Introduction to Digital Control Systems

Outline

- Brain dead digital control
- What's the picture?
- Digitization: A 1st Approximation
- Effect of delay due to sampling
- A 1st implementation of PID Control

Brain Dead Digital Control

Idea: Implement compensator $D(S)$ using a computer and numerical integration.

Most physical control systems tend to behave as a form of low pass filters (why?).

The system *bandwidth* (ω_{BW}) is defined to be the maximum frequency for which the system output will "track a sinusiod input in a satisfactory manner".

Provided the computer samples the system at *at least* 30 times the system bandwidth, and the A/D and D/A conversion is suffciently accurate you can expect the computer control system to closely approximate the original continuous control.

System Bandwidth ω_{BW}

By "satisfactory" tracking we roughly mean that the power from the input to the output is reduced by no more than $\frac{1}{2}$.

Since power varies as the square of the ampitude of the signal we have at the bandwidth frequency ω_{BW}:

$$\frac{1}{2} = \frac{P_Y}{P_U} = \frac{|Y(jw_{BW})|^2}{|U(jw_{BW})|^2} = |G(jw_{BW})|^2$$

So we must have $|G(jw_{BW}| = \frac{1}{\sqrt{2}} \cong 0.707$. Measuring the gain (reduction) of power in decibels (dB) we have

$$\begin{aligned} -3 \text{ dB} &= 10 \log_{10}(1/2) \\ &= 10 \log_{10} |G(jw_{BW})|^2 \\ &= 20 \log_{10} |G(jw_{BW})| \end{aligned}$$

A/D Resolution

Analog to digital (A/D) conversion often uses 10, 12 or even 16 bits:

A/D Converter Bits	Resolution
10	0.1%
12	0.024%
16	0.0015%

With a sampling rate $> 30\omega_{BW}$ of the **closed loop**(!) system and 16-bit A/D resolution, the following straight forward digital approximations closely approximate their continuous counterparts.

So why not use 16-bit samples at $> 30\omega_{BW}$ samples/sec all the time? Cost & hardware limitations.

Fast 16-bit hardware costs more. Also some control systems are too fast (i.e., $30\omega_{BW}$ is faster than the fastest A/D boards).

What's the Picture?

See Fig 3.1. What do each of the signals look like?

Assume we do A/D conversion of $y(t)$ every T seconds then:

T is the **sample period**

$1/T$ and $2\pi/T$ corresponds to the **sample rate** in Hz (sample per second) and radians/second respectively.

$y(kT)$: for $k \in \mathbb{Z}$ is the sampled or **discrete signal** as opposed to the continuous signal $y(t)$. For a fixed T we often write $y(k)$ and it is understood to denote $y(t)_{|t=kT}$.

$u(kT)$ aka $u(k)$ is the result of difference equations approximating $D(s)$ which becomes continuous signal $u(t)$ via D/A and a **zero-order hold** (ZOH).

Similarly $r(kT)$ (aka $r(k)$) is discrete reference signal and $e(k) = y(k) - r(k)$.

A 1st Approximation of $D(s)$

Here we use Euler's method in the **forward difference** method as follows:

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

So for sufficently small T

$$\frac{dx}{dt} \cong \frac{x(k+1) - x(k)}{T} \qquad (3.2)$$

Thus differentiation is replaced by a difference equation.

We will use t_k to denote sampling time kT.

Example Approximation of $D(s)$

Suppose

$$D(s) = \frac{U(s)}{E(s)} = K_o \frac{s+a}{s+b} \quad (3.3)$$

then $(s+b)U(s) = K_o(s+a)E(s)$ so

$$\frac{du}{dt} + bu(t) = K_o(\frac{de}{dt} + ae(t))$$

Using (3.2) to approximate we obtain:

$$\frac{u(k+1)-u(k)}{T} + bu(k) = K_o[\frac{e(k+1)-e(k)}{T} + ae(k)]$$

which can be rearrange to:

$$u(k+1) = (1-bT)u(k) + K_o(aT-1)e(k) + K_o e(k+1)$$

Let $\alpha_1 := 1 - bT$ and $\alpha_2 := K_o(aT-1)$ we have

$$u(k+1) = \alpha_1 u(k) + \alpha_2 e(k) + K_o e(k+1)$$

which is a more computationally efficient difference equation.

Digital Implementation of $D(s)$

Besides precomputing α_1 and α_2 we can implement controller as follows:

$x = 0$; $\alpha_1 = 1 - bT$; $\alpha_2 = K_o(aT - 1)$; */* initialization */*

while True do

Read A/D to get y and r;

$e = r - y$;

$u = x + K_o e$;

Output u to D/A and ZOH;

$x = \alpha_1 u + \alpha_2 e$; */*compute x for next loop */*

end while

Note: This is important since we want to

1. minimize overall computation time to increase maximum possible sampling rate,
2. minimize the time between A/D and D/A.

Effects of Sampling

Even without the computational delay associated with u between A/D and D/A, the average value of the digital version of $u(t)$ roughly becomes a $T/2$ lagged version of the continuous $u(t)$.

Why? Values of $u(kT)$ are held constant over sample period T (See Fig. 3.3).

What is the effect of the delay?

Suppose $g_1(t) = g(t - \lambda)$ for some $\lambda > 0$ (i.e. g_1 is g delayed by λ).

If $g(t) = 0$ for $t < 0$, then $G_1(s) = e^{-s\lambda}G(s)$ (why?)

Result: Delay tends to reduce stability and damping of the system.

Effects of Sampling (cont.)

Why? Consider phase margin (PM) of system (see Sec 2.4.4):

$$\angle G_1(j\omega) = \angle G(j\omega) + \angle e^{-j\omega\lambda} = \angle G(j\omega) - \omega\lambda$$

So a delay of $T/2$ will reduce the phase of the system by $\Delta = -\frac{\omega T}{2}$.

Thus for a cross-over frequency ω_c i.e.,

$$|D(j\omega_c)G(j\omega_c)| = 1$$

the phase is roughly reduced by $\frac{\omega_c T}{2}$.

Approximating PID Control

Recall the transfer function for PID controller:

$$C(s) = \frac{U(s)}{E(s)} = K_p + \frac{K_I}{s} + K_d s$$

which results in the control law:

$$u(t) = K_p e(t) + K_I \int_0^t e(\eta)d\eta + K_d \frac{de}{dt}(t)$$

If we use the *backward rectangular* version of Euler's method approximation:

$$\frac{dx}{dt} \cong \frac{x(k) - x(k-1)}{T}$$

then for proportional

$$u(k) = K_p e(k) \quad (3.14)$$

for integral where $u(t) = K_I \int_0^t e(\eta)d\eta$ take derivative of both sides to get

$$u(k) = u(k-1) + K_I T e(k) \quad (3.15)$$

and derivative then for proportional

$$u(k) = \frac{K_d}{T}[e(k) - e(k-1)]. \quad (3.16)$$

Approximating PID Control (cont.)

Considering each component separately, we would approximate PID control as:

$$u(k) = u(k-1) + (K_p + K_I T + \frac{K_d}{T})e(k) - \frac{K_d}{T}e(k-1)$$

However, considering PID all together we have:

$$\frac{du}{dt}(t) = K_p \frac{de}{dt}(t) + K_I e(t) + K_d \frac{d^2 e}{dt^2}(t)$$

which, using the backward rectangular approx results in:

$$\begin{aligned} u(k) &= u(k-1) + (K_p + K_I T + \frac{K_d}{T})e(k) \\ &\quad -(K_p + \frac{2K_d}{T})e(k-1) + \frac{K_d}{T}e(k-2) \end{aligned}$$