CAS 704 - Assignment 1

Due date: Tuesday January 29, 2008 at 10:00am

1. **Solutions of Equations** (25 marks) Consider the system shown below:

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (1)$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x} \quad (2)$$

   a) (10 marks) Find the state transition matrix,
   b) (5 marks) the weighting matrix, and
   c) (5 marks) the transfer function.
   d) (5 marks) Is the realization minimal? Justify your answers.

2. **Statespace Models & Stability I** (50 marks)

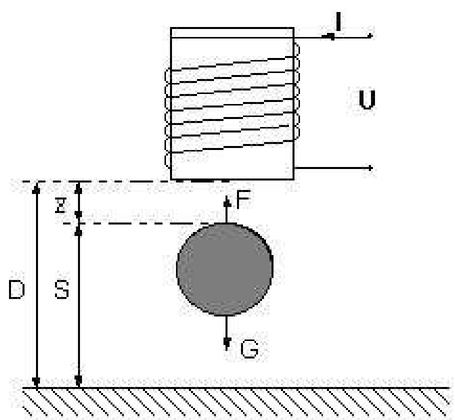


Figure 1: Magnetic suspension of a metal ball of mass M

Consider the system shown in Fig. 1 that uses an electro-magnet to suspend a metal ball of mass M. The equation of motion for the system is given by

$$M\ddot{z} = Mg - \frac{1}{2a} L_o i^2 e^{-z/a} \quad (3)$$

where z is the vertical displacement of the ball and i is the electro-magnet current and g, L_o and a are constants.

Suppose for a fixed current I we an equilibrium at $z = d$ so that $\ddot{z} = \dot{z} = 0$. We will consider a small perturbation about the equilibrium $i = I$ and $z = d$.

Let $\mathbf{x} = [x_1, x_2]^T = [z, \dot{z}]^T$. Take the control input $u = i$

a) (5 marks) Find $\mathbf{f}$ such that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$.

b) (5 marks) Justify the ommision of t as an argument of $\mathbf{f}$ by showing that the system time invariant.

c) (15 marks) As mentioned above, for open loop control $u(t) = I$, the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$ has a stable operating point at $\mathbf{x}_0 = [d, 0]^T$. Find the $\mathbf{A}, \mathbf{B}, \mathbf{C}$ matrices corresponding to the linearization of the system at $\mathbf{x}_0$. Assume that the system only senses the position of the ball (i.e. $y = z$).

d) (5 marks) Find the transfer function of the linear system from u to z.

e) (10 marks) What can you say about the stability of the linear system (assume $L_o, a, g > 0$) without feedback control? How long would you expect the ball to remain suspended without any feedback control?

f) (10 marks) If the system only senses the position of the ball z, is the system stabilizble using a feedback control policy? Justify your answer.

3. **Statespace Models & Stability II**

Let $\mathbf{x} = [x_1, x_2, x_3, x_4]^T = [\theta, l, \dot{\theta}, \dot{l}]^T$ for the nonlinear system governed by the following equations:

$$\ddot{\theta} = -2\frac{\dot{l}}{l}\dot{\theta} - \frac{g}{l}\sin\theta \tag{4}$$

$$\ddot{l} = \dot{\theta}^2 l + g\cos\theta \tag{5}$$

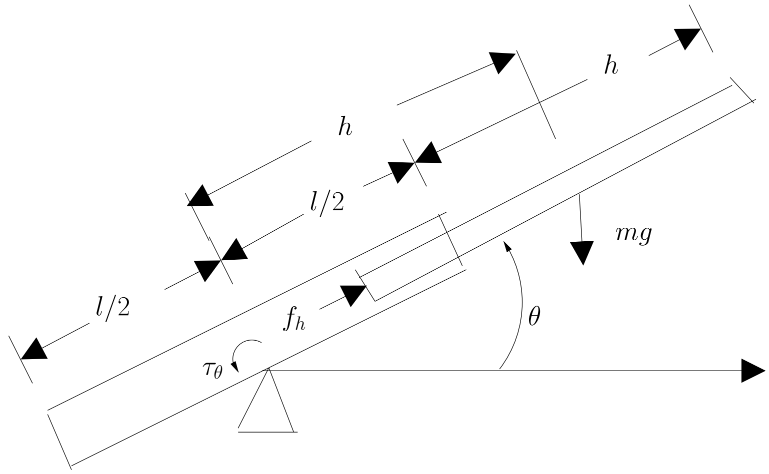
a) (5 marks) Is the system time invariant? Justify your answer.

b) (10 marks) Find $\mathbf{f}$ such that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. For open loop control $\mathbf{u}(t) = [0, 0, 0, -g]^T$, the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \mathbf{u}$ has a stable operating point at $\mathbf{x}_0 = [0, l_0, 0, 0]^T$. Find the $\mathbf{A}$ matrix corresponding to the linearization of the system at $\mathbf{x}_0$.

c) (10 marks) What can you say about the stability of the linear system for $l_0, g > 0$?

4. **State-space Models & Stability III** (70 marks)

Consider the simplified model of links 2 and 3 of the Stanford Manipulator robot shown below.



The distance of the end of the arm from the end of its guide is given by h and the angle of the arm relative to the ground is given by θ. Changes in h and θ are achieved by applying force f_h to the other end of the arm and torque τ_θ at the pivot point, respectively. The equations of motion of the system are given by:

$$
\begin{aligned}
[\frac{1}{6}ml^2 + mh^2(t)]\ddot{\theta}(t) + 2mh(t)\dot{h}(t)\dot{\theta}(t) + mgh(t)\cos\theta(t) &= \tau_\theta(t) \qquad (6)\\
m\ddot{h}(t) - mh(t)\dot{\theta}^2(t) + mg\sin\theta(t) &= f_h(t) \qquad (7)
\end{aligned}
$$

where m, g, and l are constants.

Let the state vector be $\mathbf{x} = [x_1, x_2, x_3, x_4]^T = [\theta, \dot{\theta}, h, \dot{h}]^T$, output vector $\mathbf{y} = [\theta, h]^T$, and the control vector $\mathbf{u} = [u_1, u_2]^T = [\tau_\theta, f_h]^T$.

a) (10 marks) Find $\mathbf{f}$ and $\mathbf{g}$ such that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ and $\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u})$.

b) (5 marks) Find the equilibrium points of the system where a constant input vector $\mathbf{u} = \mathbf{u}^o = [\tau_\theta^o, f_h^o]^T$, results in a constant state $\mathbf{x}^o = [x_1^o, x_2^o, x_3^o, x_4^o]^T$ such that

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}^o, \mathbf{u}^o) = 0$$

c) (20 marks) We will consider a small perturbation about the equilibrium $\mathbf{x} = [0, 0, 0, 0]^T$ and $\mathbf{u} = [0, 0]^T$.

Find the $\mathbf{A}, \mathbf{B}, \mathbf{C}$ matrices corresponding to the linearization $\mathbf{f}$ and $\mathbf{g}$ at this point.

d) For the remainder of this question let $l = m = 1$ and $g = 10$.

(i) (10 marks) What are the open loop poles (eigenvalues) of the linearized system? What can you say about the stability of the linearized system without feedback control?

(ii) (10 marks) Is the system stabilizable using a feedback control policy? Justify your answer.

(iii) (15 marks) Find the transfer function from input f_h to output θ for the linearized system.

5. Do the following questions from Aplevich:

a) Chapter 1 exercises p. 23-26 1, 4, 5, 6

b) Chapter 2 exercises p. 54-58 6, 10, 11, 12, 13, 16, 20, 21

c) Chapter 3 exercises p. 70-72 1, 2, 6, 7