Software Engineering 3DX3

Slides 8: Root Locus Techniques

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Material based on *Control Systems Engineering* by N. Nise.

Introduction

- The root locus technique shows graphically how the closed-loop poles change as a system parameter is varied.
- Used to analyze and design systems for stability and transient response.
- Shows graphically the effect of varying the gain on things like percent overshoot, and settling time.
- Also shows graphically how stable a system is; shows ranges of stability, instability, and when system will start oscillating.

The Control System Problem

- The poles of the open-loop transfer function are typically easy to find and do not depend on the gain, K.
- It is thus easy to determine stability and transient response for an open-loop system.
- Let $G(s) = \dfrac{N_G(s)}{D_G(s)}$ and $H(s) = \dfrac{N_H(s)}{D_H(s)}$.

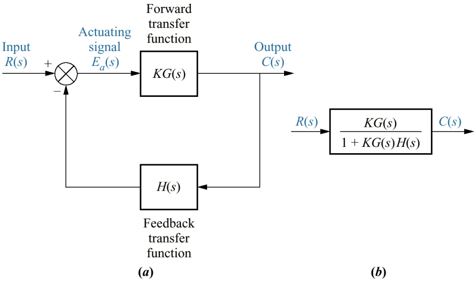


Figure 8.1.

The Control System Problem - II

- Our closed transfer function is thus

$$T(s) = \frac{K\dfrac{N_G(s)}{D_G(s)}}{1 + K\dfrac{N_G(s)}{D_G(s)}\dfrac{N_H(s)}{D_H(s)}} \tag{1}$$

$$= \frac{K N_G(s) D_H(s)}{D_G(s)D_H(s) + K N_G(s) N_H(s)} \tag{2}$$

- We thus see that we have to factor the denominator of $T(s)$ to find the closed-loop poles, and they will be a function of K.

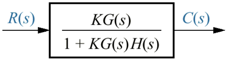


Figure 8.1(b).

The Control System Problem - III

- For example, if $G(s) = \frac{s+1}{s(s+2)}$ and $H(s) = \frac{s+3}{s+4}$, our closed-loop transfer function is:

$$T(s) = \frac{K(s+1)(s+4)}{s(s+2)(s+4) + K(s+1)(s+3)} \tag{3}$$

$$= \frac{K(s+1)(s+4)}{s^3 + (6+K)s^2 + (8+4K)s + 3K} \tag{4}$$

- To find the poles, we would have to factor the polynomial for a specific value of K.

- The root-locus will give us a picture of how the poles will vary with K.

Vector Representation of Complex Numbers

- Any complex number, $\sigma + j\omega$, can be represented as a vector.
- It can be represented in polar form with magnitude M, and an angle θ, as $M\angle\theta$.
- If $F(s)$ is a complex function, setting $s = \sigma + j\omega$ produces a complex number. For $F(s) = (s + a)$, we would get $(\sigma + a) + j\omega$.

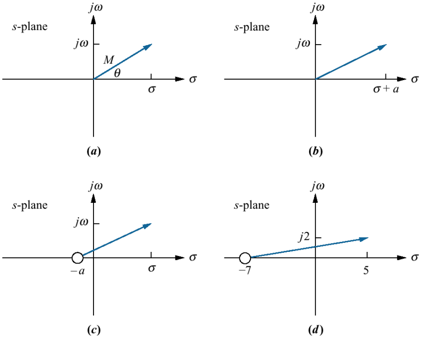


Figure 8.2.

Vector Representation of Complex Numbers - II

- If we note that function $F(s) = (s + a)$ has a zero at $s = -a$, we can alternately represent $F(\sigma + j\omega)$ as originating at $s = -a$, and terminating at $\sigma + j\omega$.
- To multiply and divide the polar form complex numbers, $z_1 = M_1\angle\theta_1$ and $z_2 = M_2\angle\theta_2$, we get

$$z_1 z_2 = M_1 M_2 \angle(\theta_1 + \theta_2) \qquad \frac{z_1}{z_2} = \frac{M_1}{M_2}\angle(\theta_1 - \theta_2) \tag{5}$$

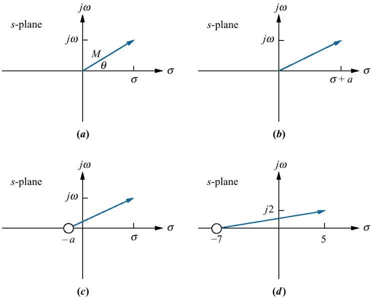


Figure 8.2.

Polar Form and Transfer Functions

- For a transfer function, we have:

$$G(s) = \frac{(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)} = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)} = M_G\angle\theta_G \tag{6}$$

where

$$M_G = \frac{\prod_{i=1}^{m}|(s+z_i)|}{\prod_{i=1}^{n}|(s+p_i)|} = \frac{\prod_{i=1}^{m} M_{z_i}}{\prod_{i=1}^{n} M_{p_i}} \tag{7}$$

and

$$\theta_G = \Sigma \textit{zero angles} - \Sigma \textit{pole angles} \tag{8}$$

$$= \Sigma_{i=1}^{m}\angle(s+z_i) - \Sigma_{j=1}^{n}\angle(s+p_j) \tag{9}$$

Polar Form and Transfer Functions eg.

- Use Equation 6 to evaluate $F(s) = \dfrac{(s+1)}{s(s+2)}$ at $s = -3 + j4$.

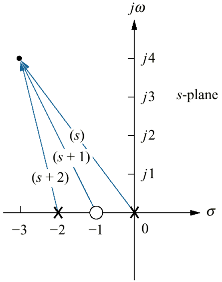
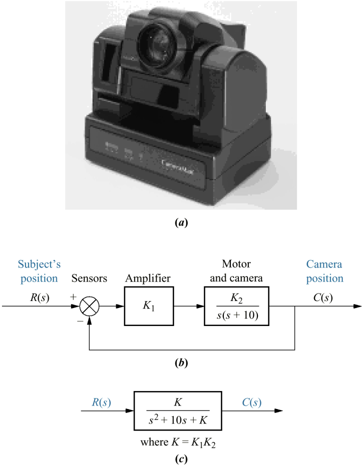


Figure 8.3.

Root Locus Introduction

- System below can automatically track subject wearing infrared sensors.

- Solving for the poles using the quadratic equation, we can create the table below for different values of K.



(a) (b) (c)

Table 8.1.

K	Pole 1	Pole 2
0	−10	0
5	−9.47	−0.53
10	−8.87	−1.13
15	−8.16	−1.84
20	−7.24	−2.76
25	−5	−5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

Figure: 8.4

Root Locus Introduction - II

- We can plot the poles from Table 8.1. labelled by their corresponding gain.

Table 8.1.

K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

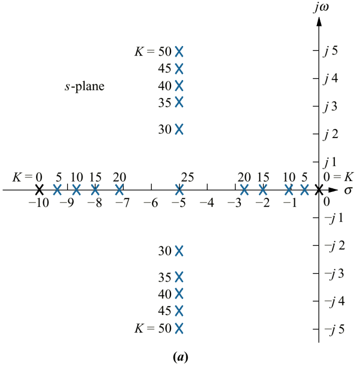


Figure: 8.5

Root Locus Introduction - III

- We can go a step further, and replace the individual poles with their paths.
- We refer to this graphical representation of the path of the poles as we vary the gain, as the root locus.
- We will focus our discussion on $K \geq 0$.
- For pole $\sigma_D + j\omega_D$, $T_s = \frac{4}{\sigma_D}$, $T_p = \frac{\pi}{\omega_D}$, and $\zeta = \frac{|\sigma_D|}{\omega_n}$.

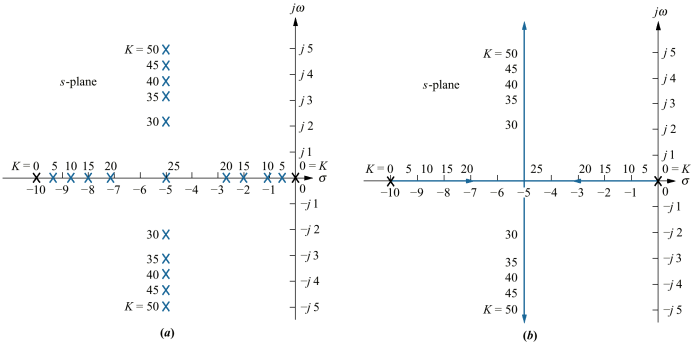


Figure 8.5.

Root Locus Properties

- For second-order systems, we can easily factor a system and draw the root locus.
- We do not want to have to factor for higher-order systems (5th, 10th etc.) for multiple values of K!.
- We will develop properties of the root locus that will allow us to rapidly sketch the root locus of higher-order systems.
- Consider the closed-loop transfer function below:

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

- A pole exists when

$$KG(s)H(s) = -1 = 1\angle(2k+1)180^o \qquad k = 0, \pm 1, \pm 2, \ldots \tag{10}$$

Root Locus Properties - II

- Equation 10 is equivalent to

$$|KG(s)H(s)| = 1 \tag{11}$$

  and

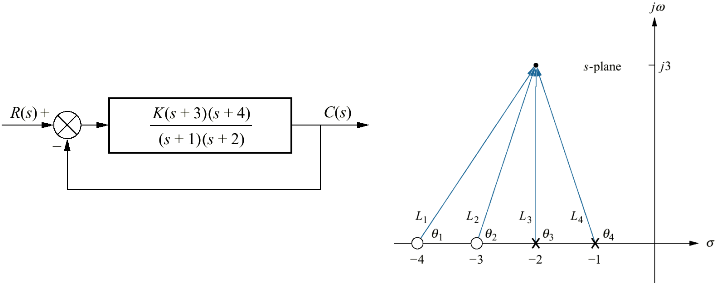
$$\angle KG(s)H(s) = (2k+1)180^o \tag{12}$$

- Equation 12 says that any s' that makes the angle of $KG(s)H(s)$ be an odd multiple of 180^o is a pole for some value of K.
- Given s' above, the value of K that s' is a pole of $T(s)$ for is found from Equation 11 as follows:

$$K = \frac{1}{|G(s)||H(s)|} \tag{13}$$

Root Locus Properties eg.

- For system below, consider $s = -2 + j3$ and $s = -2 + j(\sqrt{2}/2)$.



Figures 8.6 and 8.7.

Sketching Root Locus

- Now give a set of rules so that we can quickly sketch a root locus, and then we can calculate exactly just those points of particular interest.
  1. **Number of branches:** a branch is the path a single pole traverses. *The number of branches thus equals the number of poles.*
  2. **Symmetry:** As complex poles occur in conjugate pairs, *a root locus must be symmetric about the real axis*.

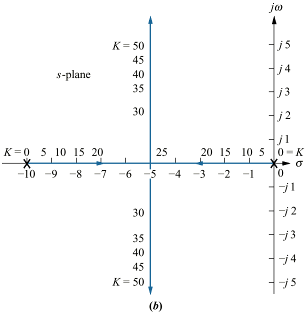


Figure 8.5.

Sketching Root Locus - II

3. **Real-axis segments:** *For $K > 0$, the root locus only exists on the real axis to the left of an odd number of finite open-loop poles and/or zeros, that are also on the real axis.*

   Why? By Equation 12, the angles must add up to an odd multiple of 180.
   - A complex conjugate pair of open-loop zeros or poles will contribute zero to this angle.
   - An open-loop pole or zero on the real axis, but to the left of the respective point, contributes zero to the angle.
   - The number must be odd, so they add to an odd multiple of 180, not an even one.

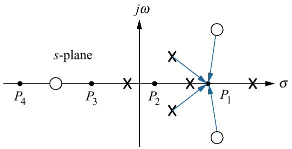


Figure 8.8.

Sketching Root Locus - III

4. **Starting and ending points:** *The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.*

   Why? Consider the transfer function below

   $$T(s) = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)}$$

   - The root locus begins at zero gain, thus for small K, our denominator is

     $$D_G(s)D_H(s) + \epsilon \tag{14}$$

   - The root locus ends as K approaches infinity, thus our denominator becomes

     $$\epsilon + KN_G(s)N_H(s)$$

Infinite Poles and Zeros

- Consider the open-loop transfer function below

$$KG(s)H(s) = \frac{K}{s(s+1)(s+2)} \tag{15}$$

- From *point 4*, we would expect our three poles to terminate at three zeros, but there are no finite zeros.
- A function can have an infinite zero if the function approaches zero as s approaches infinity. ie. $G(s) = \frac{1}{s}$.
- A function can have an infinite pole if the function approaches infinity as s approaches infinity. ie. $G(s) = s$.
- When we include infinite poles and zeros, every function has an equal number of poles and zeros

$$\lim_{s\to\infty} KG(s)H(s) = \lim_{s\to\infty} \frac{K}{s(s+1)(s+2)} \approx \frac{K}{s \cdot s \cdot s} \tag{16}$$

How do we locate where these zeros at infinity are so we can terminate our root locus?

Sketching Root Locus - IV

5. **Behavior at Infinity:** *As the locus approaches infinity, it approaches straight lines as asymptotes.*

   The asymptotes intersect the real-axis at σ_a*, and depart at angles* θ_a*, as follows:*

$$\sigma_a = \frac{\Sigma\text{finite poles} - \Sigma\text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \tag{17}$$

$$\theta_a = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}} \tag{18}$$

   where $k = 0, \pm 1, \pm 2, \pm 3$*, and the angle is in radians relative to the positive real axis.*

Sketching Root Locus eg. 1

- Sketch the root locus for system below.

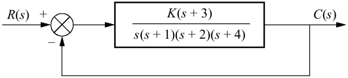


Figure 8.11.

Real-axis Breakaway and Break-in Points

- Consider root locus below.
- We want to be able to calculate at what points on the real axis does the locus leave the real-axis (breakaway point), and at what point we return to the real-axis (break-in point).
- At breakaway/break-in points, the branches form an angle of $180^o/n$ with the real axis where n is number of poles converging on the point.

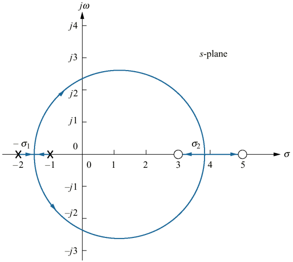


Figure 8.13.

Real-axis Breakaway and Break-in Points - II

- Breakaway points occur at maximums in the gain for that part of the real-axis.
- Break-in points occur at minimums in the gain for that part of the real-axis.
- We can thus determine the breakaway and break-in points by setting $s = \sigma$, and setting the derivative of equation below equal to zero:

$$K = \frac{-1}{G(\sigma)H(\sigma)} \tag{19}$$

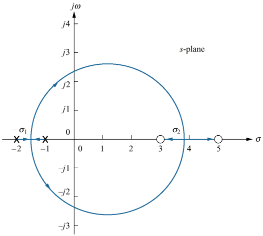


Figure 8.13.

The $j\omega$-Axis Crossings

- For systems like the one below, finding the $j\omega$-axis crossing is important as it is the value of the gain where the system goes from stable to unstable.

- Can use the Routh-Hurwitz criteria to find crossing:
  1. Force a row of zeros to get gain
  2. Determine polynomial for row above to get ω, the frequency of oscillation.

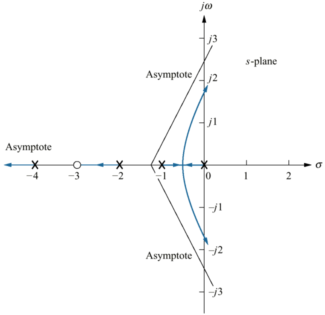
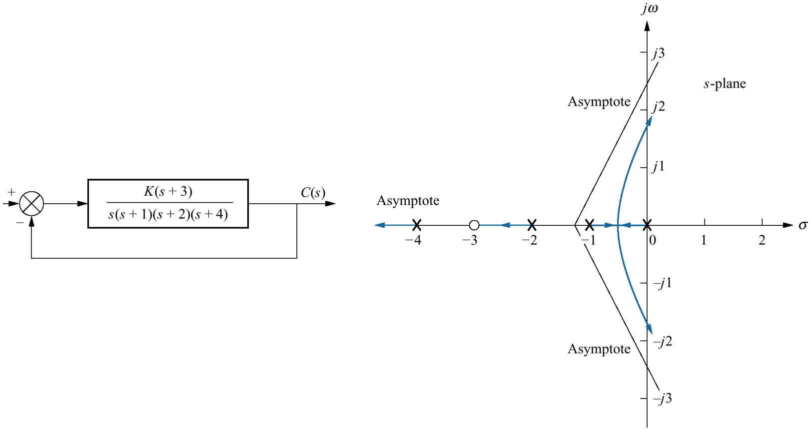


Figure 8.12.

The $j\omega$-Axis Crossing eg.

- For system below, find the frequency and gain for which the system crosses the $j\omega$-axis.



Figures 8.11 and 8.12.

Angles of Departure and Arrival

- We can refine our sketch by determining at what angles we depart from complex poles, and arrive at complex zeros.
- Net angle from all open-loop poles and zeros to a point on root access must satisfy:

$$\Sigma \textit{zero angles} - \Sigma \textit{pole angles} = (2k+1)180^o \qquad (20)$$

- To find angle θ_1, we choose a point ϵ on root locus near complex pole, and assume all angles except θ_1 are to the complex pole instead of ϵ. Can then use Equation 20 to solve for θ_1.

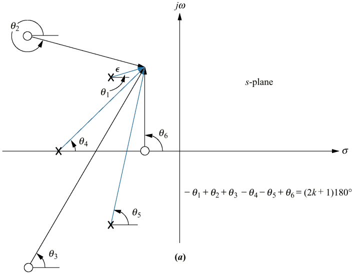


Figure 8.15.

Angles of Departure and Arrival - II

- For example in Figure 8.15a, we can solve for θ_1 in equation below:

$$\theta_2 + \theta_3 + \theta_6 - (\theta_1 + \theta_4 + \theta_5) = (2k+1)180^o \qquad (21)$$

- Similar approach can be used to find angle of arrival of complex zero in figure below.

- Simply solve for θ_2 in Equation 21.

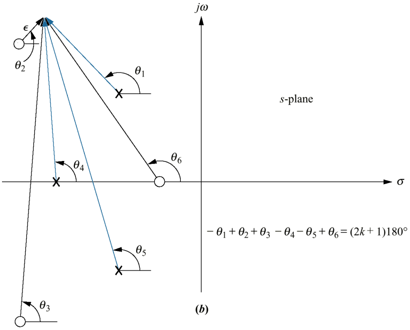
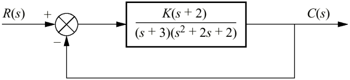
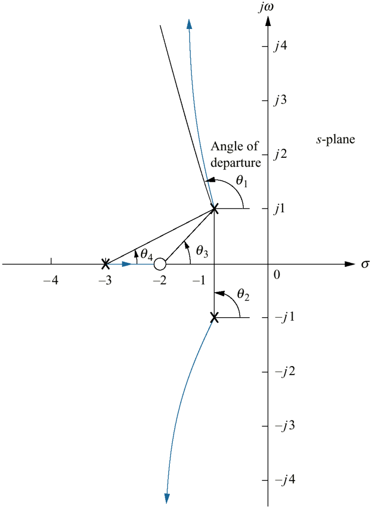


Figure 8.15.

Angles of Departure and Arrival eg.

- Find angle of departure for complex poles, and sketch root locus.





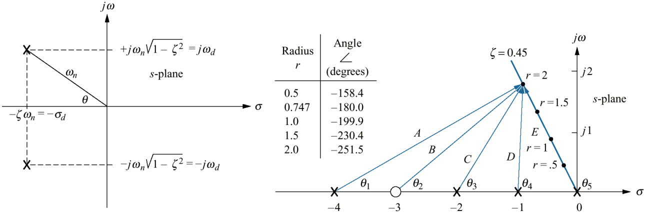
Figures 8.16 and 8.17.

Plotting and Calibrating Root Locus

- Once sketched, we may wish to accurately locate certain points and their associated gain.

- For example, we may wish to determine the exact point the locus crosses the 0.45 damping ratio line in figure below.

- From Figure 4.17, we see that $\cos(\theta) = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{\zeta\omega_n}{\omega_n} = \zeta.$
- We then use computer program to try sample radiuses, calculate the value of s at that point, and then test if point satisfies angle requirement.



Figures 4.17 and 8.18.

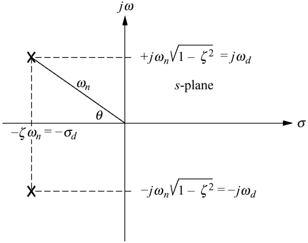
Plotting and Calibrating Root Locus - II

- Once we have found our point we can use the equation below to solve for the required gain, K.

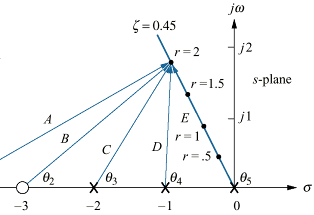
$$K = \frac{1}{|G(s)||H(s)|} = \frac{\prod_{i=1}^{m} M_{p_i}}{\prod_{i=1}^{n} M_{z_i}} \qquad (22)$$

- Uses labels in Figure 8.18, we would have for our example:

$$K = \frac{ACDE}{B} \qquad (23)$$



Radius r	Angle ∠ (degrees)
0.5	–158.4
0.747	–180.0
1.0	–199.9
1.5	–230.4
2.0	–251.5



Figures 4.17 and 8.18.

Transient Response Design via Gain Adjustment

- We want to be able to apply our transient response parameters and equations for second-order underdamped systems to our root locuses.
- These are only accurate for second-order systems with no finite zeros, or systems that can be approximated by them.
- In order that we can approximate higher-order systems as second-order systems, the higher-order closed-loop poles must be more than five times farther to the left than the two dominant poles.
- In order to approximate systems with zeros, the following must be true:
  1. The closed-loop zeros near the two dominant closed-loop poles must be nearly canceled by higher-order poles near them.
  2. Closed-loop zeros not cancelled, must be far away from the two dominant closed-loop poles.

Transient Response Design via Gain Adjustment - II

- Let $G(s) = \frac{N_G(s)}{D_G(s)}$ and $H(s) = \frac{N_H(s)}{D_H(s)}$.
- We saw earlier, that our closed-loop transfer equals:

$$T(s) = \frac{K N_G(s) D_H(s)}{D_G(s) D_H(s) + K N_G(s) N_H(s)} \tag{24}$$

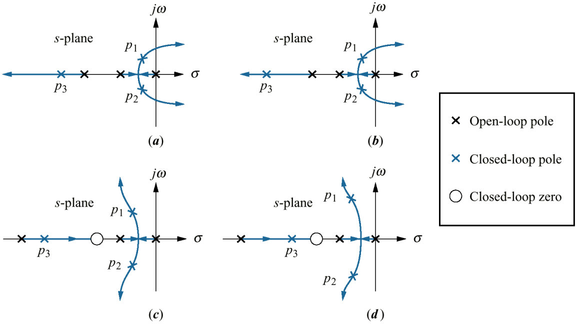


Figure 8.20.

Defining Parameters on Root Locus

- We have already seen that as $\zeta = \cos\theta$, vectors from the origin are lines of constant damping ratio.
- As percent overshoot is solely a function of ζ, these lines are also lines of constant %OS.
- From diagram we can see that the real part of a pole is $\sigma_d = \zeta\omega_n$, and the imaginary part is $\omega_d = \omega_n\sqrt{1-\zeta^2}$.
- As $T_s = \dfrac{4}{\zeta\omega_n} = \dfrac{4}{\sigma_d}$, vertical lines have constant values of T_s.

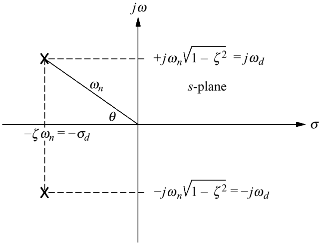


Figure 4.17.

Defining Parameters on Root Locus - II

- As $T_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{\pi}{\omega_d}$, horizontal lines thus have constant peak time.

- We thus choose a line with the desired property, and test to find where it intersects our root locus.

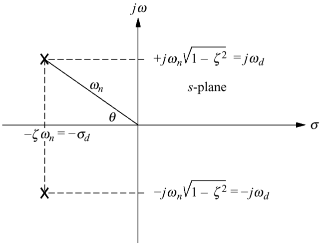


Figure 4.17.

Design Procedure For Higher-order Systems

1. Sketch root locus for system.

2. Assume system has no zeros and is second-order. Find gain that gives desired transient response.

3. Check that systems satisfies criteria to justify our approximation.

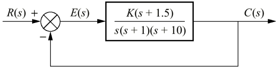
4. Simulate system to make sure transient response is acceptable.

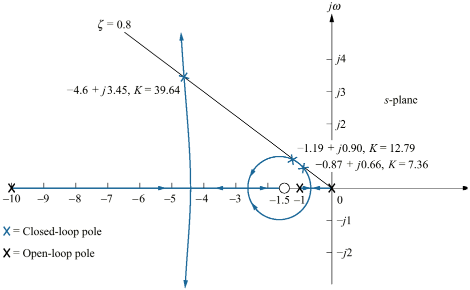
Third-order System Gain Design eg.

- For system below, design the value of gain, K, that will give 1.52% overshoot. Also estimate the settling time, peak time, and steady-state error.

- First step is to sketch the root locus below.

- We next assume system can be approximated by second-order system, and solve for ζ using $\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}}$.

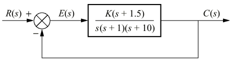


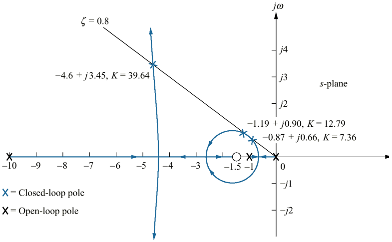


Figures 8.21 and 8.22.

Third-order System Gain Design eg. - II

- This gives $\zeta = 0.8$. Our angle is thus $\theta = \cos^{-1}(0.8) = 36.87^o$.
- We then use root locus to search values along this line to see if they satisfy the angle requirement.
- The program finds three conjugate pairs on the locus and our $\zeta = 0.8$ line. They are $-0.87 \pm j0.66$, $-1.19 \pm j0.90$, $-4.6 \pm j3.45$ with respective gains of $K = 7.36, 12.79$, and 39.64.
- We will use $T_p = \dfrac{\pi}{\omega_d}$, and $T_s = \dfrac{4}{\sigma_d}$.





Figures 8.21 and 8.22.

Third-order System Gain Design eg. - III

- For steady-state error, we have:

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s \frac{K(s+1.5)}{s(s+1)(s+10)} = \frac{K(1.5)}{(1)(10)} \quad (25)$$

- To test to see if our approximation of a second-order system is valid, we calculate the location of the third pole for each value of K we found.

- The table below shows the results of our calculations.

Case	Closed-loop poles	Closed-loop zero	Gain	Third closed-loop pole	Settling time	Peak time	K_v
1	$-0.87 \pm j0.66$	$-1.5 + j0$	7.36	-9.25	4.60	4.76	1.1
2	$-1.19 \pm j0.90$	$-1.5 + j0$	12.79	-8.61	3.36	3.49	1.9
3	$-4.60 \pm j3.45$	$-1.5 + j0$	39.64	-1.80	0.87	0.91	5.9

Table 8.4.

Third-order System Gain Design eg. - IV

- We now simulate to see how good our result is:

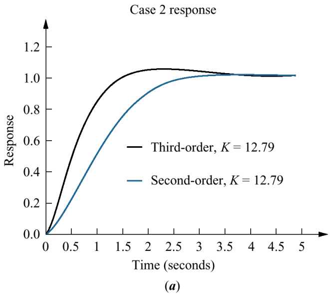


Figure 8.23.