *Natural Deduction* is one of the most useful systems.

It formalizes rules of mathematical proof you are already familiar with, e.g.,

> *To prove condition ϕ implies situation ψ, assume ϕ is true and show that ψ follows.*

While there are variations in natural deduction proof systems (e.g., see Rubin vs. Huth+Ryan), they all have the same basic elements:
- Rules to eliminate operators, and
- Rules to introduce operators.

Proof Rules: $\rightarrow e$

We know that $\phi, \phi \rightarrow \psi \models \psi$, so we can use this as a proof rule.

If $\phi_1, \ldots \phi_n \vdash \phi$ and $\phi_1, \ldots \phi_n \vdash \phi \rightarrow \psi$, then $\phi_1, \ldots \phi_n \vdash \psi$.

This rule is known as *modus ponens* or "implies (arrow) elimination" which we will abbreviate by $\rightarrow e$ and summarize as follows:

$$\frac{\phi \qquad \phi \rightarrow \psi}{\psi} \rightarrow e$$

Proof Rules: $\rightarrow i$

Another useful rule follow from the following useful fact:

$$\phi \models \psi \text{ iff } \models \phi \rightarrow \psi$$

Thus $\phi_1 \ldots \phi_n, \phi \vdash \psi$ iff $\phi_1 \ldots \phi_n \vdash \phi \rightarrow \psi$.

This rule is a form of the *Deduction Theorem*, a.k.a. conditional premise or "implies (arrow) introduction", denoted by $\rightarrow i$, and summarized as follows:

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \rightarrow \psi} \rightarrow i$$

A First Formal Proof

Example: Show that $p \to q, q \to r \vdash p \to r$

Proof:

row	Premises		Deduce	Rule
1	$p \to q, q \to r$	$\vdash$	$p \to q$	Premise
2	$p \to q, q \to r, p$	$\vdash$	p	Assumption
3	$p \to q, q \to r, p$	$\vdash$	q	$\to e1, 2$
4	$p \to q, q \to r, p$	$\vdash$	$q \to r$	Premise
5	$p \to q, q \to r, p$	$\vdash$	r	$\to e3, 4$
6	$p \to q, q \to r$	$\vdash$	$p \to r$	$\to i2-5$

Note: We could show that:

$$p \to q, q \to r, r \to s \vdash p \to s$$

in just 8 rows, not $2^4 = 16$ rows of truth table. Try it.

Proof Rules: $\wedge i$

Its getting pretty tedious writing $\phi_1, \ldots, \phi_n$ and I'm basically lazy so henceforth I'll write Γ ("Big gamma"), to represent this sequence of premises.

Assume (1) $\Gamma \models \phi$ and (2) $\Gamma \models \psi$.
Then we must have $\Gamma \models \phi \wedge \psi$. (why?)

So if (1) $\Gamma \vdash \phi$ and (2) $\Gamma \vdash \psi$ then $\Gamma \vdash \phi \wedge \psi$.

This result is known as part of the Rules of Adjunction, a.k.a. "conjunction (and) introduction", denoted by $\wedge i$, and summarized as follows:

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i$$

Proof Rules: $\wedge e_1, \wedge e_2$

Consider the following pair of valid arguments:

(1) $\phi \wedge \psi \models \phi$ and (2) $\phi \wedge \psi \models \psi$

To paraphrase these arguments:
(1) says *"When $\phi \wedge \psi$ is true, then ϕ is true."*
(2) says *"When $\phi \wedge \psi$ is true, then ψ is true."*

So if $\Gamma \vdash \phi \wedge \psi$ then (1)$\Gamma \vdash \phi$ and (2)$\Gamma \vdash \psi$.

This and the previous result are known as the Rules of Adjunction. We will call this part "conjunction (and) elimination", denoted by $\wedge e_1$ and $\wedge e_2$ respectively, and summarized as follows:

$$\frac{\phi \wedge \psi}{\phi} \wedge e_1 \qquad \frac{\phi \wedge \psi}{\psi} \wedge e_2$$

Proof Rules: $\leftrightarrow e_1, \leftrightarrow e_2$

The rules we will use for dealing with $\leftrightarrow$ are all based upon the following semantics equivalent:

$$\phi \leftrightarrow \psi \equiv (\phi \to \psi) \wedge (\psi \to \phi) \qquad (1)$$

The rules for "if and only if elimination" follow from the valid argument:

$$\phi \leftrightarrow \psi \models (\phi \to \psi) \wedge (\psi \to \phi)$$

which in turn provides the valid arguments:
(1)$\phi \leftrightarrow \psi \models \phi \to \psi$, and
(2)$\phi \leftrightarrow \psi \models \psi \to \phi$.

We denote $\leftrightarrow$ elimination by $\leftrightarrow e_1$ and $\leftrightarrow e_2$ respectively, and summarized as follows:

$$\frac{\phi \leftrightarrow \psi}{\phi \to \psi} \leftrightarrow e_1 \qquad \frac{\phi \leftrightarrow \psi}{\psi \to \phi} \leftrightarrow e_2$$

Proof Rules: $\leftrightarrow i$

Let us now consider the other valid argument that follows from (1), namely

$$(\phi \to \psi) \land (\psi \to \phi) \models \phi \leftrightarrow \psi$$

which by the defs. of $\land$ and $\models$ is the same as:

$$(\phi \to \psi), (\psi \to \phi) \models \phi \leftrightarrow \psi$$

This rule is also known as "double arrow introduction" which we will denote by $\leftrightarrow i$ and summarize as follows:

$$\frac{\phi \to \psi \qquad \psi \to \phi}{\phi \leftrightarrow \psi} \leftrightarrow i$$

So if $\Gamma \vdash \phi \to \psi$ and $\Gamma \vdash \psi \to \phi$ then $\Gamma \vdash \phi \leftrightarrow \psi$.

Example: Show $\vdash (p \to (q \to r)) \leftrightarrow (p \land q \to r)$

Proof Rules: $\neg\neg e$ and $\neg\neg i$

It is easy to check that $\neg\neg\phi \equiv \phi$. Hence:
(1)$\neg\neg\phi \models \phi$ and (2)$\phi \models \neg\neg\phi$

So we know:
(1) if $\Gamma \vdash \neg\neg\phi$ then $\Gamma \vdash \phi$, and
(2) if $\Gamma \vdash \phi$ the $\Gamma \vdash \neg\neg\phi$.

This results are known as "double negation elimination" and "double negation introduction", denoted by $\neg\neg e$ and $\neg\neg i$ respectively, and summarized as follows:

$$\frac{\neg\neg\phi}{\phi}\neg\neg e \qquad \frac{\phi}{\neg\neg\phi}\neg\neg i$$

Proof Rules: MT

Recall the proof rule $\to e$.

$$\frac{\phi \qquad \phi \to \psi}{\psi} \to e$$

If $\Gamma \vdash \phi$ and $\Gamma \vdash \phi \to \psi$, then $\Gamma \vdash \psi$.

Suppose it is still the case that $\Gamma \vdash \phi \to \psi$, but instead of $\Gamma \vdash \phi$ we know that $\Gamma \vdash \neg\psi$. If ϕ were true, then by $\to e$ we would have ψ, but since we have $\phi \to \psi$ *and* $\neg\psi$, we must have $\neg\phi$.

This reasoning is borne out by the valid argument:

$$\phi \to \psi, \neg\psi \models \neg\phi$$

The resulting proof rule is known as *modus tollens* or MT and summarized as:

$$\frac{\phi \to \psi \qquad \neg\psi}{\neg\phi}\text{MT}$$

Example: Show $p \to q, q \to r \vdash \neg r \to \neg p$

Proof Rules: $\lor i$

Consider the following pair of valid arguments:

(1) $\phi \models \phi \lor \psi$ and (2) $\psi \models \phi \lor \psi$

To paraphrase these arguments:
(1) says *"When ϕ is true, then ϕ or ψ is true."*
(2) says *"When ψ is true, then ϕ or ψ is true."*

So if (1)$\Gamma \vdash \phi$ or (2)$\Gamma \vdash \psi$ then, either way, $\Gamma \vdash \phi \lor \psi$.

We will call this "disjunction (OR) introduction" denoted $\lor i_1$ or $\lor i_2$ respectively and summarized as follows:

$$\frac{\phi}{\phi \lor \psi} \lor i_1 \qquad \frac{\psi}{\phi \lor \psi} \lor i_2$$

Proof Rules: $\vee e_1, \vee e_2$

Suppose we want to check if the following argument is valid

$$\phi \vee \psi \models \chi$$

"When ϕ is true or ψ is true, then χ is true"

Then there are two cases that we must consider:

i) When ϕ is true, then χ is true ($\phi \models \chi$)

ii) When ψ is true, then χ is true ($\psi \models \chi$)

This covers all possible case when $\phi \vee \psi$ is true, including the case when both are true.

Proof Rules: $\vee e_1, \vee e_2$ (cont)

This rule is a form of the *Rule of Alternative Proof*, a.k.a. "disjunction (or) elimination" denoted $\vee_e$ and summarized as follows:

$$\frac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi} \vee e$$

How can we use it? To show $\Gamma, \phi \vee \psi \vdash \chi$ split proof. Show:

1. $\Gamma, \phi \vdash \chi$
2. $\Gamma, \psi \vdash \chi$

Then we are done since $\Gamma, \phi \vee \psi \vdash \phi \vee \psi$. Thus

$$\Gamma, \phi \vee \psi \vdash \chi \text{ iff } \Gamma, \phi \vdash \chi \text{ and } \Gamma, \psi \vdash \chi$$

Proof Rules: $\bot e$

Consider the following valid argument:

$$\bot \models \phi$$

$\bot$ is a valid argument for any formula! Why? Because you can't find a counter example in the truth table where $\bot$ is true and ϕ is false.

Ex falso quod libet!
(From false all things are possible!)

This rule is called "bottom elimination", denoted $\bot e$ and summarized as:

$$\frac{\bot}{\phi} \bot e$$

Thus if $\Gamma \vdash \bot$ then $\Gamma \vdash \phi$ for any ϕ.

Proof Rules: $\neg e$

Consider the following semantic equivalence:

$$\phi \wedge \neg\phi \equiv \bot$$

which represents that $\bot$ is itself a contradiction. If both ϕ and $\neg\phi$ are true, then we have an inconsistent set of premises. In fact when ever ϕ is both true and false (since $\neg\phi$ is true), the $\bot$ is also valid.

Although one might think that the following rule should be called "bottom introduction", to logicians perhaps this is too close to the idea of "introducing inconsistencies" which is a "Bad Thing$^{\text{Tm}}$", hence they call it "negation elimination" denoted $\neg e$ and summarized as:

$$\frac{\phi \quad \neg\phi}{\bot} \neg e$$

Proof Rules: $\neg i$

Let us assume that when we add ϕ to our premises Γ

$$\Gamma, \phi \models \bot$$

Then everywhere that all of the premises in Γ are true, ϕ must be false. (Why?) Therefore we must have:

$$\Gamma \models \neg\phi$$

In terms of syntactic entailment this will mean that if $\Gamma, \phi \vdash \bot$ then $\Gamma \vdash \neg\phi$.

This rule is often known as *indirect proof*, "negation (not) introduction", denoted by $\neg i$, and summarized as follows:

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg i$$

Proof Rules: Copy

This rule is a bit of a kludge, but it will save us extra effort when applied correctly.

Suppose a formula ψ appears on a previous line of our proof where we had a sequence of premises Γ, i.e., $\Gamma \vdash \psi$.

Further suppose that our current sequence of premises (formulas on the left of $\vdash$) is Γ'.

Q: Can we use ψ on the current line of our proof (i.e., does $\Gamma' \vdash \psi$)?

A: Yes, provided our current sequence of premises Γ' contains all of the formulas appearing in Γ.

E.g., $p \to q, q \to r \vdash p \to r$

Therefore $p \to q, q \to r, r \vdash p \to r$.

Proof Rules: LEM

This next rule is based upon the valid argument:

$$\models \phi \vee \neg\phi$$

which say that a formula is either true, or it is false. There is no middle ground.

Since $\phi \vee \neg\phi$ is a tautology, for any set of premises $\Gamma \models \phi \vee \neg\phi$ (why?). Thus in using this rule in proofs we have $\Gamma \vdash \phi \vee \neg\phi$.

It is an example of a *derived rule*, i.e., we can prove it from the rules we have already seen ($\vdash p \vee \neg p$ - try it!).

The "Law of the Excluded Middle" denoted LEM is summarized as:

$$\frac{}{\phi \vee \neg\phi}\text{LEM}$$

Example: Show $\vdash \neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$.

Proof Rules: RAA

Two of the other rules we have seen so far, MT and $\neg\neg i$, are also derived rules. These derived rules are short cuts that represent patterns of the other rules.

An additional derived rule that is useful is *reductio ad absurdum* (RAA) that can be viewed as another form of $\neg i$.

It says that if $\Gamma, \neg\phi \vdash \bot$ then $\Gamma \vdash \phi$.

We summarize RAA as follows

$$\frac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \text{RAA}$$

Go back and compare this to the $\neg i$ summary. You can use RAA to shorten the previous example proof.

Proof Rules: Adding Your Own

In practice, people use many more proof rules than those we have shown here. Provided they are based upon valid arguments, they will result in valid proof rules.

E.g., It is easy to show $p \to Q \equiv \neg p \vee q$. This could be used to add the two proof rules:

$$\frac{\phi \to \psi}{\neg\phi \vee \psi} \to 2\vee \qquad \frac{\neg\phi \vee \psi}{\phi \to \psi} \vee 2 \to$$

Thus $\Gamma \vdash \phi \to \psi$ iff $\Gamma \vdash \neg\phi \vee \psi$.

This is actually another example of a derived rule (i.e., we could prove it from our existing rules and then just use it as a notational short cut for that proof pattern.

Do we have "enough" proof rules (i.e. are they complete)? Are any other valid proof rules derived rules? Do our rules "work correctly" (i.e., are they consistent)?

Soundness:

Def: A proof system is *sound* (or consistent) if whenever $\Gamma \vdash \psi$, then $\Gamma \models \psi$.

Our system of proof rules given above is sound. We can show this via induction on the length of our proofs. This is an immediate result of the fact that all of our proof rules are based upon valid arguments.

Stating this more formally:

Theorem (Soundness): Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be a propositional formulas. If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ then $\phi_1, \phi_2, \ldots, \phi_n \models \psi$.

If we let Γ represent $\phi_1, \ldots, \phi_n$ then the above says: "If $\Gamma \vdash \psi$ then $\Gamma \models \psi$."

Completeness:

Def: A proof system is *complete* if whenever $\Gamma \models \psi$, then $\Gamma \vdash \psi$.

Our system of proof rules given thus far is complete. We can show this via induction on the height of the parse tree of ψ.

Stating this formally:

Theorem (Completeness): Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be a propositional formulas. If $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ then $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$.

If we let Γ represent $\phi_1, \ldots, \phi_n$ then the above result together with the previous one gives us the following corollary:

Corollary: $\Gamma \vdash \psi$ iff $\Gamma \models \psi$.