Software Engineering/Mechatronics 3DX4

Slides 7: Steady-State Errors

Dr. Ryan Leduc

Department of Computing and Software
McMaster University

Material based on lecture notes by P. Taylor and M. Lawford, and *Control Systems Engineering* by N. Nise.

Introduction

- We now focus on the third design specification, *steady-state error.*
- We define steady-state error to be the difference between input and ouput as $t \to \infty$.
- We will see that control system design typically means we will have to make trade-offs between the desired transient, steady-state, and stability specifications.

Test Inputs

- Table below shows the standard test inputs typically used for evaluating steady-state error.

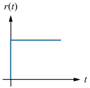
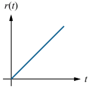
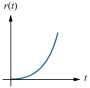
Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Table 7.1.

Choosing a Test Inputs

- The test inputs we will choose for our steady-state analysis and design depends on our target application.

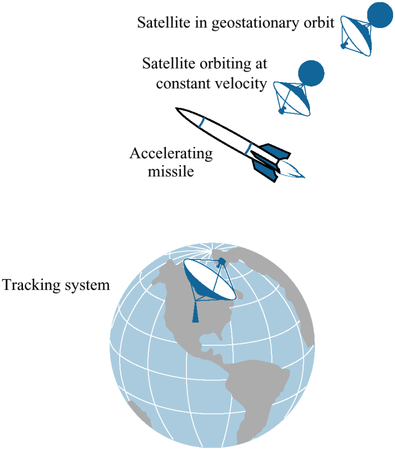


Figure 7.1.

Steady-State Error and Stable Systems

- The calculations we will be deriving for steady-state apply only to stable systems.
- Unstable systems represent loss of control in steady-state as the transient response swamps the forced response.
- As we analyze and design a system for steady-state error, we must constantly check the system for stability.

Steady-State Error and Step Inputs

- With step inputs, we can get two types of steady-state errors:
  1. Zero error.
  2. A constant error value.

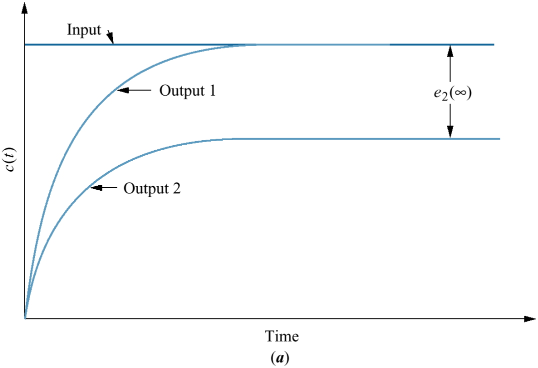


Figure 7.2.

Steady-State Error and Ramp input

- With ramp inputs, we can get three types of steady-state errors:
  1. Zero error.
  2. A constant error value.
  3. Infinite error.

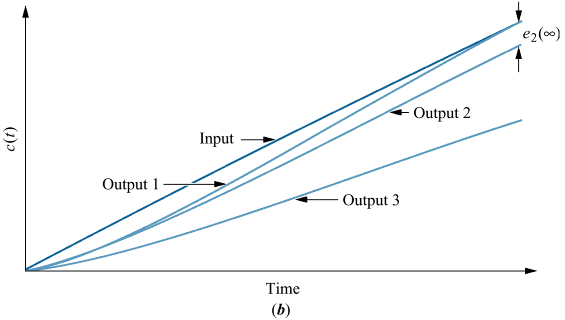


Figure 7.2.

Steady-State Error and Block Diagrams

- If we have a closed-loop transfer function $T(s)$, we can represent our error signal, $E(s)$, as in figure (a).

- We are interested in the time domain signal, $e(t) = \mathcal{L}^{-1}\{E(s)\}$, as $t \rightarrow \infty$.

- If we have a unity feedback system, we already have $E(s)$ as part of our diagram, as shown in figure (b).

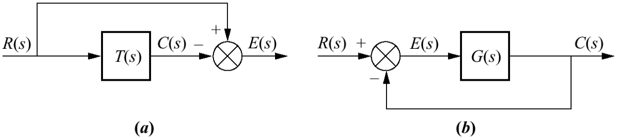


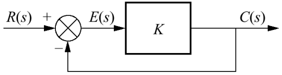
Figure 7.3.

Sources of Steady-State Error

- Steady-state errors can arise from nonlinear sources, such as backlash in gears or motors requiring a minimum input voltage before it starts to move.
- Steady-state errors can also arise from configuration of system and the input we apply.
- Consider a step input applied to the system below which has constant gain.
- If a unity feedback system has a feedforward transfer function $G(s)$, then we can derive the transfer function $\frac{E(s)}{R(s)}$ as follows:

$$C(s) = E(s)G(s) \qquad (1)$$

$$E(s) = R(s) - C(s) \qquad (2)$$



(a)

Figure 7.4.

Sources of Steady-State Error - II

- Substituting equation 1 into equation 2 gives:

$$E(s) = R(s) - E(s)G(s)$$
$$E(s)[1 + G(s)] = R(s)$$
$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)} \quad (3)$$

- For $G(s) = K$, we get

$$\frac{E(s)}{R(s)} = \frac{1}{1 + K} \quad (4)$$

- For $R(s) = \frac{1}{s}$ (unit step), we get $E(s) = \frac{1}{s(1+K)}$.
- We thus have $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{1}{1+K}$

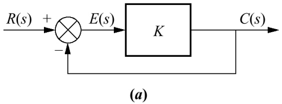


Figure 7.4.

Sources of Steady-State Error - III

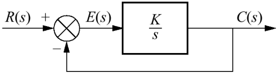
$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)}$$

- If we add an integrator to the forward-path gain, we get $G(s) = \frac{K}{s}$ giving

$$\frac{E(s)}{R(s)} = \frac{1}{1+\frac{K}{s}} = \frac{s}{s+K} \qquad (5)$$

- For $R(s) = \frac{1}{s}$ (unit step), we get $E(s) = \frac{1}{(s+K)}$.
- We thus have

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{0}{0+K} = 0 \qquad (6)$$



(b)

Figure 7.4.

Steady-State Error and $T(s)$

- In Diagram below, we have $E(s) = R(s) - C(s)$.
- We also have:

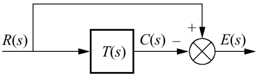
$$C(s) = R(s)T(s) \tag{7}$$

- Combining the two we get

$$E(s) = R(s) - R(s)T(s) = R(s)[1 - T(s)] \tag{8}$$

- We thus have

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s\,E(s) \\ &= \lim_{s\to 0} s\,R(s)[1 - T(s)] \end{aligned} \tag{9}$$



(a)

Figure 7.3.

Steady-State Error and $G(s)$

- From equation 3, we have

$$E(s) = \frac{R(s)}{1 + G(s)} \tag{10}$$

- We thus have

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} s\, E(s) \\ &= \lim_{s \to 0} s\, \frac{R(s)}{1 + G(s)} \end{aligned} \tag{11}$$

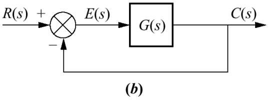


Figure 7.3.

Steady-State Error, $G(s)$, and Step Input

- For input $R(S) = \frac{1}{s}$, we get

$$e_{ss} = \lim_{s \to 0} s \, \frac{1/s}{1 + G(s)} = \frac{1}{1 + \lim_{s \to 0} G(s)} \qquad (12)$$

- We refer to the term $\lim_{s \to 0} G(s)$ as dc gain of the forward transfer function.

- To have zero steady-state error we need

$$\lim_{s \to 0} G(s) = \infty \qquad (13)$$

- For $G(s)$ of form below, we thus need $n \geq 1$

$$G(s) \equiv \frac{(s + z_1)(s + z_2)\cdots}{s^n (s + p_1)(s + p_2)\cdots} \qquad (14)$$

- If $n = 0$, we get

$$\lim_{s \to 0} G(s) = \frac{(0 + z_1)(0 + z_2)\cdots}{(0 + p_1)(0 + p_2)\cdots} = \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots} \qquad (15)$$

Steady-State Error, $G(s)$, and Ramp Input

- For input $R(S) = \frac{1}{s^2}$, we get

$$e_{ss} = \lim_{s \to 0} \frac{s(1/s^2)}{1 + G(s)} = \lim_{s \to 0} \frac{1}{s + sG(s)} = \frac{1}{\lim_{s \to 0} sG(s)} \quad (16)$$

- To have zero steady-state error for ramp input, we need

$$\lim_{s \to 0} s\,G(s) = \infty \quad (17)$$

- For $G(s)$ of form below, we thus need $n \geq 2$

$$G(s) \equiv \frac{(s + z_1)(s + z_2)\cdots}{s^n (s + p_1)(s + p_2)\cdots} \quad (18)$$

- If $n = 1$, we get

$$\lim_{s \to 0} s\,G(s) = \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots} \quad (19)$$

- If $n = 0$, we get

$$\lim_{s \to 0} s\,G(s) = \frac{s(s + z_1)(s + z_2)\cdots}{(s + p_1)(s + p_2)\cdots} = 0 \quad (20)$$

Steady-State Error, $G(s)$, and Parabolic Input

- For input $R(S) = \frac{1}{s^3}$, we get

$$e_{ss} = \lim_{s \to 0} \frac{s(1/s^3)}{1 + G(s)} = \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s)} = \frac{1}{\lim_{s \to 0} s^2 G(s)} \quad (21)$$

- To have zero steady-state error for ramp input, we need

$$\lim_{s \to 0} s^2 \, G(s) = \infty \quad (22)$$

- For $G(s)$ of form below, we thus need $n \geq 3$

$$G(s) \equiv \frac{(s + z_1)(s + z_2) \cdots}{s^n (s + p_1)(s + p_2) \cdots} \quad (23)$$

- If $n = 2$, we get

$$\lim_{s \to 0} s^2 \, G(s) = \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots} \quad (24)$$

- If $n = 1$, we get

$$\lim_{s \to 0} s^2 \, G(s) = \frac{s(s + z_1)(s + z_2) \cdots}{(s + p_1)(s + p_2) \cdots} = 0 \quad (25)$$

Steady-State Error eg.

- Find the steady state errors for inputs $5u(t)$, $5tu(t)$, and $5t^2u(t)$.

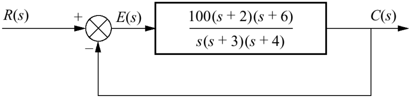


Figure 7.6.

Static Error Constants

- We now define steady state-error performance specifications called static error constants.

1. **Position Constant:** $K_p = \lim_{s \to 0} G(s)$, thus

$$e_{step}(\infty) = \frac{1}{1 + K_p}$$

2. **Velocity Constant:** $K_v = \lim_{s \to 0} sG(s)$, thus

$$e_{ramp}(\infty) = \frac{1}{K_v}$$

3. **Aceleration Constant:** $K_a = \lim_{s \to 0} s^2 G(s)$, thus

$$e_{parabola}(\infty) = \frac{1}{K_a}$$

System Type

- The static error constants are determined by the structure of $G(s)$.
- They are mostly determined by the number of integrators in $G(s)$.
- The system type is the number of integrators in the forward path, thus the value of n in figure below.

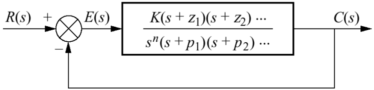


Figure 7.8.

Steady-State Error Summary

- Table shows relationship between input type, system type, static error constants, and steady-state errors.

		Type 0		Type 1		Type 2	
Input	Steady-state error formula	Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

Table 7.2.

Tight Steady-State Error Specifications

- Example of a system requiring tight steady-state error specifications to be useful.

Figure 7.9.

Steady-State Error Specifications eg.

- For system below, find value of K such there is 10% error in steady state.

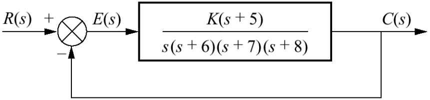


Figure 7.10.

Steady-State Error and Disturbances

- Can use feedback systems to handle unwanted disturbances to the systems.
- By using feedback, we can design systems that follow the input signal with small or zero error, *despite* these disturbances.
- Consider feedback system below with disturbance, $D(S)$, added between plant and controller.
- The system output is

$$C(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s) \tag{26}$$

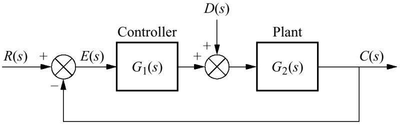


Figure 7.11.

Steady-State Error and Disturbances - II

- However

$$E(s) = R(s) - C(s) \Rightarrow C(s) = R(s) - E(s) \tag{27}$$

- Using Equations 27 and 26 and solving for $E(s)$ gives

$$E(s) = \frac{R(s)}{1 + G_1(s)G_2(s)} - \frac{D(s)G_2(s)}{1 + G_1(s)G_2(s)} \tag{28}$$

- Using final-value theorem, the steady-state error is

$$e_{ss} = \lim_{s \to 0} sE(s) \tag{29}$$

$$= \lim_{s \to 0} \frac{sR(s)}{1 + G_1(s)G_2(s)} - \lim_{s \to 0} \frac{sD(s)G_2(s)}{1 + G_1(s)G_2(s)} \tag{30}$$

$$= e_R(\infty) + e_D(\infty) \tag{31}$$

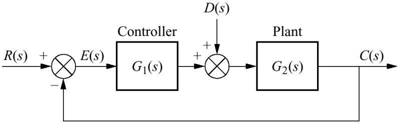


Figure 7.11.

Steady-State Error and Disturbances - III

- The $e_R(\infty)$ term is the steady-state error due to input $R(s)$ that we have already seen.
- The $e_D(\infty)$ term is the steady-state error due to $D(s)$.
- If $D(s) = 1/s$ (step input), we have

$$e_D(\infty) = -\frac{1}{\lim_{s\to 0} \frac{1}{G_2(s)} + \lim_{s\to 0} G_1(s)} \qquad (32)$$

- If we set $R(s) = 0$, we get from Eqn28 the transfer function:

$$\frac{E(s)}{D(s)} = -\frac{G_2(s)}{1 + G_1(s)G_2(s)} \qquad (33)$$

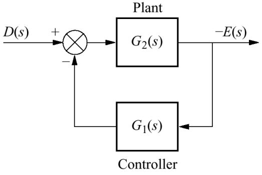


Figure 7.12.

Steady-State Error and State Space

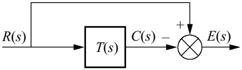
- We now consider how to evaluate steady-state error for a system represented in state-space.

- As we saw in Section 3.6 of the text, we can convert a single-input single-ouput state-space representation to an equivalent closed-loop transfer function using

$$T(s) = \frac{Y(s)}{U(s)} = \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B} \tag{34}$$

- In Diagram below, we have $E(s) = R(s) - C(s)$.
- We also have:

$$C(s) = R(s)T(s) \tag{35}$$



(a)

Figure 7.3.

Steady-State Error and State Space - II

- Combining the two we get

$$E(s) = R(s) - R(s)T(s) = R(s)[1 - T(s)] \tag{36}$$

- We thus have

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} s\, E(s) \\ &= \lim_{s \to 0} s\, R(s)[1 - T(s)] \end{aligned} \tag{37}$$

- Substituting in for $T(s)$ gives

$$e_{ss} = \lim_{s \to 0} s\, R(s)[1 - \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B}] \tag{38}$$