Review of Continuous Control Systems

Outline

- Basic concepts and definitions
- Useful Laplace & complex number facts
- Block diagram manipulation
- Response vs. pole location
- Steady-state errors and the internal model principle
- PID Control
- Root Locus
- Frequency Response

Control Basics

Feedback control: The use of information to produce the desired behavior from a dynamical system.

See Fig 2.8 for basic unity feedback diagram.

regulation : The process of holding output $y(t)$ close to the reference signal $r(t)$ (i.e. $y(t)$ *tracking* $r(t)$).

disturbance rejection : Good regulation in the presence of disturbance signals.

low sensitivity : A system has low sensitivity to some plant parameters if it has good regulation in the face of peturbation to these parameters.

robust : good disturbance rejection + low sensitivity

Useful Facts

For signal $x(t), t > 0$, denote the Laplace transform of $x(t)$ by $\mathcal{L}(x(t)) = X(s)$.

Then $\mathcal{L}(\frac{dx}{dt}) = sX(s)$ and $\mathcal{L}^{-1}(\frac{A}{s+a}) = Ae^{-at}$.

A system is *strictly stable* if all poles are in the open LHP. In this case the Final Value Theorem (FVT) states:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

From complex analysis: $e^{j\omega} = \cos\omega + j\sin\omega$.

Any complex number $z = \sigma + j\omega$ can also be represented as a radius r and angle θ such that $z = re^{j\theta}$ (how?) which in "phasor" notation is written $r\angle\theta$.

Then for complex numbers z_1 and z_2, we have $z_1 \cdot z_2 = r_1 \cdot r_2\angle(\theta_1 + \theta_2)$.

Block Diagram Manipulation

For Fig. 2.1 we have:

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + H(s)G(s)} \qquad (2.10)$$

We can use this to obtain other transfer functions (TFs). E.g. for Fig. 2.8:

$$\frac{Y(s)}{R(s)} = \frac{D(s)G(s)}{1 + D(s)G(s)}$$

Just take $H(s) := 1$ and $G(s) := D(s)G(s)$ in (2.10)

Typically unity feedback configuration is used in digital control with error formed in the computer.

What is the difference between these configurations? Consider $G(s) = \frac{1}{s(s+1)}$ and $D(s) = H(s) = K$.

Response vs. Pole Location

Consider TF $H(s) = \frac{b(s)}{a(s)}$.

The **poles** (**zeros**) of $H(s)$ are the values of s s.t. $a(s) = 0$ ($b(s) = 0$).

Let $\delta(t)$ denote the unit impulse. Then $\mathcal{L}(\delta(t)) = 1$ The *impulse response* of $H(s)$ is $\mathcal{L}^{-1}(H(s))$.

Each pole of $H(s)$ can be identified with a particular response (see Fig. 2.5).

Note that complex poles (or zeros) appear in pairs as complex conjugates

$$s = -\sigma \pm j\omega_d$$

which will result in partial fractions expansions of the form:

$$\frac{\alpha + j\beta}{s + \sigma + j\omega_d} + \frac{\alpha - j\beta}{s + \sigma - j\omega_d}$$

resulting in time domain response of the form:

$$e^{-\sigma t}[2\alpha \cos \omega_d t + 2\beta \sin \omega_d t]$$

Time Domain Specifications

For Fig. 2.6. w.r.t. a system's unit step response:

rise time t_r: Time it takes the system to go from 10% to 90% of final value.

settling time t_s: time it takes system transients to decay to within, e.g., 1% of the final value

overshoot M_p: maximum amount systems overshoots its final value divide by the final value.

steady state error e_{ss}:

$$e_{ss} := \lim_{t\to\infty} e(t)$$

PID Control

Transfer function for PID controller:

$$C(s) = \frac{U(s)}{E(s)} = K_p + \frac{K_I}{s} + K_d s$$

where

- K_p = Proportional gain
- K_I = Integral gain
- K_d = Derivative gain

which results in the control law:

$$u(t) = K_p e(t) + K_I \int_0^t e(\eta) d\eta + K_d \frac{de}{dt}(t)$$

Characteristics of P, I, and D controllers

In general w.r.t. $G_{cl}(s)$ system step response:

proportional controller (K_p) will reduce the rise time and will reduce ,but never eliminate, the steady-state error.

integral control (K_I) will eliminate the steady-state error, but may make the transient response worse.

derivative control (K_d) will increase the stability of the system, reduce the overshoot, and improve the transient response.

Effects of K_p, K_I and K_d Gains

Effects of each of controllers Kp, Kd, and Ki on a closed-loop system G_{cl} are summarized in the table shown below.

$Gain$	t_r	M_p	t_s	e_{ss}
K_p	$\downarrow$	$\uparrow$	Δ	$\downarrow$
K_I	$\downarrow$	$\uparrow$	$\uparrow$	0
K_d	Δ	$\downarrow$	$\downarrow$	Δ

where $\Delta =$"small change", $\uparrow =$ "increase" & $\downarrow =$ "decrease"

Note: These correlations may not always be accurate, because K_p, K_I, and K_d are dependent of each other. Therefore the table should only be used as a reference when you are determining the values for the gains.

General tips for designing a PID controller

When you are designing a PID controller for a given system, follow the steps shown below to obtain a desired response.

1. Obtain an open-loop response and determine what needs to be improved
2. Add a proportional control to improve the rise time
3. Add a derivative control to improve the overshoot
4. Add an integral control to eliminate the steady-state error
5. Adjust each of K_p, K_i, and K_d until you obtain a desired overall response.

Root Locus

Shows how system's open loop dynamics influence closed loop. Usually used to study effect of loop (proportional) gain but can be used to study effect of any parameter of $C(s)$.

For block diagram Fig. 2.8, we know that the closed loop TF is:

$$\frac{Y(s)}{R(s)} = \frac{D(s)G(s)}{1 + D(s)G(s)}$$

Typically $D(s)G(s) = K\frac{b(s)}{a(s)}$ where

$$\begin{aligned} b(s) &= b_1 s^m + b_2 s^{m-1} + \ldots + b_{m+1} \\ a(s) &= a_1 s^n + a_2 s^{n-1} + \ldots + a_{n+1} \end{aligned}$$

Thus the poles of the closed loop system are values of s such that:

$$1 + K\frac{b(s)}{a(s)} = 0 \quad (2.26)$$

Root Locus (cont)

Consider the case where $K > 0$. Eqn. 2.26 can be rewritten as:

$$a(s) + Kb(s) = 0$$

so as $K \to 0$ the poles of the closed-loop system are a(s) = 0 or the poles of $D(s)G(s)$.

Rewriting Eqn. 2.26 again we obtain:

$$\frac{a(s)}{K} + b(s) = 0$$

so as $K \to \infty$, the poles of the closed-loop system are b(s) = 0 or the zeros of $D(s)G(s)$.

But there are n roots for all the above so when $n > m$ where do the other $n - m$ poles go?
To ∞ (and beyond! ;-).

Root Locus (cont)

Let's be a little more specific. Since

$$1 + K\frac{b(s)}{a(s)} = 0$$

we know that $K\frac{b(s)}{a(s)} = -1$. Therefore for $K > 0$

1. $|\frac{b(s)}{a(s)}| = \frac{1}{K}$, and

2. $\angle\frac{b(s)}{a(s)} = \pi + 2l\pi$ rad for $l \in \mathbb{Z}$

Assume that

$$\frac{b(s)}{a(s)} = \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}$$

Then

$$\angle\frac{b(s)}{a(s)} = \sum_{i=1}^{m}\angle(s - z_i) - \sum_{i=1}^{n}\angle(s - p_i)$$

Root Locus (cont)

For some s_1 on the real axis, consider $\angle\frac{b(s_1)}{a(s_1)}$:

- zeros (poles) to the right of s_1 on the axis contribute $+\pi$ ($-\pi$) rads.

- zeros & poles to the left of s_1 on the axis contribute 0 rads.

- angles of complex pairs of poles or zeros cancel each other out.

If s_1 lies on the root locus for $K > 0$, we must have $\angle\frac{b(s_1)}{a(s_1)} = \pi + 2l\pi$ for some $l \in \mathbb{Z}$.

Therefore, there must be an odd number of poles and zeros to the right of s_1.

Drawing the Root Locus

To sketch a root locus for a give $D(s)G(s) = \frac{b(s)}{a(s)}$, use the follow steps:

1. Mark the n open loop poles (roots of $a(s)$) with a $\times$ and the m open loop zeros (roots of $b(s)$) with a $\circ$.

2. Draw the locus on the real axis where there are an odd number of poles and zeros to the right.

3. If there are two or more poles than zeros (i.e. $n - m \geq 2$), we will have this many asymptotes to ∞, centered at α and at angles ϕ_l where:

$$\alpha = \frac{\sum p_i - \sum z_i}{n - m}$$
$$\phi_l = \frac{\pi + 2l\pi}{n - m}, \text{ for } l = 0, 1, \ldots n - m - 1$$

Frequency Response Design

Suppose we have a strictly stable (all poles in open LHP) linear time invariant (LTI) system given by:

$$\frac{Y(s)}{U(s)} = G(s)$$

Q: What is the steady state response to a sinusoidal input?:

$$u(t) = U_o \sin \omega_1 t$$

A: In steady state transient response from stable poles $\to 0$ so we are left with

$$y(t) = AU_o \sin(\omega_1 t + \phi)$$

where

$$\begin{aligned} A &= |G(j\omega_1)| \\ \phi &= \angle G(j\omega_1) = \tan^{-1} \frac{\mathrm{Im}[G(j\omega_1)]}{\mathrm{Re}[G(j\omega_1)]} \end{aligned}$$

Frequency Response (cont)

So for a stable LTI system with TF $G(s)$ with sinusoidal input, the steady state *magnitude* and *phase* of the output w.r.t. the input are functions of the frequency ω, respectively denoted:

$$A(\omega) = |G(j\omega)| \text{ and } \phi(\omega) = \angle G(j\omega)$$

Plotting the frequency response $A(\omega)$ vs. ω and $\phi(\omega)$ vs. ω are often referred to as Bode plots.

They can be used to help design the system. In general a stable *closed loop* system (cascade configuration) has TF:

$$\frac{Y(s)}{R(s)} = G_{cl} = \frac{D(s)G(s)}{1 + D(s)G(s)}$$

Typically the output follows the input at low frequency ($|G_{cl}| = 1$) and fails to track at higher frequencies ($|G_{cl}| < 1$).

What the hell is a decibel?

In systems we are often interested in how system affects a signal from its input to its output. One important measure is the power gain from the input to the output $\frac{P_{out}}{P_{in}}$. A *bel* is the $\log_{10}$ of this ratio. A *decibel* (dB) is one tenth of a bel. Thus the power gain in decibels is

$$10 \log_{10} \frac{P_{out}}{P_{in}} = 10 \log_{10} \frac{P_Y}{P_U}$$

for our system $G(s) = \frac{Y(s)}{U(s)}$.

For a voltage or a current, power varies as the square of the amplitude of the signal. E.g.,

$$\frac{P_{out}}{P_{in}} = \frac{v_{out}^2/R}{v_{in}^2/R} = \left(\frac{v_{out}}{v_{in}}\right)^2 \text{ or } \frac{P_{out}}{P_{in}} = \frac{i_{out}^2 R}{i_{in}^2 R} = \left(\frac{i_{out}}{i_{in}}\right)^2$$

Thus the power gain $\frac{P_Y}{P_U}$ in decibels of $G(s)$ for sinusoidal input at frequency ω_1 is:

$$10 \log_{10} \left(\frac{Y(j\omega_1)}{U(j\omega_1)}\right)^2 = 20 \log_{10} |G(j\omega_1)| \ \text{ dB}$$

System Bandwidth ω_{BW}

The *bandwidth* of the system (ω_{BW}) is defined to be the maximum frequency at which the system will satisfactorily track a sinusoidal input.

By "satisfactory" tracking we roughly mean that the power from the input to the output is reduced by no more than $\frac{1}{2}$.

Since power varies as the square of the amplitude of the signal we have at the bandwidth frequency ω_{BW}:

$$\frac{1}{2} = \frac{P_Y}{P_U} = \frac{|Y(jw_{BW})|^2}{|U(jw_{BW})|^2} = |G(jw_{BW})|^2$$

So we must have $|G(jw_{BW}| = \frac{1}{\sqrt{2}} \cong 0.707$. Measuring the power gain in decibels (dB) we have

$$\begin{aligned} 20 \log_{10} |G(jw_{BW})| &= 20 \log_{10}(\frac{1}{\sqrt{2}}) \\ &= -3 \text{ dB} \end{aligned}$$

Gain& Phase Stability Margins

Recall from the section on Root Locus that the roots of $1 + KD(s)G(s) = 0$ occur when $KD(s)G(s) = -1$ or

$$|KD(s)G(s)| = 1 \text{ and } \angle KD(s)G(s) = \pi + 2l\pi$$

As we change the gain K, the closed loop system becomes stable or unstable when the root locus crosses the imaginary axis, i.e.,

$$|KD(j\omega)G(j\omega)| = 1 \text{ and } \angle KD(j\omega)G(j\omega) = \pi + 2l\pi$$

Case 1: If the system becomes unstable as K increases, the stability condition is

$$|KD(j\omega)G(j\omega)| < 1 \text{ when } \angle KD(j\omega)G(j\omega) = -\pi$$

Case 2: If the system becomes stable as K increases, the stability condition is

$$|KD(j\omega)G(j\omega)| > 1 \text{ when } \angle KD(j\omega)G(j\omega) = -\pi$$

Gain Margin & Phase Margin

To provide an idea of just how stable a system is, we can talk about the **Gain Margin** (GM), the factor by which $|KD(j\omega)G(j\omega)|$ differs from 1 when

$$\angle KD(j\omega)G(j\omega) = -\pi$$

And the **Phase Margin**, the factor by which $\angle KD(j\omega)G(j\omega)$ differs from π when

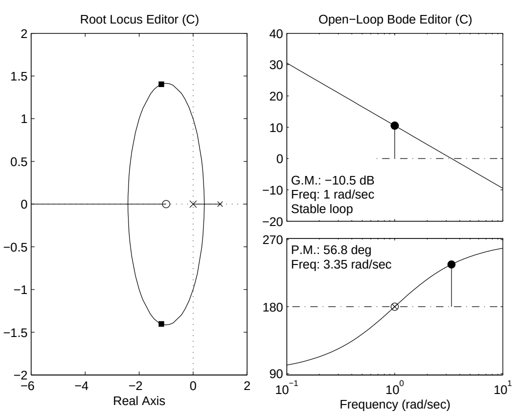
$$|KD(j\omega)G(j\omega)| = 1$$

These values can be determined from the Bode plot of the system with the help of a quick root locus sketch.

Example: Consider the system:

$$D(s)G(s) = \frac{K(s+1)}{s(s-1)}$$

From the root locus we see that the system corresponds to Case 2. For $K = 3.35$ we get GM=-10.5dB and PM=56.8 degrees.



Example (cont.): For $D(s)G(s) = \frac{K(s+1)}{s(s-1)}$ with $K = 5.83$, the GM=-15.3dB and PM=70.5 degress.