Software Engineering 4A03

DAY CLASS — Dr. Mark Lawford
DURATION OF EXAMINATION: 1 Hour 50 min
McMaster University Midterm Examination — October 25, 2002

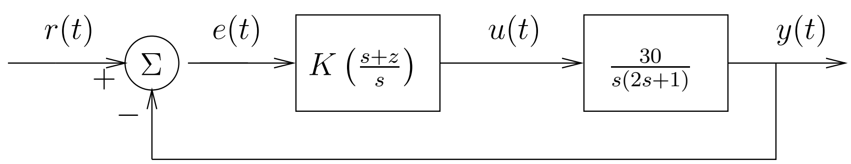
THIS EXAMINATION PAPER INCLUDES 7 PAGES AND 3 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

Special Instructions: The use of notes, and text books is not permitted during this exam. You may use the McMaster Standard Calculator. Answer all questions in the provided answer booklets. Fill in your name and student number and sign each booklet you use.

Tables of Transforms will be handed out separately.

1. Root Locus Design & Basic Discrete Equivalents (35 marks)

Consider the feedback control system below where the controller zero $z \in \mathbb{R}$ will be chosen by the system designer:



a) (5 marks) Compute the closed loop transfer function $G_{cl}(s) = \frac{Y(s)}{R(s)}$.

$$
\begin{aligned}
G_{cl}(s) &= \frac{D(s)G(s)}{1 + D(s)G(s)} \\
&= \frac{K\frac{s+z}{s}\frac{30}{s(2s+1)}}{1 + K\frac{s+z}{s}\frac{30}{s(2s+1)}} \\
&= \frac{30K(s+z)}{2s^3 + s^2 + 30Ks + 30Kz} \\
&= \frac{15K(s+z)}{s^3 + \frac{1}{2}s^2 + 15Ks + 15Kz}
\end{aligned}
$$

Part marks if not simplified.

b) (10 marks) For what values of z is the closed loop systems stable for all values of $K > 0$? Justify your answer with Root Locus sketches. (Hint: Think asymptotes.)

The system $D(s)G(s)$ has one open loop zero at $s = -z$ and open loop poles $s = 0, 0, -\frac{1}{2}$.

In this case we have $n - m = 3 - 1 = 2$ (i.e. two zeros at infinity) so there are asymptotes at $\pm\pi/2$ centered at:

$$
\begin{aligned}
\alpha &= \frac{\sum p_i - \sum z_i}{n - m} \\
&= \frac{0 + 0 + (-\frac{1}{2}) - (-z)}{2} = \frac{z - \frac{1}{2}}{2}
\end{aligned}
$$

Continued on page 2

Since the root locus starts at open loop poles and ends up at open loop zeros, the when $s = -z$ is a zero in the right half plane, (i.e., $z < 0$). Therefore, for the root locus to be in the LHP for all values of K we must have

$$z \geq 0 \tag{1}$$

then for some value of $K > 0$ the locus will cross into the right half plane.

When the zero at $s = -z$ is in the left half plane (i.e., $z > 0$), then the root locus will have have one branch ending at $s = -z$ and two others going to infinty along the asymptotes. Therefore the asymptote must be in the LHP for all parts ofn the root locus to be in the LHP. This means that $\alpha < 0$ which translates into:

$$\frac{z - \frac{1}{2}}{2} < 0 \Rightarrow z < \frac{1}{2} \tag{2}$$

Putting (1) and (2) together we have:

$$0 \leq z < \frac{1}{2} \tag{3}$$

[1 marks for the poles and zeros, 2 marks for α, 3 marks for (1), 3 marks for (2), 1 mark for putting them together to get (3)]

c) (5 marks) What is the steady state error $e(t)$ in response to a unit ramp input $r(t) = t$ when z and K are chosen so that $G_{cl}(s)$ is strictly stable?

The error transfer function in this case is:

$$\begin{aligned}
\frac{E(s)}{R(s)} &= \frac{1}{1 + D(s)G(s)} \\
&= \frac{1}{1 + K\frac{s+z}{s}\frac{30}{s(2s+1)}} \\
&= \frac{s^2(2s+1)}{2s^3 + s^2 + 30Ks + 30Kz}
\end{aligned}$$

The Laplace transform of the input $r(t)$ is $\mathcal{L}\{t\} = \frac{1}{s^2}$. Thus the steady state error for a ramp input is:

$$\begin{aligned}
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} sE(s) \\
&= \lim_{s\to 0} s\left(\frac{s^2(2s+1)}{2s^3 + s^2 + 30Ks + 30Kz}\right) \cdot \frac{1}{s^2} \\
&= 0
\end{aligned}$$

[1 mark for knowing that the FVT can be applied since $sE(s)$ is strictly stable (poles are closed loop system's poles). 2 marks for computing $E(s)$ and 2 marks for final answer. If someone says that error is zero since $y(\infty) - u(\infty) = \infty - \infty = 0$ then only 1 or 2 marks out of 5.]

d) (8 marks) Find a difference equation to implement a digital version of the controller $D(s) = K\frac{s+z}{s}$ using the Trapezoid Rule with sampling period T.

$$\frac{U(s)}{E(s)} = D(s) = K\frac{s + z_1}{s}$$

Continued on page 3

Therefore

$$\begin{aligned}
U(s) &= KE(s) + Kz_1\frac{E(s))}{s} \\
&\quad \text{Taking Laplace Inverse of both sides} \\
u(t) &= Ke(t) + Kz_1\int_0^t e(\tau)d\tau \\
u(kT) &= Ke(kT) + Kz_1\int_0^{kT} e(\tau)d\tau \\
&= Ke(kT) + Kz_1\int_{(k-1)T}^{kT} e(\tau)d\tau + Kz_1\int_0^{(k-1)T} e(\tau)d\tau \\
&= Ke(kT) + Kz_1\int_{(k-1)T}^{kT} e(\tau)d\tau + u((k-1)T) - Ke((k-1)T)) \\
&\quad \text{Given we are using Trapezoid appox.} \\
u(kT) &\approx Ke(kT) + K\frac{T}{2}z_1[e(kT) + e((k-1)T)] + u((k-1)T) - Ke((k-1)T)) \\
&\quad \text{Let } u_k, e_k \text{ denote } u(kT), e(kT) \text{ respectively, we get CCDE} \\
u_k &= Ke_k + K\frac{T}{2}z_1[e_k + e_{k-1}] + u_{k-1} - Ke_{k-1} \\
&\quad \text{Grouping like terms} \\
u_k &= u_{k-1} + K(1+\frac{T}{2}z_1)e_k + K(\frac{T}{2}z_1 - 1)e_{k-1}
\end{aligned}$$

Method 2: Using answer to next part

$$\begin{aligned}
\frac{U(z)}{E(z)} &= D_T(z) = K\frac{(1+\frac{T}{2}z_1)z + \frac{T}{2}z_1 - 1}{z-1} \\
&= K\frac{(1+\frac{T}{2}z_1) + (\frac{T}{2}z_1 - 1)z^{-1}}{1-z^{-1}} \\
(1-z^{-1})U(z) &= K[(1+\frac{T}{2}z_1) + (\frac{T}{2}z_1 - 1)z^{-1}]E(z) \\
u_k &= u_{k-1} + K(1+\frac{T}{2}z_1)e_k + K(\frac{T}{2}z_1 - 1)e_{k-1}
\end{aligned}$$

e) (7 marks) Assuming zero initial conditions, find the transfer function $D(z)$, for this difference equation.

Method 1: Take $\mathcal{Z}$ transform of CCDE (d)

$$\begin{aligned}
u_k &= u_{k-1} + K(1+\frac{T}{2}z_1)e_k + K(\frac{T}{2}z_1 - 1)e_{k-1} \\
u_k - u_{k-1} &= K(1+\frac{T}{2}z_1)e_k + K(\frac{T}{2}z_1 - 1)e_{k-1} \\
&\quad \text{Take } \mathcal{Z} \text{ transform of both sides} \\
(1-z^{-1})U(z) &= K[(1+\frac{T}{2}z_1) + (\frac{T}{2}z_1 - 1)z^{-1}]E(z) \\
\frac{U(z)}{E(z)} &= K\frac{(1+\frac{T}{2}z_1) + (\frac{T}{2}z_1 - 1)z^{-1}}{1-z^{-1}}
\end{aligned}$$

Continued on page 4

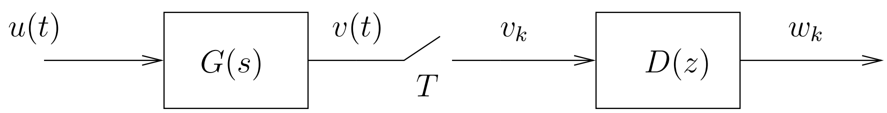
$$= K\frac{(1+\frac{T}{2}z_1)z + \frac{T}{2}z_1 - 1}{z-1}$$

Method 2: Using Trapezoid approx for s *For the trapezoid rule* $s = \frac{2}{T}\frac{z-1}{z+1}$. *The discrete equivalent is obtained as:*

$$\begin{aligned} D_T(z) &= D(s)|_{s=\frac{2}{T}\frac{z-1}{z+1}} = K\tfrac{s+z_1}{s}|_{s=\frac{2}{T}\frac{z-1}{z+1}} \\ &= K\frac{\frac{2}{T}\frac{z-1}{z+1}+z_1}{\frac{2}{T}\frac{z-1}{z+1}} \\ &= K\frac{z-1+\frac{T}{2}z_1(z+1)}{z-1} \\ &= K\frac{(1+\frac{T}{2}z_1)z+\frac{T}{2}z_1-1}{z-1} \end{aligned}$$

2. Frequency Response (30 marks)

Consider the system shown below:



a) (5 marks) Assume that $G(s)$ is a stable, strictly causal, linear, time invariant transfer function of a continuous system. Use the Final Value Theorem to prove that for a step input $u(t) = 1(t)$ then

$$\lim_{t\to\infty} v(t) = G(0)$$

The final value theorem says that if $sV(s)$ *has all poles in the open LHP then*

$$\lim_{t\to\infty} v(t) = \lim_{s\to 0} sV(s)$$

In this case using the fact that $\mathcal{L}\{1(t)\} = \frac{1}{s}$ *we have*

$$sV(s) = sG(s)U(s) = s\frac{G(s)}{s} = G(s)$$

Since $G(s)$ *is stable then all poles of* $G(s)$ *are in the LHP (note: I am assuming the system* $G(s)$ *is BIBO stable so no poles on the* $j\omega$ *axis). Therefore we can apply the FVT to get:*

$$\lim_{t\to\infty} v(t) = \lim_{s\to 0} sV(s) = \lim_{s\to 0} G(s) = G(0)$$

b) (10 marks) Assume the sampling period $T = \frac{1}{2}$ second, input $u(t) = 1(t)$,

$$G(S) = \frac{1}{s+3} \text{ and } D(z) = \frac{W(z)}{V(z)} = \frac{z}{z+0.5}$$

Consider only the **steady state** behaviour of the system

i) What is $v(t)$ in steady state?
From part (a) $v^{ss}(t) = \lim_{t\to\infty} v(t) = G(0) \approx \frac{1}{3}$, *where the ss superscript denotes steady state.*

Continued on page 5

ii) What is v_k in steady state?
v_k is just the sampling of $v(t)$ so in steady state $v_k^{ss} = \frac{1}{3}$

iii) What is w_k in steady state?
The discrete frequency response to input $v_k = \sin \omega kT$ in steady state is

$$w_k^{ss} = |D(e^{j\omega T})| \sin(\omega kT + \angle D(e^{j\omega T}))$$

is given by

$$D(z)|_{z=e^{j0}} v_k = D(1) \cdot \frac{1}{3} = \frac{1}{1+0.5} \cdot \frac{1}{3} = \frac{2}{9} \approx 0.22222$$

c) (15 marks) Repeat the previous question with $u(t) = 3 - 2\sin(2\pi t)$.
Since $G(s)$, the sample operator and $D(z)$ are all linear, then for

$$u(t) = A_1 u_1(t) + A_2 u_2(t)$$

the response at $v(t)$ will be

$$v(t) = A_1 v_1(t) + A_2 v_(t)$$

where $v_i(t)$ is response for input $u_i(t)$, for $i = 1, 2$ and
Here $A_1 = 3$ and $u_1(t) = 1(t)$ while $A_2 = -2$ and $u_2(t) = \sin(2\pi t)$. We know the response due to $1(t)$ from part (b). We just have to scale it by a factor or 3 at every step. Therefore it remains to compute the response due to $\sin(2\pi t)$ at every stage and then we can scale it by -2.
[2 marks for knowing how to use linearity]

i) What is $v(t)$ in steady state? *[5 marks]*
For an input $u(t) = \sin(\omega t)$ the steady state response at $v(t)$ is given by

$$v(t) = |G(j\omega)| \sin(\omega t + \angle G(j\omega))$$

Therefore for $u_2(t) = \sin(2\pi t)$ we have $\omega = 2\pi$ so:

$$\begin{aligned} G(j2\pi) &= \frac{1}{3 + j2\pi} \\ |G(j2\pi)| &= \frac{1}{\sqrt{9 + 4\pi^2}} \approx 0.21533 \\ \angle G(j2\pi) &= -\tan^{-1}(\frac{2\pi}{3}) \approx -1.1253 \end{aligned}$$

Therefore the steady state response for $u(t) = 3 - 2\sin(2\pi t)$ is

$$\begin{aligned} v(t) &= 3(\frac{1}{3}) - 2\frac{1}{\sqrt{9 + 4\pi^2}} \sin(2\pi t - \tan^{-1}(\frac{2\pi}{3})) \\ &= 1 - \frac{2}{\sqrt{9 + 4\pi^2}} \sin(2\pi t - \tan^{-1}(\frac{2\pi}{3})) \\ &\approx 1 - 0.43067 \sin(2\pi t - 1.1253) \end{aligned}$$

ii) What is v_k in steady state? *[3 marks]*
v_k is just the sampling of $v(t)$ so in steady state

$$\begin{aligned} v_k &= v(kT) = v(\frac{k}{2}) \\ &= 1 - \frac{2}{\sqrt{9 + 4\pi^2}} \sin(k\pi - \tan^{-1}(\frac{2\pi}{3})) \\ &\approx 1 - 0.43067 \sin(k\pi - 1.1253) \end{aligned}$$

Continued on page 6

Using the trig identity

$$\sin(s-t) = \sin s \cos t - \cos s \sin t$$

with $s = k\pi$ *and* $t = \tan^{-1}(\frac{2\pi}{3})$ *we get*

$$\begin{aligned}
\sin(k\pi - \tan^{-1}(\frac{2\pi}{3})) &= \sin(k\pi)\cos(\tan^{-1}(\frac{2\pi}{3})) - \cos(k\pi)\sin(\tan^{-1}(\frac{2\pi}{3})) \\
&= (-1)^{k+1}\sin(\tan^{-1}(\frac{2\pi}{3})) \\
&\approx (-1)^{k+1}\sin(1.1253) \\
&= 0.90241(-1)^{k+1}
\end{aligned}$$

So the best answer is:

$$v_k \approx 1 - 0.90241(-1)^{k+1} = 1 + 0.90241(-1)^k$$

iii) What is w_k in steady state? *[5 marks]*
The steady state frequency response to an input $\sin(k\pi - \tan^{-1}(\frac{2\pi}{3}))$

$$\begin{aligned}
D(z)|_{z=e^{j\pi}}\frac{1}{3} &= D(z)|_{z=e^{j\pi}}\frac{1}{3} \\
&= D(1)\frac{1}{3} == \frac{1}{1+.5}\cdot\frac{1}{3}\frac{2}{9}
\end{aligned}$$

Therefore for $u_2(t) = \sin(2\pi t)$ *we have* $\omega T = 2\pi \cdot \pi = \pi$ *so:*

$$\begin{aligned}
D(e^{j\pi}) &= D(-1) = \frac{-1}{-1+0.5} = 2 \\
|D(e^{j\pi})| &= 2 \\
\angle D(e^{j\pi}) &= 0
\end{aligned}$$

Therefore we simply scale by a factor of 2 *the component of the steady state behavior due to the* $u_2 = \sin(k\pi - \tan^{-1}(\frac{2\pi}{3}))$ *to obtain*

$$w_k = \frac{2}{3} - \frac{4}{\sqrt{9+4\pi^2}}\sin(k\pi - \tan^{-1}(\frac{2\pi}{3}))$$

$$\begin{aligned}
w_k &\approx D(1)\cdot 1 + D(-1)\cdot 0.90241(-1)^k \\
&= \frac{2}{3} + 1.8048(-1)^k
\end{aligned}$$

3. Sampled Data Systems (35 marks)

You have been assigned to work on the controls software for a new car that is powered by an electric motor. After considerable effort the automotive engineers have come up with a simplified model for the engine:

$$G(s) = \frac{3}{s(s+3)}$$

You have been asked to do a digital control design for the setting shown in Figure 1.

Continued on page 7

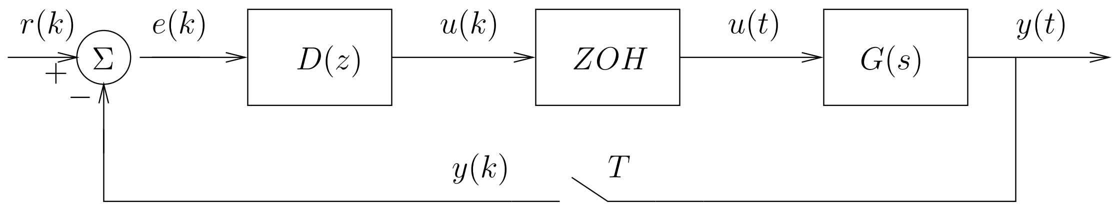


Figure 1: Block diagram for Question 3.

a) (10 marks) Assuming a sampling rate of T, find the ZOH discrete equivalent $G_{ZOH}(z)$ system for $G(s)$ where:

$$G_{ZOH}(z) = (1 - z^{-1})\mathcal{Z}\{\frac{G(s)}{s}\}$$

This represents the discrete transfer function from $u(k)$ to $y(k)$.

b) (5 marks) What is the transfer function for $G_{ZOH}(z)$ when $T = \frac{1}{30}$?

c) (10 marks) What is the (open loop) impulse response of $G_{ZOH}(z)$ when $T = \frac{1}{30}$?

d) (5 marks) What is the (open loop) response of the system $G_{ZOH}(z)$ for sampling rate $T = \frac{1}{30}$ when the input is:

$$u_k = \begin{cases} 2, & k = 1 \\ -2, & k = 2 \\ 0, & \text{otherwise} \end{cases}$$

(HINT: The discrete system we are considering is LTI!)

e) (5 marks) Let $D(z) = K$. Compute the closed loop transfer function for the discrete system transfer function from $r(k)$ to $y(k)$ for sampling rate $T = \frac{1}{30}$?

"Where's Mo?" - A mistaken 4A03 lecturer

The End