Supervisory Control of Probabilistic Discrete Event Systems

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Abstract

In this paper the Supervisory Control Problem (SCP) for discrete event systems (DES) is generalized to a class of probabilistic discrete event systems (PDES). Necessary and sufficient conditions for the existence of a solution to the probabilistic SCP for a class of nonterminating PDES are developed. Methods of representing probabilistic supervisors for PDES are described and the computation of supervisors is briefly discussed. Finally, we discuss how the results may be extended to terminating probabilistic languages.

I. Introduction

Ramadge and Wonham first developed the Supervisory Control Problem (SCP) for Discrete Event Systems (DES) in [7]. The main concern of the supervisory control problem is to ensure that only acceptable strings or sequences of events occur. Although the deterministic language framework applied in [7] allows for nondeterminism in the sense that there may be more than one possible continuation of a string, there is no effort made to quantify this randomness. It is assumed that the choice of a possible continuation of a string is made by some internal structure unmodeled by the systems designer.

As pointed out in [2], many DES have noise associated with them that we may be able to model by assigning probabilities to the possible one step continuations of a string. This results in the use of probabilistic languages over a set of events as the underlying model for these discrete event systems. In [2] and [3] the author develops an algebra to model probabilistic languages along the lines of Milner's CCS [5].

The treatment in [2] and [3] differs from the optimal control theory of Markov chains [1] and the recent application of supervisory control theory to Semi-Markov Decision problems [4], in its emphasis on sequences of events rather than states and state sequences. The work also allows for the possibility of termination of a system after completing a finite sequence of events. The algebraic theory is used to build complex models of probabilistic DES by combining subsystems using the defined algebraic operators but possible control mechanisms to alter the probabilistic behavior of the systems are not explicitly discussed.

Currently a theory of stochastic supervisory control is being developed. Our probabilistic supervisors differ from the stochastic supervisors of [6] in that the probabilistic supervisors enable or disable a controllable event in the underlying deterministic automata with a certain probability according to the string observed thus far. The stochastic supervisors of [6] can disable a controllable event, or enable it and choose the probabilities with which the occurrence of the controllable event causes the system to move from the current state to any other state of the system in the underlying nondeterministic automata. Probabilistic controls can simply randomly enable or disable controllable transitions. They cannot change the underlying plant dynamics. The probabilistic effect of this random disablement is entirely determined by the plant.

In what follows we use the same event disablement control technology as employed in the control of DES, but we generalize the supervisors of [7] to probabilistic supervisors, allowing them to perform "random disablement". Random disablement is the key feature that allows one to significantly alter the probabilistic behavior of a system. Random disablement will be described in detail in the next section that defines the class of probabilistic generators before introducing the class of probabilistic DES we will consider. Section III develops the Probabilistic Supervisory Control Problem (PSCP), providing necessary and sufficient conditions for the existence of a solution for nonterminating systems. Finally Section IV discusses extensions of these results to terminating systems.

II. Probabilistic Discrete Event Systems

A. Probabilistic Generators

To define a probabilistic discrete event system we first present a generalization of the generator of a formal language. A *probabilistic generator* is a 6-tuple

$$G = (Q, \Sigma, \delta, q_0, Q_m, \rho)$$

where Q is the set of *states* q, Σ is the set of *events* σ, $\delta : Q \times \Sigma \to Q$ is the (partial) transition function, $q_0 \in Q$ is the *initial state*, $Q_m \subset Q$ is the set of *marker states* and $\rho : Q \times \Sigma \to [0,1]$ is the *statewise event probability distribution*. Formally G is equivalent to a directed graph, with node set Q and edges $q \xrightarrow{\sigma} q'$ whenever $\delta(q,\sigma) = q'$, along with the statewise event probability distribution ρ.

The generator G starts in q_0 and executes state transitions, generating sequences of events by following the graph. Upon reaching state q, the generator will execute event σ with probability $\rho(q,\sigma)$. For G to be well defined we require that $\rho(q,\sigma) = 0$ iff $\delta(q,\sigma)$ is not defined. In general $\sum_{\sigma\in\Sigma} \rho(q,\sigma) \leq 1$ and, as in [2] and [3], we interpret $1 - \sum_{\sigma\in\Sigma} \rho(q,\sigma) \leq 1$ to be the probability that upon entering q, the generator terminates in that state. In much of the remainder of this paper we will consider a special class of nonterminating probabilistic generators.

Definition 1: The probabilistic generator G is nonterminating if

$$(\forall q \in Q) \sum_{\sigma\in\Sigma} \rho(q,\sigma) = 1$$

If we let Σ^* denote the set of all finite strings s over Σ including the empty string ϵ, one can easily construct the extended transition function (in fact a partial function) $\delta : Q \times \Sigma^* \to Q$ as follows:

$$\begin{aligned} \delta(q,\epsilon) &= q \\ \delta(q,s\sigma) &= \delta(\delta(q,s),\sigma) \end{aligned}$$

whenever $q' = \delta(q,s)$ and $\delta(q',\sigma)$ are both defined. The *language generated by* G is then

$$L(G) := \{s \in \Sigma^* : \delta(q_0,s) \text{ is defined}\}$$

While $L(G)$ is the set of all finite sequences of events that G can execute, the statewise event probability function ρ implicitly determines a function $L_p(G) : \Sigma^* \to [0,1]$, where

$$\begin{aligned} L_p(G)(\epsilon) &:= 1 \\ L_p(G)(s\sigma) &:= \begin{cases} L_p(G)(s) \cdot \rho(\delta(q_0,s),\sigma) & \text{if } \delta(q_0,s) \text{ is defined} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

As Garg demonstrates, $L_p(G)$ is a p-language (probabilistic language) in the rigorously defined sense of [3]. Hence we call $L_p(G)$ the *probabilistic language generated by* G. The interpretation is that $L_p(G)(s)$ is the probability that G executes string s. Note that $L_p(G)(s) > 0$ iff $s \in L(G)$.

B. Control of PDES

Using identical control technology to that of [7] we introduce an equivalent mathematical description of the supervisors of [7] that is easily generalized to the probabilistic supervisors employed later in the paper. In [7] the systems designer is given a generator G of the language $L(G) \subseteq \Sigma^*$, a partitioning of the event set $\Sigma = \Sigma_c \dot{\cup} \Sigma_u$ into disjoint sets of controllable and uncontrollable events, and a specification language $E \subset \Sigma^*$. It is assumed that controllable events can be disabled, or prevented from occurring, by a supervisor. To solve the supervisory control problem the systems designer must find, if possible, a feedback controller or "supervisor" V such that $L(V/G) = E$, where $L(V/G)$ is the language generated by G under the supervision of V. Here $V : L(G) \times \Sigma \to \{0,1\}$ such that

$$(\forall s \in L(G))(\forall \sigma \in \Sigma)V(s,\sigma) = \begin{cases} 1 & \text{if } \sigma \in \Sigma_u \text{ or } s\sigma \in E \\ 0 & \text{otherwise} \end{cases}$$

The supervisor maps the pair (s, σ) to 1 if event σ is enabled after observing string s and to 0 if the event is to be disabled. Supervisors can also be represented as generators or, equivalently, directed graphs, with σ exiting a state if for all strings s that arrive at the state we have $V(s,\sigma) = 1$.

From now on we will refer to a probabilistic generator G where the partitioning of the event set as $\Sigma = \Sigma_c \cup \Sigma_u$ is understood, as a ***probabilistic discrete event system*** or PDES. Assuming that supervisors can only disable controllable events in order to achieve their control objectives, one might wonder what effect a standard supervisory control as defined above would have upon a PDES. We give a possible interpretation which then motivates the introduction of probabilistic supervisory controls.

Figure 1 shows the simple PDES G, the supervisory control V and $L_p(V/G)$, the probabilistic language generated by G under the supervision of V. G is a nonterminating PDES with $\Sigma = \{\alpha, \beta, \gamma\}$. The evaluation

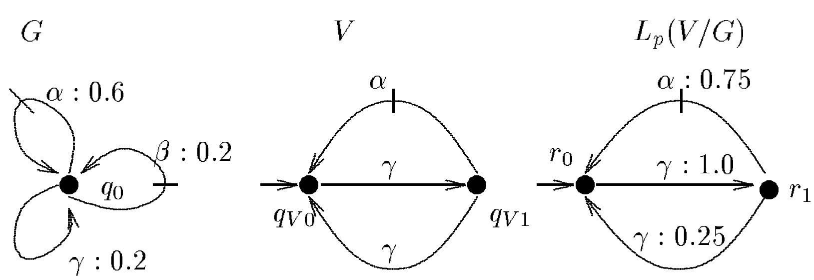

Fig. 1. PDES G and (deterministic) supervisor V and $L_p(V/G)$

of G's statewise event probability distribution ρ is indicated on G's graph by the number immediately following each event. For instance edge $\overset{\alpha:0.6}{\to}$ exiting state q indicates that G will perform α with probability 0.6 when in state q (ie. $(\forall s \in \Sigma^*)\delta(q_0, s) = q$ implies $\rho(\delta(q_0,s),\alpha) = 0.6$). The set of controllable events $\Sigma_c = \{\alpha, \beta\}$ is indicated on the graph of G by placing cross bars on controllable edges.

The arrow enter q_{V0} indicates that supervisor V begins in this state. When in q_{V0} V disables $\{\alpha, \beta\}$ and then alternately disables $\{\beta\}$ and $\{\alpha, \beta\}$. For example $V(\epsilon, \alpha) = 0$ but $V(\gamma\alpha\gamma, \alpha) = 1$. We now have to make an assumption about the behavior of PDES when under control. When an event is disabled by a supervisor then it increases the probability of the remaining enabled events in proportion to their probability in the uncontrolled system. This assumption allows us to determine the probabilistic language that G would be observed to generate when under the supervision of V.

When V disables both α and β only γ is enabled in G. In this case γ occurs with probability 1 since G terminates with probability 0 so γ is the only choice that G can make. When V just disables β, we have a slightly more complicated situation since two alternatives remain. We assume that β's probability is distributed to the enabled events α and γ in proportion to these events respective probabilities of occurrence in the uncontrolled system. More formally, if $V(s, \alpha) = 1$ and $V(s, \beta) = 0$ then upon the occurrence of s, if G is in state q then α will occur with probability

$$\begin{aligned} P(\alpha|\sigma \in \{\gamma, \alpha\}) &= \frac{\rho(q, \alpha)}{\rho(q, \alpha) + \rho(q, \gamma)} \\ &= \frac{0.6}{0.6 + 0.2} = 0.75 \end{aligned}$$

Similarly γ will occur with probability 0.25. This information is represented in the probabilistic generator representing $L_p(V/G)$ in figure 1.

When applied to PDES the class of supervisory controllers of the form $V : L(G) \times \Sigma \rightarrow \{0, 1\}$ can only cause a system to generate a very restricted class of p-languages. Consider the p-language represented by the probabilistic generator shown in figure 2. Clearly there is no (deterministic) supervisory controller V such

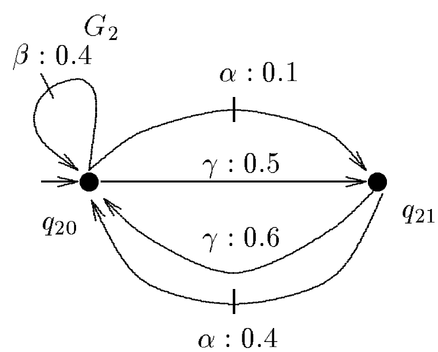

Fig. 2. Probabilistic language $L_p(G_2)$

that $L_p(V/G) = L_p(G_2)$ since a standard disabling supervisor cannot change the event occurrence probability distribution of a system without reducing the probability of one or more controllable events to 0 by disabling the events. In the case of G_2 all the events of G would be required to be initially enabled but with altered probabilities. Also, after an odd number of α and γ events, a supervisor would be required to disable β while increasing the probability of γ and reducing the probability of α. From the previous example we know that when V disables β the probability of α will become 0.75, not the required 0.4. Until now we have been trying to control a random system with a deterministic controller. For instance, upon observing $\gamma\alpha\gamma$, V of figure 1 always disables β. What happens when a supervisor uses "random disablement" to control a PDES?

We now generalize the definition of a supervisor for a DES to define a *probabilistic supervisor* for a PDES. The control technique will be called *random disablement* and can be formalized as follows.

For a PDES $G = (Q, \Sigma, \gamma, q_0, Q_m, \rho)$ a *probabilistic supervisor* is a function $V_p : L(G) \times \Sigma \rightarrow [0, 1]$ such that

$$(\forall s \in L(G))(\forall \sigma \in \Sigma) V_p(s, \sigma) = \begin{cases} 1 & \text{if } \sigma \in \Sigma_u \\ x_{s,\sigma} & \text{for some } x_{s,\sigma} \text{ otherwise} \end{cases}$$

We interpret $V_p(s, \sigma)$ as the probability that the supervisor V_p enables event σ after observing the string s. As was the case with deterministic supervisory controllers, we can represent probabilistic supervisors as labeled directed graphs. Whenever $V_p(s, \sigma) > 0$ then there is an edge labeled by $\sigma : V_p(s, \sigma)$ exiting the current graph state. Figure 3 shows the graph representation of a controller that will later be applied to G of figure 1. Upon

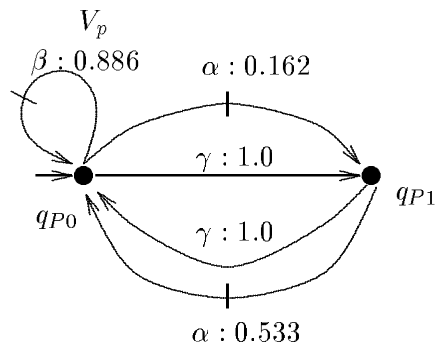

Fig. 3. Probabilistic supervisor V_p

starting in q_{P0} the probabilistic supervisor enables α and β with probabilities 0.162 and 0.886 respectively, while γ is enabled with probability 1.0 (ie. all the time) since it is an uncontrollable event. After an odd number of α and γ events have been observed the supervisor is in state q_{P1}. The absence of a β transition exiting this state indicates that in this case $V_p(s, \beta) = 0$ (β is always disabled). Note that although the graph of V_p is similar to the probabilistic generators used to represent PDES, it is not a probabilistic generator since in general $\sum_{\sigma \in \Sigma} V_p(s, \sigma) > 1$. This is because the number next to an event in the graph is the probabilities that the event is *enabled*, not the probability that the event *occurs* in the system. For each $\sigma \in \Sigma$ $V_p(s, \sigma)$ is independent of the values V_p assigns to other events.

Instead of always enabling or disabling the same events after observing the string s, we suppose that upon observing s the supervisor flips $|\Sigma|$ appropriately biased coins, one for each possible event, and then the supervisor enables or disables the events according to the outcome of their individual coin tosses. Once the choice of event enablements has been made, it is then as if a deterministic supervisory controller with the chosen enablement pattern has been applied and the system acts accordingly. Thus the probability of event α occurring after s in the controlled system can be computed as

$$P(\alpha \text{ in } V/G|s) = \sum_{\Gamma \in 2^{\Sigma}} P(V_p \text{ enables } \Gamma|s) \cdot P(\alpha|V_p \text{ enables } \Gamma) \tag{1}$$

where

$$\begin{aligned}
P(V_p \text{ enables } \Gamma|s) &= \prod_{\sigma \in \Gamma} V_p(s, \sigma) \cdot \prod_{\sigma \in (\Sigma - \Gamma)} (1 - V_p(s, \sigma)) \\
&\text{and} \\
P(\alpha|V_p \text{ enables } \Gamma) &= P(\alpha|\sigma \in \Gamma) \\
&= \frac{\rho(s, \alpha)}{\sum_{\sigma \in \Gamma} \rho(s, \sigma)}
\end{aligned}$$

We now illustrate the effect of probabilistic supervisor V_p of figure 3 upon the PDES G. Note that by the definition of probabilistic supervisor, the only enablement patterns that occur with non-zero probability are those patterns such that $\Sigma_u \subseteq \Gamma$ since for $\sigma \in \Sigma_u$ $V(s, \sigma) = 1$. Thus for the G under consideration, potential enablement patterns are $\Gamma = \Sigma$, $\Gamma = \{\gamma, \alpha\}$ and $\Gamma = \{\gamma, \beta\}$. If we assume that s is the string that G has executed under the supervision of V_p, then for $s = \epsilon$, or s containing an even number of α and γ transitions, V_p will enable each event with the probabilities given by the events exiting q_{P0} in figure 3. Thus letting P_σ

be the probability of σ continuing s in the controlled system V_p/G we have

$$\begin{aligned} P_\alpha &= \rho(q,\alpha)V_p(s,\alpha)V_p(s,\beta) + P(\alpha|\sigma \in \{\gamma,\alpha\})V_p(s,\alpha)(1-V_p(s,\beta)) \\ &= 0.6(0.162)(0.886) + 0.75(0.162)(1-0.886) \\ &= 0.1 \end{aligned} \tag{2}$$

$$\begin{aligned} P_\beta &= \rho(q,\beta)V_p(s,\alpha)V_p(s,\beta) + P(\beta|\sigma \in \{\gamma,\beta\})(1-V_p(s,\alpha))V_p(s,\beta) \\ &= 0.2(0.162)(0.886) + 0.5(1-0.162)(0.886) \\ &= 0.4 \end{aligned} \tag{3}$$

$$\begin{aligned} P_\gamma &= \rho(q,\gamma)V_p(s,\alpha)V_p(s,\beta) + P(\gamma|\sigma \in \{\gamma,\alpha\})V_p(s,\alpha)(1-V_p(s,\beta)) \\ &\quad +P(\gamma|\sigma \in \{\gamma,\beta\})(1-V_p(s,\alpha))V_p(s,\beta) \\ &\quad +P(\gamma|\sigma \in \{\gamma\})(1-V_p(s\alpha))(1-V_p(s,\beta)) \\ &= 0.2(0.162)(0.886) + 0.25(0.162)(1-0.886) \\ &\quad +0.5(1-0.162)(0.886) + 1.0(1-0.162)(1-0.886) \\ &= 0.5 \end{aligned} \tag{4}$$

After observing s containing an odd number of α's and γ's V_p will enable events with the probabilities given by state q_{P1}. By a calculation identical to the above with these new values of $V_p(s,\sigma)$ substituted we obtain $P_\alpha = 0.4$, $P_\beta = 0.0$ and $P_\gamma = 0.6$. Note that $L_p(V_p/G) = L_p(G_2)$, thus the introduction of probabilistic supervisors allows us to generate p-languages that no deterministic supervisor could cause the system to generate. Clearly any deterministic supervisor can be viewed as a special case of a probabilistic supervisor.

The above example serves to illustrate some interesting properties of controlled PDES. First, through random disablement the probability of an uncontrollable event with nonzero occurrence probability is always increased when controllable events are randomly disabled. Second, it is possible to increase the probability of some controllable events at the expense of other controllable events. In our example β's probability in one state was increased from 0.2 to 0.4 while α's probability was reduced from 0.6 to 0.1. Now that we have a general idea of how a PDES's behavior may be altered it is logical to ask: What probabilistic languages can a PDES generate under probabilistic supervisory control? We will answer this question for the above system, and for nonterminating PDES in general, in the next section, when we develop necessary and sufficient conditions for the solution of the probabilistic supervisory control problem.

III. The Probabilistic Supervisory Control Problem

We begin this section with a formal statement of the Probabilistic Supervisory Control Problem (PSCP), followed by some results on how probabilistic supervisors affect PDES, culminating in Theorem 5 giving necessary and sufficient conditions for the existence of a solution to the PSCP for nonterminating PDES.

Probabilistic Supervisory Control Problem (PSCP). Given PDES G_1 and G_2, construct, if possible, a probabilistic supervisor V_p such that $L_P(V_p/G_1) = L_p(G_2)$.

The main result of this section will follow immediately from two propositions regarding the behavior of nonterminating PDES under probabilistic supervisory control. In the remainder of the section we consider only nonterminating PDES.

For a nonterminating PDES $G = (Q,\Sigma,\delta,q_0,Q_m,\rho)$ we illustrate a typical state q in figure 4. Let $Pos(q) := \{\sigma \in \Sigma | \rho(q,\sigma) > 0\}$, the set of events that are possible next transitions from state q in G. $Pos(q) \cap \Sigma_c = \{\sigma_1,\ldots,\sigma_n\}$ represents the set of controllable events with nonzero occurrence probability in state q (ie. $\rho(q,\sigma_i) > 0$) and similarly $\{\sigma_{u1}\ldots\sigma_{um}\} = Pos(q) \cap \Sigma_u$. We now assume that G is under the supervision of

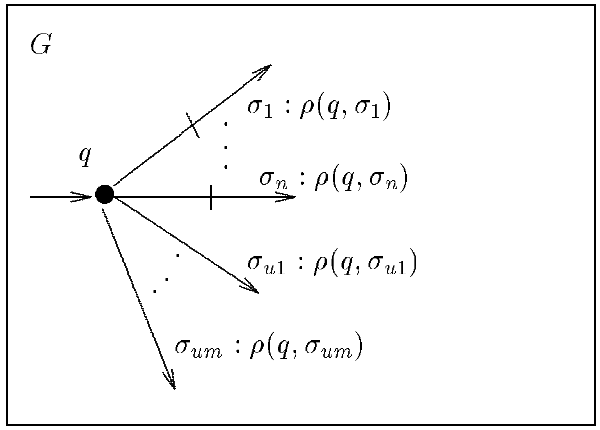

Fig. 4. Typical state q in nonterminating PDES G

some V_p such that for string $s \in \Sigma^*$ $\delta(q_0, s) = q$ and $L_p(V_p/G)(s) > 0$. In this case we will let P_σ be the probability that σ continues s in V_p/G, that is

$$P_\sigma := L_p(V_p/G)(s\sigma)/L_p(V_p/G)(s)$$

The first proposition states that the ratio of the probabilities of uncontrollable events remains constant under supervision.

Claim 2: For G and q as given above, if there exists $\sigma_{ui} \in \Sigma_u$ such that $\rho(q, \sigma_{ui}) > 0$ then for any probabilistic supervisor V_p we have for all $\sigma_u \in \Sigma_U$

$$\frac{P_{\sigma_u}}{P_{\sigma_{ui}}} = \frac{\rho(q, \sigma_u)}{\rho(q, \sigma_{ui})}$$

Proof: Let $k = \frac{\rho(q,\sigma_u)}{\rho(q,\sigma_{ui})}$. From the previous section we know that

$$\begin{aligned}
P_{\sigma_u} &= \sum_{\Gamma \in 2^\Sigma} P(V_p \text{ enables } \Gamma | s) \cdot P(\sigma_u | V_p \text{ enables } \Gamma) \\
\text{where} & \\
P(\sigma_u | V_p \text{ enables } \Gamma) &= P(\sigma_u | \sigma \in \Gamma) \\
&= \frac{\rho(s, \sigma_u)}{\sum_{\sigma \in \Gamma} \rho(s, \sigma)} \\
&= \frac{k \cdot \rho(s, \sigma_{ui})}{\sum_{\sigma \in \Gamma} \rho(s, \sigma)} \\
\text{hence} & \\
P_{\sigma_u} &= \sum_{\Gamma \in 2^\Sigma} P(V_p \text{ enables } \Gamma | s) \cdot k P(\sigma_{ui} | V_p \text{ enables } \Gamma) \\
&= k P_{\sigma_{ui}}
\end{aligned}$$

□

The above claim informs us that we are relatively restricted in how we can influence the probability of uncontrollable events, as we might expect. The following example illustrates that the designer is free to choose the occurrence probabilities of controllable events within the bounds given in Claim 4. Once again considering the G of figure 1, equations (2) and (3) relate P_α and P_β to a supervisor's choice of $V_P(s,\alpha)$ and $V_p(s,\beta)$. Substituting G's values for $\rho(q,\alpha)$ and $\rho(q,\beta)$ we obtain:

$$\begin{aligned} P_\alpha(V_p(s,\alpha),V_p(s,\beta)) &= 0.6V_p(s,\alpha)V_p(s,\beta)+0.75V_p(s,\alpha)(1-V_p(s,\beta)) \\ &= 0.75V_p(s,\alpha)-0.15V_p(s,\alpha)V_p(s,\beta) \qquad (5)\\ P_\beta(V_p(s,\alpha),V_p(s,\beta)) &= 0.2V_p(s,\alpha)V_p(s,\beta)+0.5(1-V_p(s,\alpha))V_p(s,\beta) \\ &= 0.5V_p(s,\beta)-0.3V_p(s,\alpha)V_p(s,\beta) \qquad (6)\end{aligned}$$

For any V_p we know that $(V_p(s,\alpha),V_p(s,\beta)) \in [0,1]\times[0,1]$. Figure 5 represents this region and the resulting

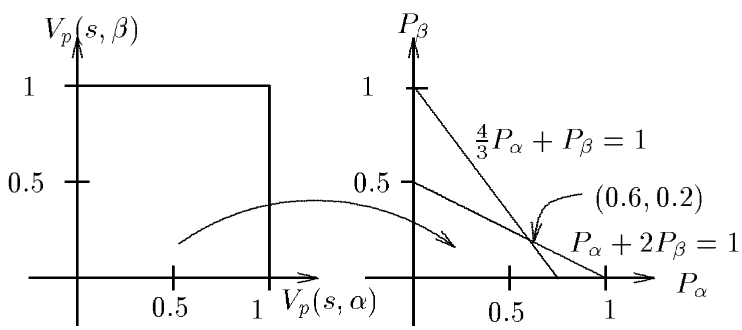

Fig. 5. Mapping from choice of V_p to resulting (P_α, P_β)

image region under the mapping using equations (5) and (6). To obtain the image of the region we map the boundaries into the (P_α,P_β) plane. When $V_p(s,\alpha)=0$ in (5) then $P_\alpha=0$ as we would expect, since in this case α is always disabled, while $P_\beta(0,V_p(s,\beta))=0.5V_p(s,\beta)$. Thus if $V_p(s,\beta)\in[0,1]$ with $V_p(s,\alpha)=0$ then $P_\alpha=0$ and $P_\beta\in[0,0.5]$. Similarly when $V_p(s,\beta)=0$ then $P_\alpha\in[0,0.75]$ and $P_\beta=0$. Substituting $V_p(s,\alpha)=1$ in (6) we can solve for $V_p(s,\beta)$ in terms of P_β and then substituting into (5) to obtain $\frac{4}{3}P_\alpha+P_\beta=1$. Setting $V_p(s,\beta)=1$ and eliminating $V_p(s,\alpha)$ gives $P_\alpha+2P_\beta=1$. Note that these two lines intersect at $(P_\alpha,P_\beta)=(0.6,0.2)$, the values of the uncontrolled event probabilities in G and hence the probabilities when $V_p(s,\alpha)=V_p(s,\beta)=1$.

It is possible to derive the linear bounds on P_α and P_β by another method. For a nonterminating PDES G, upon arriving at a state q after executing string s, if $(\exists\sigma\in Pos(q))V_p(s,\sigma)=1$ then the uncontrolled system exits q with probability 1. This follows from the fact that the uncontrolled system terminates in q with probability 0 and the supervisor never disables all the possible events. In this case we conclude that

$$\sum_{\sigma\in\Sigma} P_\sigma = 1$$

For G of figure 1 $\rho(q,\gamma)=0.2$ and $\gamma\in\Sigma_u$ so by definition, if V_p is a probabilistic supervisor then $V_p(s,\gamma)=1$. Thus in the case of G

$$P_\alpha + P_\beta + P_\gamma = 1 \qquad (7)$$

In the one boundary case $V_p(s,\alpha)=1$ so α is, in effect, uncontrollable. Thus for any choice of $V_p(s,\beta)$ with $V_p(s,\alpha)=1$ it follows from Claim 2 that

$$\frac{P_\gamma}{P_\alpha} = \frac{\rho(q,\gamma)}{\rho(q,\alpha)} = \frac{1}{3}$$

But any reduction of $V_p(s,\alpha)$ will decrease P_α while increasing P_γ. Hence we conclude that

$$P_\gamma \geq \frac{1}{3} P_\alpha$$

with equality holding when $V_p(s,\alpha) = 1$. Using this result in (7) we have

$$\begin{aligned} P_\alpha + P_\beta + P_\gamma \quad &\geq \quad P_\alpha + P_\beta + \frac{1}{3} P_\alpha \\ \text{hence} \\ \frac{4}{3} P_\alpha + P_\beta \leq 1 \end{aligned}$$

Upon noting that $\rho(q,\beta) = \rho(q,\gamma)$ a similar computation yields

$$P_\alpha + 2P_\beta \leq 1$$

The next result shows that this bounding of assignable probabilities extends to cases where there are an arbitrary number of controllable and uncontrollable events exiting state q, each with nonzero probability.

Definition 3: Given a PDES G with q as shown in figure 4 with $\{\sigma_1, \ldots \sigma_n\} = Pos(q) \cap \Sigma_c$ and $\{\sigma_{u1}, \ldots, \sigma_{um}\} = Pos(q) \cap \Sigma_u$. Suppose $s \in L(G)$ such that $\delta(q_0, s) = q$ and that $V_p : L(G) \times \Sigma \to [0,1]$ is an arbitrary probabilistic supervisor for G. Then we define the *set of assignable probabilities for* Σ_c, $SAP_c(q)$, to be the following:

$$SAP_c(q) := \{(x_1, \ldots, x_n) \in [0,1]^n : (\exists V_p) \text{ for } i = 1..n, x_i = P_{\sigma_i}\}$$

Claim 4: For G and $q = \delta(q_0, s)$ with $Pos(q) \cap \Sigma_c = \{\sigma_1, \ldots, \sigma_n\}$ and $Pos(q) \cap \Sigma_u = \{\sigma_{u1}, \ldots, \sigma_{um}\}$ as in Definition 3 it follows that

$$SAP_c(q) = \bigcap_{i=1}^{n} \{(x_1, \ldots, x_n) \in [0,1]^n : c_i x_i + \sum_{j=1}^{n} x_j \leq 1\}$$

where

$$c_i = \frac{\sum_{\sigma \in \Sigma_u} \rho(q,\sigma)}{\rho(q,\sigma_i)}$$

Proof: (Sketch) The proof relies upon considering the boundary case $V_p(s,\sigma_i) = 1$ and then applying Claim 2 as above to obtain $\sum_{j=1}^{m} P_{\sigma_{uj}} \geq \sum_{\sigma \in \Sigma_u} \frac{\rho(q,\sigma)}{\rho(q,\sigma_i)} P_{\sigma_i}$. $\Box$

The two preceding claims can be easily used to develop necessary and sufficient conditions for the existence of a solution to the PSCP for nonterminating PDES.

Theorem 5: Given nonterminating PDES $G_1 = (Q, \Sigma, \delta_1, q_0, Q_m, \rho_1)$ and $G_2 = (R, \Sigma, \delta_2, r_0, R_m, \rho_2)$ with disjoint state sets Q and R. There exists a probabilistic supervisor V_p such that $L_p(V_p/G_1) = L_p(G_2)$ iff for all $s \in L(G_2)$ there exists $q \in Q$ such that $\delta_1(q_0, s) = q$ and, letting $r = \delta_2(r_0, s)$, the following two conditions hold:

(i) .
$$Pos(q) \cap \Sigma_u = Pos(r) \cap \Sigma_u$$

and if $\sigma_{ui} \in Pos(r) \cap \Sigma_u$ then $(\forall \sigma_u \in \Sigma_u)$

$$\rho_2(r, \sigma_u) = \frac{\rho_1(q, \sigma_u)}{\rho_1(q, \sigma_{ui})} \rho_2(r, \sigma_{ui})$$

(ii) .
$$Pos(r) \cap \Sigma_c \subseteq Pos(q) \cap \Sigma_c$$

and $(\forall \sigma_i \in Pos(q) \cap \Sigma_c)$

$$c_i \rho_2(r, \sigma_i) + \sum_{\sigma \in Pos(q) \cap \Sigma_c} \rho_2(r, \sigma) \leq 1$$

where

$$c_i = \frac{\sum_{\sigma \in \Sigma_u} \rho_1(q, \sigma)}{\rho_1(q, \sigma_i)}$$

Proof: The statement is an immediate consequence of Claims 2 and 4.

□

Given G_1 and G_2, when $\delta_1(q_0, s) = q$, solving for the values $V_p(s, \sigma)$ to make $L_p(V_p/G_1)(s\sigma) = L_p(G_2)(s\sigma)$ involves solving $|Pos(q)|$ simultaneous polynomial equations in $|Pos(q) \cap \Sigma_c|$ variables. The main result of this section says, in effect, that a solution to these equations exists iff the $|Pos(q) \cap \Sigma_u| - 1$ linear equations of condition (i) and the $|Pos(q) \cap \Sigma_c|$ linear inequalities of condition (ii) are satisfied. By tracing through the graphs of G_1 and G_2 while checking conditions (i) and (ii) it is possible to determine if a supervisor exists that solves the given case of the PSCP. If a solution exists, one can then solve the systems of nonlinear equations derived from formula (1).

Consider nonterminating PDES G_1 of figure 6. Here $\Sigma_c = \{\alpha, \beta, \mu\}$ while $\Sigma_u = \{\gamma, \tau\}$. Suppose we want

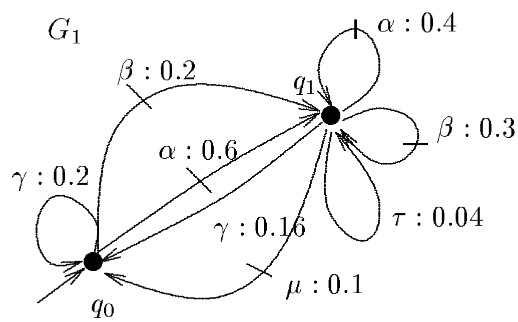

Fig. 6. Example to illustrate Theorem 5

to solve the PSCP for some G_2. Letting $V_p(s, \sigma) = x_\sigma$ then for $s \in L(G_2)$ such that $\delta_1(q_0, s) = q_0$ and $\delta_2(r_0, s) = r$ for some $r \in R$, the set of equations developed from (1) that have to be solved to determine the values of x_α and x_β are given below.

$$\begin{aligned} 0.75x_\alpha - 0.15x_\alpha x_\beta &= \rho_2(r, \alpha) \\ 0.5x_\beta - 0.3x_\alpha x_\beta &= \rho_2(r, \beta) \\ 1 - 0.75x_\alpha - 0.5x_\beta + 0.45x_\alpha x_\beta &= \rho_2(r, \gamma) \end{aligned}$$

Note that the values of $\rho_1(q_0, \cdot)$ are identical to those of $\rho(q_0, \cdot)$, so these equations are obtained by straight forward substitution in equations (5), (6) and (7) respectively. It also follows that these equations have a solution iff the event set conditions of (i) and (ii) hold and the linear inequalities below are satisfied.

$$\begin{aligned} \frac{4}{3}\rho_2(r, \alpha) + \rho_2(r, \beta) &\leq 1 \\ \rho_2(r, \alpha) + 2\rho_2(r, \alpha) &\leq 1 \end{aligned}$$

For the case when $\delta_1(q_0, s) = q_1$ then the equations resulting from (1) to be solved are given by

$$\frac{1}{315}[16x_\alpha x_\beta x_\mu - 70x_\alpha x_\beta - 30x_\alpha x_\mu + 210x_\alpha] = \rho_2(r, \alpha) \tag{8}$$

$$\frac{1}{30}[2x_\alpha x_\beta x_\mu - 8x_\alpha x_\beta - 3x_\beta x_\mu + 18x_\beta] = \rho_2(r,\beta) \tag{9}$$

$$\frac{1}{210}[26x_\alpha x_\beta x_\mu - 40x_\alpha x_\mu - 35x_\beta x_\mu + 70x_\mu] = \rho_2(r,\mu) \tag{10}$$

for the controllable transitions and, letting LS(8) denote the left side of equation (8), the equations for the uncontrollable transitions are

$$\tfrac{4}{5}[1-\text{LS(8)-LS(9)-LS(10)}] = \rho_2(r,\gamma)$$
$$\tfrac{1}{5}[1-\text{LS(8)-LS(9)-LS(10)}] = \rho_2(r,\tau)$$

These equations have a solution iff the other constraints of (i) and (ii) are met and

$$\begin{aligned}
\rho_2(r,\tau) &= \frac{1}{4}\rho_2(r,\gamma) \\
\frac{3}{2}\rho_2(r,\alpha) + \rho_2(r,\beta) + \rho_2(r,\mu) &\leq 1 \\
\rho_2(r,\alpha) + \frac{5}{3}\rho_2(r,\beta) + \rho_2(r,\mu) &\leq 1 \\
\rho_2(r,\alpha) + \rho_2(r,\beta) + 3\rho_2(r,\mu) &\leq 1
\end{aligned}$$

IV. Extension to Terminating PDES

With some minor modifications Theorem 5 can be extended to cover terminating PDES by a technique employed in [3]. A terminating PDES G with event set Σ can be transformed to a nonterminating PDES G' with the extended event set $\Sigma \cup \{\zeta\}$ where $\zeta \notin \Sigma$. From terminating $G = (Q, \Sigma, \delta, q_0, Q_m, \rho)$ we obtain nonterminating $G' = (Q \cup \{q_\zeta\}, \Sigma \cup \{\zeta\}, \delta', q_0, Q_m, \rho')$ where

$$\begin{aligned}
\rho'(q,\sigma) &= \rho(q,\sigma) \\
\rho'(q,\zeta) &= 1 - \sum_{\sigma\in\Sigma} \rho(q,\sigma) \\
\rho'(q_\zeta,\zeta) &= 1 \\
&\text{and} \\
\delta'(q,\sigma) &= \delta(q,\sigma) \\
\delta'(q,\zeta) &= q_\zeta \text{ if } \rho'(q,\zeta) > 0 \\
\delta'(q_\zeta,\zeta) &= q_\zeta
\end{aligned}$$

Figure 7 shows a simple terminating PDES and the nonterminating PDES G' obtained by the above transformation. One of the subtleties in the terminating case is illustrated here. If we are to consider the event ζ in G' to be equivalent to termination in G then we have to consider the special case when all the events exiting a state q are controllable. In this case a supervisor can cause the system G to terminate by disabling all the controllable events. Upon entering q the probability of termination in the controlled system is equal to the probability that the supervisor disables all events in $Pos(q) \cap \Sigma_c$. This situation is indicated in figure 7 by the dashed edge from q to q_ζ.

V. Conclusions

In this paper we have introduced the class of probabilistic discrete event systems. The use of random disablement generalizes the (deterministic) supervisors of [7] to probabilistic supervisors. The Supervisory Control Problem for DES then generalizes in a straight forward way to the Probabilistic Supervisory Control

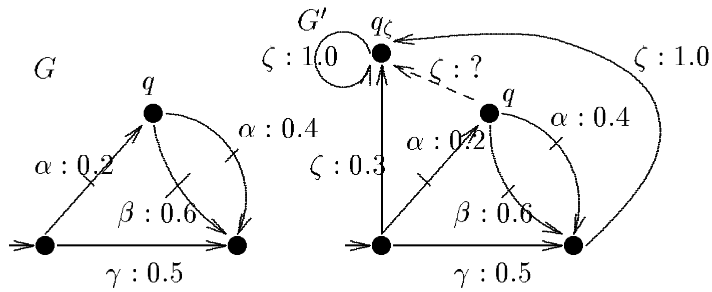

Fig. 7. Terminating PDES G and resulting nonterminating PDES G'

Problem (PSCP). The paper has focused on nonterminating PDES with the development of necessary and sufficient conditions for the existence of a solution to the PSCP in the case when both the PDES to be controlled and the specification PDES (or equivalently the specification p-language) are nonterminating. A method for extending the results on the PSCP to terminating PDES is outlined.

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