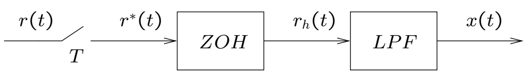
Sampled Data Systems

Example

Consider the system shown below:



Let the sampling period be $T = 0.01$ sec and assume that the input is:

$$r(t) = \frac{3}{\pi}\cos(50t + \frac{\pi}{6})$$

Here LPF is an ideal low pass filter such that:

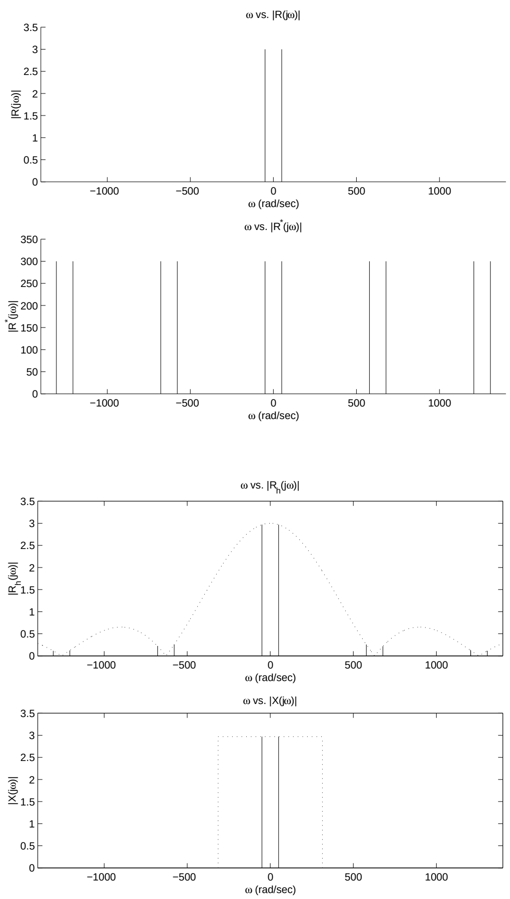
$$LPF(j\omega) = \begin{cases} 1, & |\omega| < \frac{\pi}{T} \\ 0, & \text{otherwise} \end{cases}$$

First recall:

$$\mathcal{F}\{A\cos(\omega_o t + \phi)\} = \pi A \begin{bmatrix} e^{j\phi}\delta(\omega - \omega_o) \\ +e^{j\phi}\delta(\omega + \omega_o) \end{bmatrix} \quad (1)$$

$$ZOH(jw) = Te^{-j\omega T/2}\frac{\sin(\omega T/2)}{\omega T/2} \quad (2)$$

We can sketch the spectrum of all of the signals, $|R(j\omega)|, |R^*(j\omega)|, |R_h(j\omega)|$, and $|X(j\omega)|$.



Where do these come from? From eqn. (1) we have

$$R(j\omega) = 3[e^{j\frac{\pi}{6}}\delta(\omega - 50) + e^{j\frac{\pi}{6}}\delta(\omega + 50)]$$

Hence,

$$|R(j\omega)| = 3[\delta(\omega - 50) + \delta(\omega + 50)]$$

Sampling $r(t)$ to obtain $r^*(t)$ produces copies of $R(jw)$ shifted by integer multiples of the sampling frequency $\omega_s = \frac{2\pi}{T} = 200\pi$ rad/sec that are scaled by a factor of $\frac{1}{T} = 100$. Thus

$$|R^*(j\omega)| = \frac{1}{T}|\sum_{k=-\infty}^{\infty} R(j(\omega - k\omega_s))|$$

$$= 300 \sum_{k=-\infty}^{\infty} [\delta(\omega - k\omega_s - 50) + \delta(\omega - k\omega_s + 50)]$$

$$= 300 \sum_{k=-\infty}^{\infty} [\delta(\omega - 200\pi k - 50) + \delta(\omega - 200\pi k + 50)]$$

We obtain $R_h(j\omega) = ZOH(j\omega)R^*(j\omega)$ using eqn. (2) to get:

$$R_h(j\omega) \;=\; ZOH(j\omega)R^*(j\omega)$$

$$= [Te^{-j\omega T/2}\frac{\sin(\omega T/2)}{\omega T/2}][\frac{1}{T}\sum_{k=-\infty}^{\infty} R(j(\omega - k\omega_s))]$$

$$= [e^{-j\omega T/2}\frac{\sin(\omega T/2)}{\omega T/2}][\sum_{k=-\infty}^{\infty} R(j(\omega - k\omega_s))]$$

$$= 3\frac{\sin(\omega T/2)}{\omega T/2}e^{j\frac{\pi}{6}}\sum_{k=-\infty}^{\infty}\left[\begin{array}{c}\delta(\omega - 200\pi k - 50)\\ +\delta(\omega - 200\pi k + 50)\end{array}\right]$$

$$|R_h(j\omega)| = |3\frac{\sin(\omega T/2)}{\omega T/2}|\sum_{k=-\infty}^{\infty}\left[\begin{array}{c}\delta(\omega - 200\pi k - 50)\\ +\delta(\omega - 200\pi k + 50)\end{array}\right]$$

We obtain $X(j\omega)$ by throwing away all of the signal $R_h(j\omega)$ outside of the interval $[-\omega_c, \omega_c]$ where $\omega_c = \frac{\pi}{T} = 100\pi$, the cut-off frequency of our ideal filter. Thus we are left with:

$$X(j\omega) \;=\; L(j\omega)R_h(j\omega)$$

$$= \; 3\frac{\sin(\omega T/2)}{\omega T/2}e^{j\frac{\pi}{6}}[\delta(\omega - 50) + \delta(\omega + 50)]$$

so

$$|X(j\omega)| \;=\; 3|\frac{\sin(\omega T/2)}{\omega T/2}|[\delta(\omega - 50) + \delta(\omega + 50)]$$

What is the output $x(t)$?

$$
\begin{aligned}
x(t) &= \mathcal{F}^{-}1\{X(j\omega)\} \\
&= \frac{3}{\pi}|ZOH(j50)| \cos(50t + \frac{\pi}{6} + \angle ZOH(j50)) \\
&= \frac{3}{\pi}|\frac{\sin(50T/2)}{50T/2}| \cos(50t + \frac{\pi}{6} - \frac{50T}{2}) \\
&= \frac{3}{\pi}|\frac{\sin(50T/2)}{50T/2}| \cos(50(t - \frac{T}{2}) + \frac{\pi}{6}) \\
&= \frac{3}{\pi}|\frac{\sin(.25)}{.25}| \cos(50(t - .005) + \frac{\pi}{6}) \\
&\approx 0.9896\frac{3}{\pi} \cos(50(t - .005) + \frac{\pi}{6}) \\
&\approx 0.955 \cos(50(t - .005) + \frac{\pi}{6})
\end{aligned}
$$