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CAS 745
Supervisory Control of Discrete-Event Systems

Slides 1:

Algebraic Preliminaries

*Material based on W. M. Wonham, *Supervisory Control of Discrete-Event Systems*, Department of Electrical and Computer Engineering, University of Toronto, July 2004. Lecture notes of Professor W. M. Wonham also used.

Control Systems

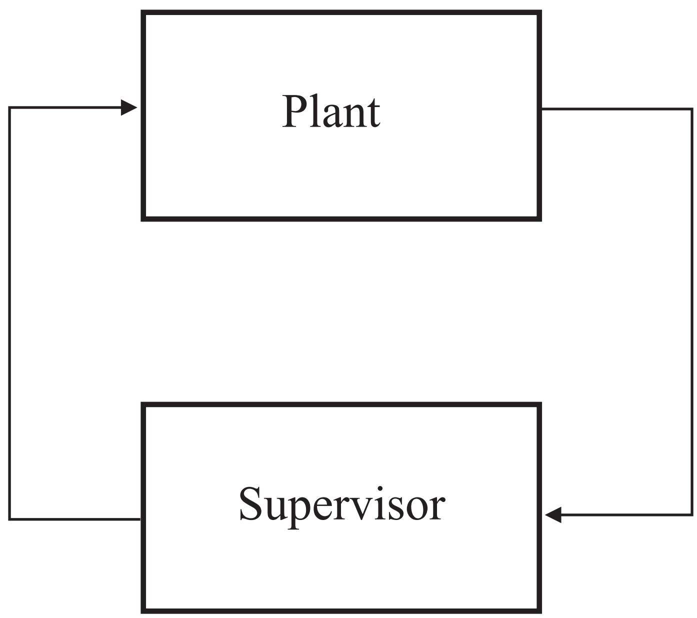
Assumptions: Students have taken a discrete math course that includes elementary logic and set theory.

The research area of *Supervisory Control* originated in systems control research.

Area deals with dynamic systems: systems that have state space, state transition structure, and inputs/outputs.

Control Systems Cont.

Feedback control in the form of a controller is usually needed to ensure the desired behaviour.



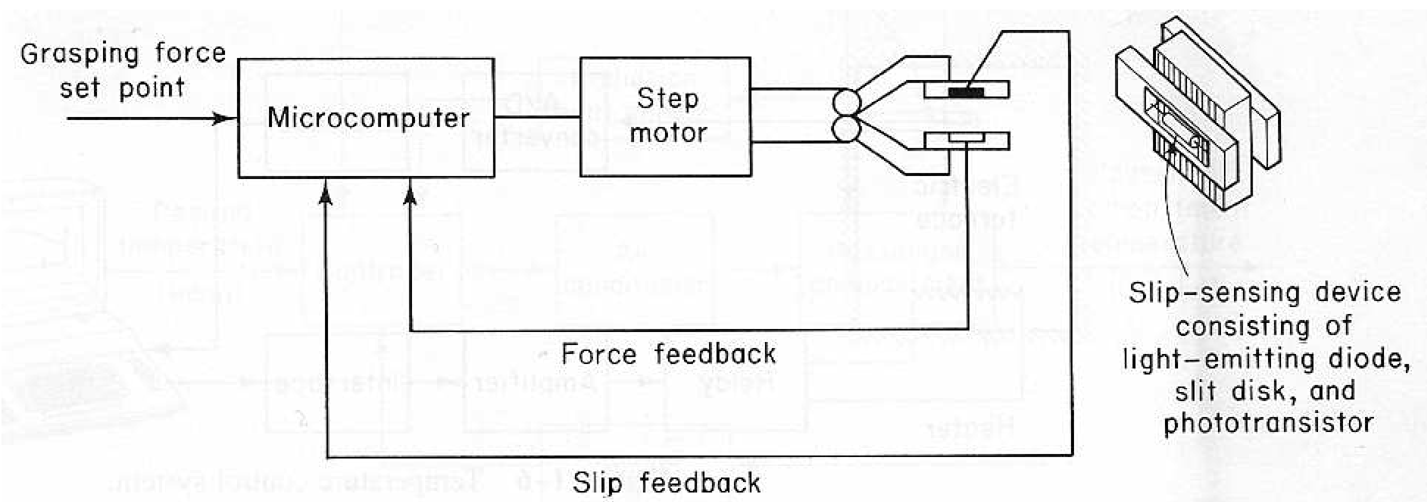
Feedback Control

Typically, the controller has its own dynamics, and its the combination of the two (called the *closed behaviour*) that provides the desired behaviour (*specification*).

A suitable controller may not always exist for a given desired behaviour and plant.

Continuous Control Systems Eg.

Here our inputs/outputs are represented as continuous or discrete time functions, as well as the system's dynamics.



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Example shows a robot hand, grasping force, control system.

*Diagram taken from Katsuhiko Ogata, *Modern Control Engineering, 2nd ed.*, Prentice Hall, 1990

Discrete-Event Systems Overview

Discrete-event (dynamic) systems (DES or DEDS) are a relatively new area of research for control systems, but is now well represented at conferences and in the literature.

Recently, continuous systems have been composed with DES to form a new research area called *Hybrid Systems.*

The parent domains of DES are *Operations Research (OR)* and *Software Engineering.*

Operations Research

OR deals with systems of *stores* and *servers* that are interconnected. Together, they process “items.”

Focus of area:

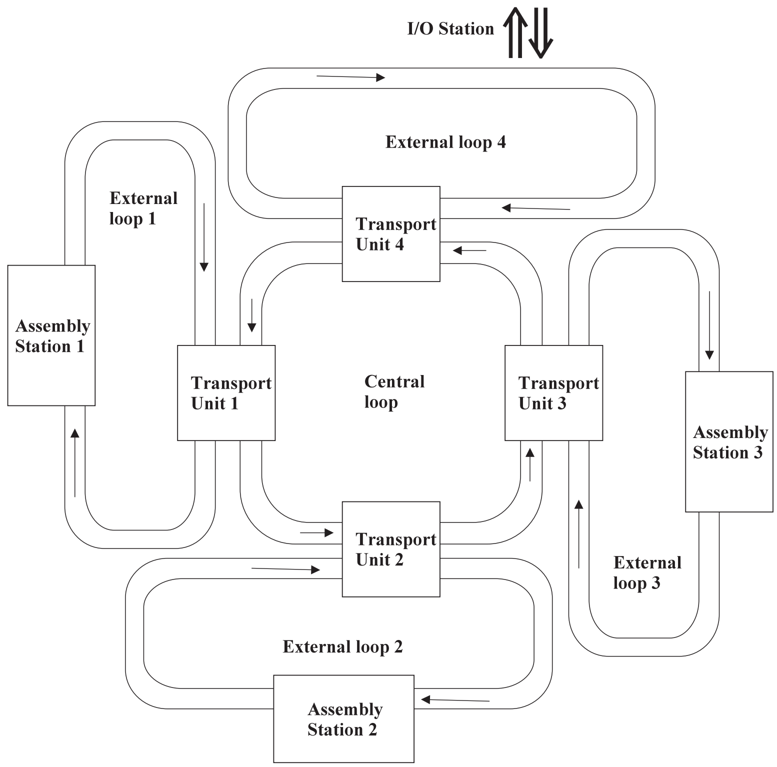
- Measure quantitative performance and tradeoffs (throughput of operation versus cost etc.).
- Optimize control parameters such as buffer size, maintenance schedules, uptime, etc.

Some examples:

Manufacturing Systems: operates on *work pieces* which are stored in *queues or buffers.*

These work pieces are then operated on by *robots, machines, automated guided vehicles (AGVs)* etc.

Manufacturing Example



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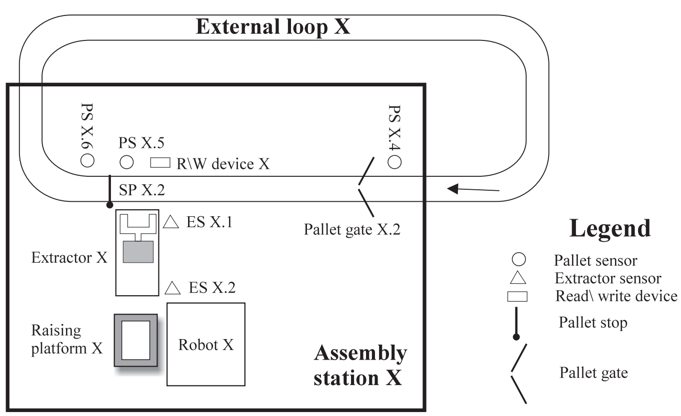


*AIP example taken from R.J. Leduc, *Hierarchical Interface-based Supervisory Control.* Doctoral Thesis, Dept. of Elec. & Comp. Engrg., Univ. of Toronto, 2002.

Operations Research Cont.

Communication Systems: operates on *messages* which are stored in queues and buffers.

These are then served by *transmitters, channels, receivers, network nodes* etc.

other examples are: **Traffic systems, Database management systems,** etc.

Software Engineering and DES

Some areas of software engineering that are relevant are operating systems (resource management etc.), concurrent computation, real-time (embedded and reactive) systems.

Example applications are synchronization algorithms that control resource sharing and mutual exclusion enforcement for concurrent entities.

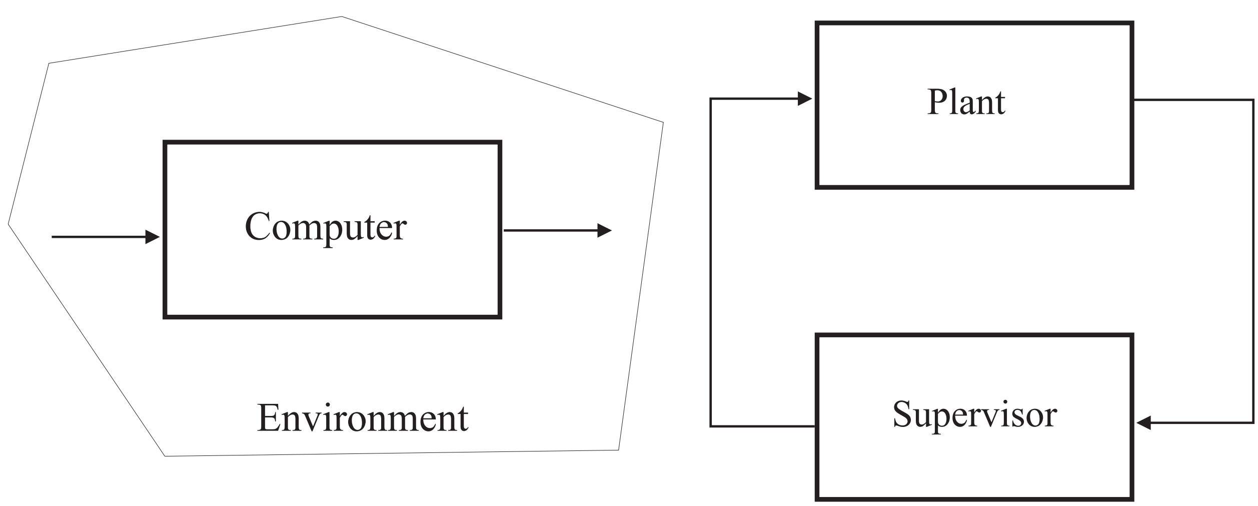
Focus of area:

- Guarantee safety ("nothing bad will happen"), mutual exclusion, deadlock prevention.

- Guarantee liveness ("something good will eventually happen"), efficiency, absence of starvation or lockout.

Software Engineering and DES Cont.

Diagram below shows different viewpoint of Software Engineering and discrete-event systems.



Discrete-Event Systems

A Discrete-event system is a dynamic system that is:

- Discrete in time and state space.

- Asynchronous or event driven. DES are driven by "events" (ie machine starts, machine stops etc.) or instantaneous "occurrences in time."

- Occurrence of events is nondeterministic. Means that the selection process that determines which event out of those currently possible to occur, is not modelled (could be chance, or some unspecified mechanism).

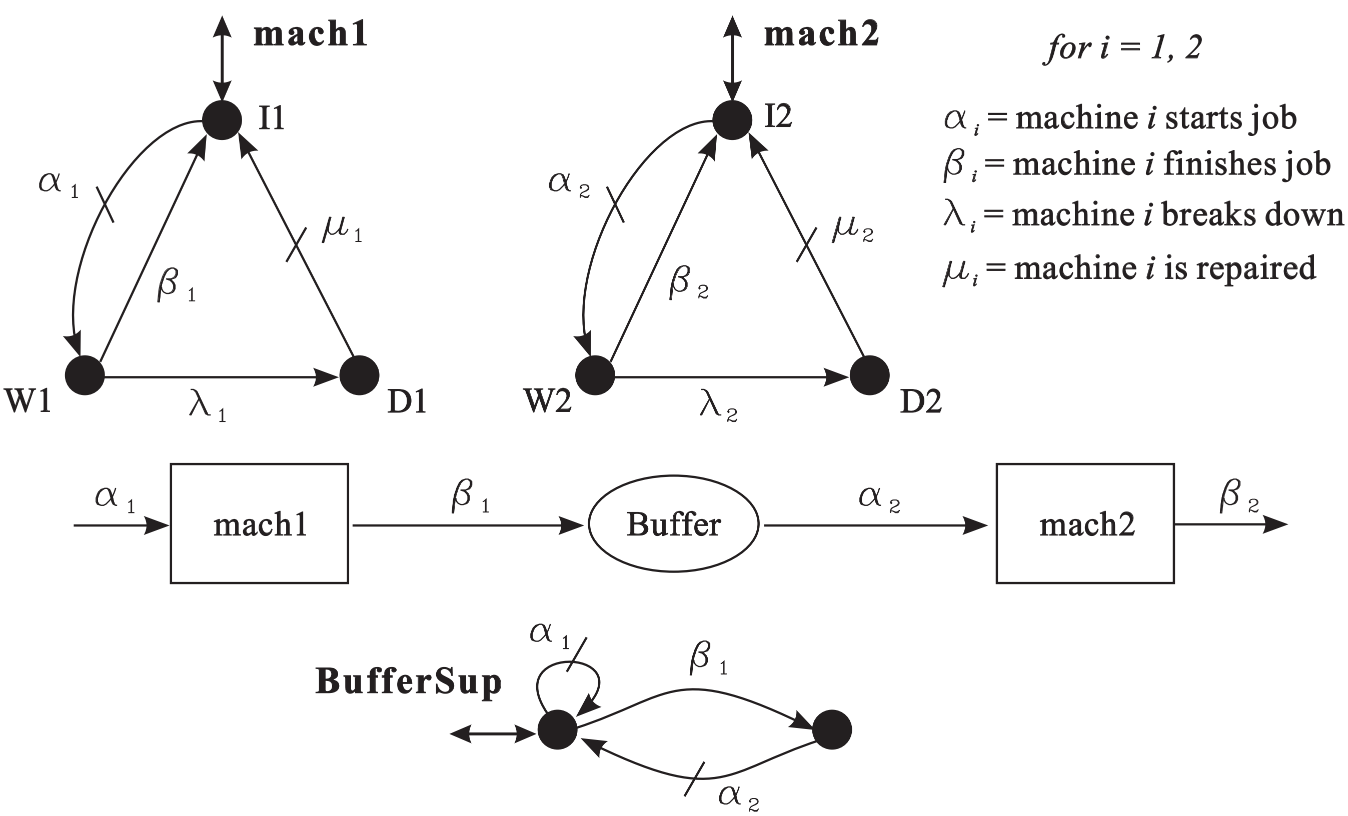
Focus of area:

- Logical correctness in presence of concurrency, timing constraints, etc.

- Quantitative performance.

- Feedback control synthesis and optimization.

Small Factory Example



Closed Loop: sync (mach1, mach2,BufferSup)

Industrial Chemical Process Eg.

Plant Components: Reaction tank, valves, heater, mixer etc.

States: Fluid level (low, medium, high, etc), temperature, pH (too low, OK, too high, etc).

X

valve states (open, closed, etc.)

X

heater and mixer states (off, on, etc.)

Transitions (events): chemical mixture reaching desired composition, temperature entering a desired range, critical variables entering a dangerous range etc.

Discrete-Event Systems Challenges

Model size: DES state space increases combinatorially in the number of model components. Can quickly make computation intractable.

Structure: How to exploit regularities of a system's structure to improve scalability. Usually combined with appropriate architecture that can take advantage of structure.

Implementation: How to implement supervisors as hardware and software, taking care of necessary concurrency issues.

Binary Relations - §1.1

Let X be an arbitrary set. The set of all ordered pairs of X is:

$$X \times X := \{(x_1, x_2) | \, x_1, x_2 \in X\}$$

Note: (x_1, x_2) an ordered pair means that

$$(x_1, x_2) = (x_1', x_2') \Leftrightarrow x_1 = x_1' \,\&\, x_2 = x_2'$$

Defn: A *binary relation* is any $R \subseteq X \times X$

We write: $(x_1, x_2) \in R$ or $x_1 R x_2$ (infix notation).

Posets

Let the symbol "$\leq$" denote a binary relation R with the properties:

1. $(\forall x \in X)\ x \leq x$ *(reflexive)*

2. $(\forall x, y, z \in X)\ x \leq y \ \&\ y \leq z \Rightarrow x \leq z$ *(transitive)*

3. $(\forall x, y \in X)\ x \leq y \ \&\ y \leq x \Rightarrow x = y$ *(antisymmetric)*

Defn: A relation $\leq$ with the properties (1)-(3) is a *partial order (p.o)* on X.

Poset stands for *partially ordered sets.*

We refer to "the poset X" if the relation $\leq$ is understood. Otherwise, we refer to the poset $(X, \leq)$.

Total Ordering

Defn: we say that $x, y \in X$ are *comparable* wrt $\leq$ if $x \leq y$ or $y \leq x$.

If two elements $x, y \in X$ are not comparable, we write $x <> y$.

A poset is a *total ordering* if every $x, y \in X$ is comparable.

How do you know if the relation is a poset? Check that it satisfies the three properties.

We can extend our notation as follows:

If $x \leq y$, may write $y \geq x$

If $x \leq y$ and $x \neq y$ then may write $x < y$ or $y > x$

NOTE: $\neg(x \leq y) \Leftrightarrow (x <> y)$ or $(x > y)$

Poset Examples

1. Let $X = \mathbb{R}$ (set of real numbers), or $X = \mathbb{N} := \{0, 1, 2, \ldots\}$ (set of natural numbers) or $X = \mathbb{Z} := \{\ldots, -1, 0, +1, \ldots\}$ (set of integers) and let $\leq$ have normal meaning.

   Thus $2 \leq 3$ but not $4 \leq 3$.

2. Let $X = \underbrace{\mathbb{Z} \times \ldots \times \mathbb{Z}}_{\text{k-fold}}$ thus:

   $x \in X \Rightarrow x = (x_1, \ldots, x_k)$ with $x_i \in \mathbb{Z}$, $i = 1, \ldots, k$.

   Defn: $(\forall x, y \in X)\ x \leq y \Leftrightarrow x_i \leq y_i$ (normal meaning) for all $i = 1, \ldots, k$.

   eg. $(5, -2) \leq (7, -1)$ but $(5, -1) <> (7, -2)$ as $5 \leq 7$ but $-1 > -2$.

Poset Examples (2)

3. Let A be an arbitrary set.

   Let $X = \text{Pwr}(A) = 2^A$.

   The powerset of A (2^A) is the set of all subsets of A thus: $x \in X \Rightarrow x \subseteq A$

   eg. $x = \emptyset$, $x = \{a, b\}$ with $a, b \in A$, $x = A$

   Defn: $(\forall x, y \in X)\, x \leq y \Leftrightarrow x \subseteq y$

   eg. $A = \{\alpha, \beta, \gamma\}$ thus:

   $X = \{\emptyset, \{\alpha\}, \{\beta\}, \{\gamma\}, \{\alpha, \beta\}, \{\alpha, \gamma\}, \{\beta, \gamma\}, \{\alpha, \beta, \gamma\}\}$

   Have $\{\beta\} \leq \{\beta, \gamma\}$ but $\{\beta\} <> \{\alpha, \gamma\}$

   Should be able to check for yourself that $(X, \leq)$ is a poset.

Meet

Let $(X, \leq)$ be a poset, and let $x, y \in X$.

Defn: Element $a \in X$ is a *lower bound* for x and y if $a \leq x$ and $a \leq y$.

eg. for $X = \text{Pwr}(A)$, this means a is a subset of both x and y.

Defn: Element $l \in X$ is a *meet (greatest lower bound)* for x and y iff

$$l \leq x \ \& \ l \leq y \ \& \ (\forall a \in X)\,(a \leq x \ \& \ a \leq y) \Rightarrow a \leq l$$

In other words, l is a lower bound and beats any other lower bound.

For a given x and y, the meet may not exist. If it does, it is unique and we denote it $x \wedge y$.

Defn: We denote the *bottom element* by $\perp$. If it exists, it satisfies: $\perp \in X$ and $(\forall x \in X) \perp \leq x$.

Join

Defn: Element $b \in X$ is an *upper bound* for x and y if $x \leq b$ and $y \leq b$.

eg. for $X = \text{Pwr}(A)$, this means x and y are a subset of b.

Defn: Element $u \in X$ is a *join (least upper bound)* of x and y iff

$$x \leq u \ \& \ y \leq u \ \& \ (\forall b \in X)\,(x \leq b \ \& \ y \leq b) \Rightarrow u \leq b$$

In other words, u is a upper bound and is lower than any other upper bound.

For a given x and y, the join may not exist. If it does, it is unique and we denote it $x \vee y$.

Defn: We denote the *top element* by $\top$. If it exists, it satisfies: $\top \in X$ and $(\forall x \in X)\, x \leq \top$.

Meet and Join Examples

1. Let $X = \text{Pwr}(A)$ and $x, y \in X$.

   We then have $x \wedge y = x \cap y$ (set intersection) and $x \vee y = x \cup y$ (set union).

   We also have $\bot = \emptyset$ and $\top = A$.

   For this X, the meet, join, bottom, and top always exist.

2. Let $X = \mathbb{Z} \times \mathbb{Z}$ and $x = (x_1, x_2), y = (y_1, y_2) \in X$. We then have:

$$
\begin{aligned}
x \wedge y &= (\min(x_1, y_1), \min(x_2, y_2)) \\
x \vee y &= (\max(x_1, y_1), \max(x_2, y_2))
\end{aligned}
$$

   As $\mathbb{Z} := \{\ldots, -1, 0, +1, \ldots\}$, neither $\bot$ or $\top$ exist.

Lattices - §1.2

Defn: A *lattice* L is a poset $(L, \leq)$ in which the meet and join is defined for all $x, y \in L$.

The binary operators $\wedge$ and $\vee$ define the functions:

$$\wedge : L \times L \rightarrow L, \;\; \vee : L \times L \rightarrow L$$

For $x, y, z \in L$ then the operators $\wedge$ and $\vee$ (denoted consistently throughout as $\star$) satisfy the following properties:

1. $x \star x = x$ ($\star$ is *idempotent*)
2. $x \star y = y \star x$ ($\star$ is *commutative*)
3. $(x \star y) \star z = x \star (y \star z)$ ($\star$ is *associative*)

eg. for $\vee$, point 2 gives: $x \vee y = y \vee x$

We also have the following relationships:

$$x \wedge (x \vee y) = x \vee (x \wedge y) = x \qquad (\textit{absorption})$$
$$x \leq y \text{ iff } x \wedge y = x \text{ iff } x \vee y = y \qquad (\textit{consistency})$$

Absorption Example

Verify: $x \wedge (x \vee y) = x$

let $a = x \wedge (x \vee y)$

Proof:

Must show 1) $x \leq a$ 2) $a \leq x$

Part 1) show $x \leq a$

Have $x \leq x$ by reflexivity of $\leq$

As $x \vee y$ is upper bound of x, y by definition of join, we have: $x \leq x \vee y$

$\Rightarrow$ x is a lower bound of x and $x \vee y$

$\Rightarrow x \leq x \wedge (x \vee y)$ by defn of meet.

Part 1 done.

Absorption Example (2)

Part 2) show $a \leq x$

By defn of meet, $a \leq x$ and $a \leq x \vee y$

Part 2 done.

By parts 1) - 2), we have $x \leq a$ and $a \leq x$.

$\Rightarrow x = a$ by antisymmetry of Posets.

QED

Infinum and Supremum

Defn: Let L be a lattice and let S be a nonempty, possibly infinite subset of L.

The *infimum* of S, denoted *inf*(S), is an element $l \in L$ with the properties:

$$(\forall y \in S)\, l \le y \;\; \& \;\; (\forall z \in L)\, ((\forall y \in S)\, z \le y) \Rightarrow z \le l$$

Note: $l \in L$ but is possible $l \notin S$

Defn: Let L be a lattice and let S be a nonempty, possibly infinite subset of L.

The *supremum* of S, denoted *sup*(S), is an element $u \in L$ with the properties:

$$(\forall y \in S)\, y \le u \;\; \& \;\; (\forall z \in L)\, ((\forall y \in S)\, y \le z) \Rightarrow u \le z$$

Note: $u \in L$ but is possible $u \notin S$

Complete Lattices

If lattice L is finite, the inf(S) and sup(S) reduce to the meet and join of a finite number of elements of L.

In this case, they always exist as L is a lattice.

If S is an infinite subset, they need not exist. If they do, they are unique.

Defn: A lattice L is *complete* if for any nonempty subset S of L, the inf(S) and sup(S) always exists.

Lattice Examples

1. Let A be an arbitrary finite set. Let $X =$ Pwr(A). Let $S \subseteq L$ and $S \neq \emptyset$.

$$\text{inf}(S) = \bigcap_{y \in S} y \quad \text{sup}(S) = \bigcup_{y \in S} y$$

   Clearly, $(L, \cap, \cup)$ is complete.

2. Let $L = \mathbb{Z} \times \mathbb{Z}$ and $x = (x_1, x_2), y = (y_1, y_2) \in L$. We then have:

$$\begin{aligned} x \wedge y &= (\text{min}(x_1, y_1), \text{min}(x_2, y_2)) \\ x \vee y &= (\text{max}(x_1, y_1), \text{max}(x_2, y_2)) \end{aligned}$$

   Let $S := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} | \, y \geq 1\}$.

   Thus S is an infinite subset, with y component able to be arbitrarily large. Clearly no max value exists for S, thus sup(S) not defined.

   Thus, L is not complete.

When S is Empty

Regardless of whether L is complete, if $\sup(L) = \top$ exists, then we can include $S = \emptyset \subseteq L$ in our definition of $\inf(S)$ by defining:

$$\inf(\emptyset) = \sup(L)$$

Similarly if $\inf(L) = \bot$ exists, we can define:

$$\sup(\emptyset) = \inf(L)$$

This is a result of "empty set logic."

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

If $p = F$ then the statement $p \Rightarrow q$ is always true for any q.

When S is Empty (2)

The statement $y \in \emptyset$ is false for any y in the universe, thus statement $y \in \emptyset \Rightarrow q$ is true for any q.

To show that inf($\emptyset$) $= \top$, we need to show:

1) $(\forall y \in \emptyset)\, \top \leq y$

Equivalent to showing: $(\forall y)\, y \in \emptyset \Rightarrow \top \leq y$

This statement would only be false if:

$$(\exists y)\, y \in \emptyset \;\;\&\;\; \neg(\top \leq y)$$

NOTE: Similarly $(\forall z \in L)(\forall y \in \emptyset)\, z \leq y$ is true.

2) $(\forall z \in L)\, z \leq \top$ true by defn of $\top$

Equivalence Relations - §1.3

Let X be a nonempty set, and let $E \subseteq X \times X$ be a binary relation on X.

Defn: The relation E is an *equivalence relation* on X if:

1. $(\forall x \in X)\, xEx$ (E is *reflexive*)
2. $(\forall x, x' \in X)\, xEx' \Rightarrow x'Ex$ (E is *symmetric*)
3. $(\forall x, x', x'' \in X)\, xEx'\ \&\ x'Ex'' \Rightarrow xEx''$ (E is *transitive*)

For xEx' we may write $x \equiv x'\,(\text{mod}E)$

Equivalence Relations Example

1. Let X be the set of all motorized vehicles.

   We could define an equivalence relation, E, as follows:

   - All two wheeled vehicles (mopeds, motorbikes etc) are equivalent, AND

   - All four wheeled vehicles (cars, trucks, buses) are equivalent, but no two wheel is equivalent to a four wheel vehicle, AND

   - All vehicles that do not fall into the above types are equivalent.

2. Any arbitrary partition (ie doesn't need any physical meaning) of X could be used to create an equivalence relation.

Equivalence Classes (Cosets)

For $x \in X$, let $[x]$ represent the subset of elements that are equivalent to x:

$$[x] := \{x' \in X |\, x' E x\} \subseteq X$$

Defn: We refer to the subset $[x]$ as the *coset* (or *equivalence class*) of x wrt to E.

By reflexivity, we have $x \in [x]$ thus all cosets are non empty.

Proposition 1.3.1

$$(\forall x, y \in X) \text{ either } \; [x] = [y] \text{ or } \; [x] \cap [y] = \emptyset$$

Proof: Assume one statement is false, and show this implies the other is true.

Partitions

The equivalence relation divides X into nonempty, non-overlapping subsets, inducing a partition of X.

Let $\mathcal{P}$ be a family of subsets of X, with index set A:

$$\mathcal{P} = \{C_\alpha \subseteq X | \, \alpha \in A\}$$

Defn: $\mathcal{P}$ is a *partition* of X if:

1. $(\forall \alpha \in A)\, C_\alpha \neq \emptyset$
2. $(\forall x \in X)(\exists \alpha \in A)\, x \in C_\alpha$
3. $(\forall \alpha, \beta \in A)\, \alpha \neq \beta \Rightarrow C_\alpha \cap C_\beta = \emptyset$

The subsets C_α are called the *cells* of $\mathcal{P}$.

Clearly, the collection of distinct cosets of an equivalence relationship is a partition.

Inducing Equivalence Relations

Given a partition $\mathcal{P}$ of X, we can define a corresponding equivalence relation E as follows:

$$(\forall x, y \in X)\, xEy \text{ iff } (\exists \alpha \in A)\, x \in C_\alpha \ \& \ y \in C_\alpha$$

Will speak of equivalence relations and partitions interchangeably.

Lattice of Partitions

Let $\mathcal{E}(X)$ (or simply $\mathcal{E}$) be the set of all equivalence relations on (partitions of) X. We will now define a partial order on $\mathcal{E}$ and then show that it is a complete lattice:

$(\forall E_1, E_2 \in \mathcal{E})\, E_1 \leq E_2$ iff $(\forall x, y \in X)\, xE_1y \Rightarrow xE_2y$

Implies every coset of E_1 is a subset of exactly one coset of E_2.

Defn: If $E_1 \leq E_2$, we say that E_1 *refines* E_2, or E_1 is *finer* than E_2, or that E_2 is *coarser* than E_1.

Existence of Meet

Proposition 1.3.2: In the poset $(\mathcal{E}, \leq)$ the meet $E_1 \wedge E_2$ always exists and is given by:

$$(\forall x, x' \in X)\, x \equiv x' (\mathrm{mod}\, E_1 \wedge E_2)$$
$$\text{iff} x \equiv x' (\mathrm{mod}\, E_1) \,\&\, x \equiv x' (\mathrm{mod}\, E_2)$$

The relation $E_1 \wedge E_2$ is the coarsest partition that is still finer than E_1 and E_2.

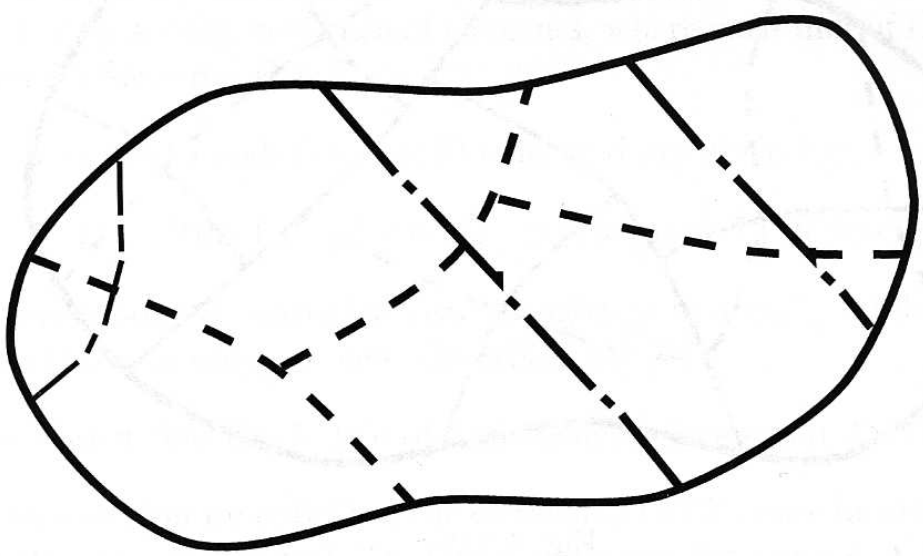


Fig. 1.3.1
Meet of Two Partitions

—— - —— - —— boundaries of E-cells
. . . . boundaries of F-cells
Any boundary line is a boundary of an $(E \wedge F)$-cell

Existence of Join

Proposition 1.3.3: In the poset $(\mathcal{E}, \leq)$ the join $E_1 \vee E_2$ always exists and is given by:

$(\forall x, x' \in X)\, x \equiv x' (\text{mod}\, E_1 \vee E_2)$
  iff $(\exists \text{ integer } k \geq 1)(\exists x_0, x_1, \ldots, x_k \in X) x_0 = x \;\&\; x_k = x'$
  $\&(\forall i) 1 \leq i \leq k \Rightarrow [x_i \equiv x_{i-1} (\text{mod}\, E_1) \text{ or } x_i \equiv x_{i-1} (\text{mod}\, E_2)]$

The relation $E_1 \vee E_2$ is the finest partition that is still coarser than E_1 and E_2.

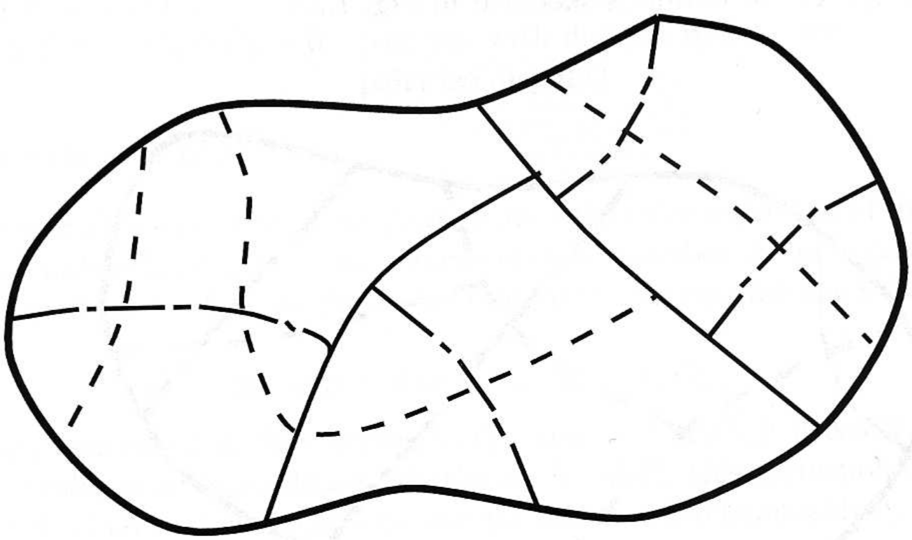


Fig. 1.3.2
Join of Two Partitions

boundaries of E-cells
boundaries of F-cells
common boundaries of E-cells and F-cells, forming boundaries of $(E \vee F)$-cells

Completeness of $\mathcal{E}$

Proposition 1.3.4: Let $\mathcal{F} \subseteq \mathcal{E}$ be a nonempty collection of equivalence relations on X. Then $\inf(\mathcal{F})$ exists; in fact:

$$(\forall x, x' \in X)\, x(\inf(\mathcal{F}))x' \text{ iff } (\forall F \in \mathcal{F})\, xFx'$$

Also, $\sup(\mathcal{F})$ exists; in fact:

$$\begin{aligned}&(\forall x, x' \in X)\, x(\sup(\mathcal{F}))x' \\ &\quad \text{iff } (\exists \text{ integer } k \geq 1)(\exists F_1, \ldots, F_k \in \mathcal{F})(\exists x_0, x_1, \ldots, x_k \in X) \\ &\quad\quad x_0 = x \;\&\; x_k = x' \;\&\; (\forall i) 1 \leq i \leq k \Rightarrow x_i \equiv x_{i-1} (\text{mod}\, F_i)\end{aligned}$$

We can also state $\perp = \inf(\mathcal{E})$ and $\top = \sup(\mathcal{E})$ as follows:

$$x \equiv x' (\text{mod}\, \perp) \text{ iff } x = x' \qquad x \equiv x' (\text{mod}\, \top) \text{ iff } \textit{true}$$

Canonical Projection - §1.4

Let X and Y be sets. Will use π, ρ etc. for elements of $\mathcal{E}(X)$.

For $\pi \in \mathcal{E}(X)$, let $\overline{X} = X/\pi$ be the set of distinct cells of the equivalence relation π.

Defn: The *canonical projection* associated with π, denoted P_π, is defined to be the surjective function:

$$P_\pi : X \to X/\pi : x \mapsto [x]$$

Equivalence Kernel

Let $f : X \to Y$ be a function with the given domain and codomain. For $A \subseteq X$, we define:

$$f(A) := \{f(x) | \, x \in A\}$$

Defn: We also associate with f the *inverse image function* f^{-1} : Pwr$(Y) \to$ Pwr(X) as follows:

$$f^{-1}(B) := \{x \in X | \, f(x) \in B\}, \quad B \in \text{Pwr}(Y)$$

Defn: The *equivalence kernel* of $f : X \to Y$, denoted ker$f$, is the equivalence relation in $\mathcal{E}(X)$, defined as follows:

$$(\forall x, x' \in X) \, x \equiv x' (\text{mod ker} f) \text{ iff } f(x) = f(x')$$

For $\pi \in \mathcal{E}(X)$ and the canonical projection $P_\pi : X \to X/\pi$, we have $\ker P_\pi = \pi$

Equivalence Kernel Example

Let the set X be a set of people in a room and let each person's age be in the integer range $\{21, 22, \ldots, 25\}$.

Let $Y = \mathbb{N} := \{0, 1, 2, \ldots\}$, and $f : X \rightarrow Y$ map a person to their corresponding age.

The equivalence relation ker$f$ groups the people based on their age.

Let $\overline{X}$ be the cells of ker$f$. Treating $\overline{X}$ as a set of labels for the cosets, we could take $\overline{X} = \{21, 22, \ldots, 25\}$, representing the five possible ages of the people.

Let $P_f : X \rightarrow \overline{X}$.

See diagram in class.

Canonical Factorization

Let $f : X \rightarrow Y$ and let $P_f : X \rightarrow X/\text{ker}f$.

$$\begin{tikzcd} X \arrow[r, "P_f"] \arrow[dr, "f"'] & X/\ker f \arrow[d, dashed, "g"] \\ & Y \end{tikzcd}$$

Defn: We refer to $f = g \circ P_f$ as the *canonical factorization* of f.

If we only care about evaluating f, we can convert X to $\overline{X}$, and then just use g to evaluate f using the typically much smaller set $\overline{X}$.

Note: $\circ$ denotes function composition. In this case, equivalent to saying:

$$(\forall x \in X)\, f(x) = g(P_f(x))$$

Canonical Factorization Proof

Claim: There exists a unique function $g : X/\text{ker}f \rightarrow Y$ such that $f = g \circ P_f$.

Proof:

Let $\overline{x} \in X/\text{ker}f = \overline{X}$.

$\Rightarrow (\exists a \in X)\, [a] = \overline{x}$ by definition of $\overline{X}$

Define $g(\overline{x}) = f(a)$ **(1)**

If there exists other $a' \in X$ such that $[a'] = \overline{x}$ then: $a \equiv a' (\text{mod ker}f)$

$\Rightarrow f(a) = f(a')$ (by definition of ker$f$), thus function g is well defined. **(2)**

Now need to verify that: $f = g \circ P_f$

Sufficient to show: $(\forall x \in X)\, f(x) = g(P_f(x))$

Canonical Factorization Proof (2)

Let $x \in X$, then $P_f(x) = [x] = \overline{x}$

$\Rightarrow g(\overline{x}) = f(x)$ by definition.

thus $f(x) = g(P_f(x))$ as required. **(3)**

Finally, we need to show that g is unique.

Let $\widehat{g} : X/\text{ker}f \rightarrow Y$ and assume that
$f = \widehat{g} \circ P_f$. **(4)**

Will show this implies $g = \widehat{g}$.

Must show: $(\forall \overline{x} \in \overline{X})\, g(\overline{x}) = \widehat{g}(\overline{x})$

Let $\overline{x} \in \overline{X}$, then $\overline{x} = [a]$ for some $a \in X$.

$\Rightarrow \overline{x} = P_f(a)$

Canonical Factorization Proof (3)

$\Rightarrow \widehat{g}(\overline{x}) = \widehat{g}(P_f(a)) = f(a)$ by **(4)**

As $g(\overline{x}) = f(a)$ by definition, we have $\widehat{g}(\overline{x}) = g(\overline{x})$

By **(1)**, **(2)**, and **(3)** we have defined a suitable function g, and by **(4)** we have shown it is unique.

QED

Other Factorizations

Proposition 1.4.1: Suppose $f : X \to Y$ and let $\rho \in \mathcal{E}(X)$ with ker$f \geq \rho$.

There exists a unique map $g : X/\rho \to Y$ such that $f = g \circ P_\rho$

$$\begin{tikzcd} X \arrow[r, "P_\rho"] \arrow[dr, "f"'] & X/\rho \arrow[d, dashed, "g"] \\ & Y \end{tikzcd} \qquad \ker f \geq \rho$$

Other Factorizations (2)

Proposition 1.4.2: Suppose $f : X \to Y$ and let $g : X \to Z$ and let ker$f \geq$ ker$g$.

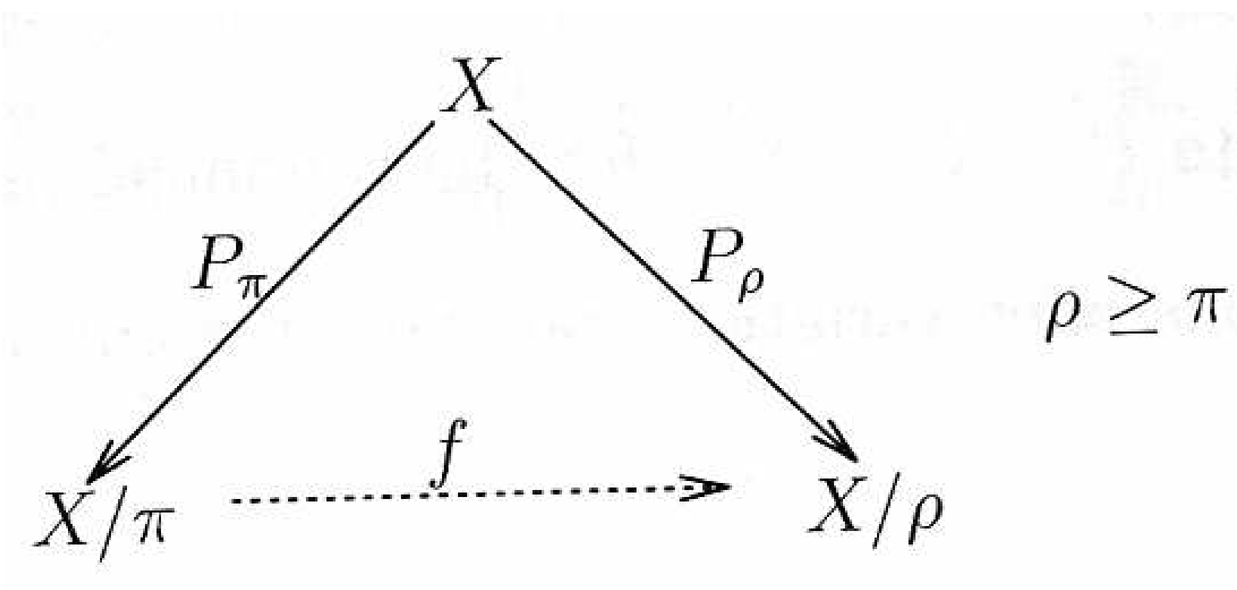
There exists a map $h : Z \to Y$ such that $f = h \circ g$.

Furthermore, h is uniquely defined on the image $g(X)$ of X in Z; that is, the restriction $h_{|g(X)}$ is unique.

$$\begin{tikzcd}
& X \arrow[dl, "g"'] \arrow[dr, "f"] & \\
Z \arrow[rr, dotted, "h"] & & Y
\end{tikzcd}
\qquad \ker f \geq \ker g$$

Other Factorizations (3)

Proposition 1.4.3: If $\pi, \rho \in \mathcal{E}(X)$ and $\pi \leq \rho$, there is a unique function $f : X/\pi \to X/\rho$ such that $P_\rho = f \circ P_\pi$.



Dynamic Systems

Defn: A *dynamic system* on a set X is a map $\alpha : X \to X$ with interpretation X are the system's *states*, and α is the *state transition function* .

We can then select an initial state $x_0 \in X$ and the system will evolve through states $x_1 = \alpha(x_0), x_2 = \alpha(x_1), \ldots$

We write α^k for the *k-fold composition* of α, taking $\alpha^0 =$ identity. ie. for $k = 2$ get $\alpha^2 = \alpha \circ \alpha$.

The *sequence* $\langle \alpha^0(x_0), \alpha^1(x_0), \alpha^2(x_0), \ldots \rangle$ is path of (X, α) with initial state x_0.

Congruences of Dynamic Systems

Defn: Let $\pi \in \mathcal{E}(X)$ with canonical projection $P_\pi : X \to \overline{X} := X/\pi$.

Relation π is a *congruence* for α if there exists a map $\overline{\alpha} : \overline{X} \to \overline{X}$ such that $\overline{\alpha} \circ P_\pi = P_\pi \circ \alpha$. ie., the diagram below commutes.

$$
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "P_\pi"'] & X \arrow[d, "P_\pi"] \\
\bar{X} \arrow[r, dashed, "\bar{\alpha}"] & \bar{X}
\end{tikzcd}
$$

We say that $\overline{\alpha}$ is the map induced by α on $\overline{X}$.

Fixing (X, α), let $\mathcal{C}(X) \subseteq \mathcal{E}(X)$ be the set of all congruences for α.

It can be shown that $\mathcal{C}(X)$ is a complete sublattice of $\mathcal{E}(X)$ that contains the elements $\perp$ and $\top$ of $\mathcal{E}(X)$.

Congruence Condition

How do we know when π is a congruence?

Proposition 1.4.4: π is a congruence for α iff

$$\ker P_\pi \leq \ker(P_\pi \circ \alpha)$$

namely

$$(\forall x, x' \in X)\,(x, x') \in \pi \Rightarrow (\alpha(x), \alpha(x')) \in \pi$$

Proof: Immediate from **Proposition 1.4.1**, with identifications $(Y, f, \rho, g) = (\overline{X}, P_\pi \circ \alpha, \pi, \overline{\alpha})$.

QED

Condition says that α maps all elements in the same cell of π to the same new cell of π.

Can take the dynamic system $(\overline{X}, \overline{\alpha})$, with initial state $\overline{x_0} = [x_0]$, as a consistent aggregated (high level) model of (X, α).

Observers

Let $\alpha : X \to X$ be an arbitrary function. Let $\omega \in \mathcal{E}(X)$.

Defn: Let $\omega \cdot \alpha \in \mathcal{E}(X)$ be defined as:

$(x, x' \in X)\, x \equiv x' (\text{mod}\, \omega{\cdot}\alpha)$ iff $\alpha(x) \equiv \alpha(x') (\text{mod}\, \omega)$

ie. $\omega \cdot \alpha = \ker(P_\omega \circ \alpha)$.

Thus, ω is a congruence for α iff $\omega \leq \omega \cdot \alpha$.

Defn: Let $\gamma : X \to Y$ be an arbitrary function. The *observer* for the triple (X, α, γ) is defined to be the equivalence relation:

$$\omega_o := \sup\{\omega \in \mathcal{E}(X) |\, \omega \leq (\ker \gamma) \wedge (\omega \cdot \alpha)\}$$

Will show the observer is the coarsest congruence for α that is finer than $\ker \gamma$.

Observers Proof

Let $\mathcal{F} = \{\omega \in \mathcal{E}(X) | \, \omega \leq (\ker \gamma) \wedge (\omega \cdot \alpha)\}$.

Thus, $\mathcal{F}$ is the set of all congruences finer than the $\ker \gamma$.

Claim: $\omega_o \in \mathcal{F}$

To prove this, we must show that ω_o is a congruence for α and that $\omega_o \leq \ker \gamma$.

Subclaim 1: ω_o is a congruence for α.

Proof: Clearly, $\mathcal{F} \subseteq \mathcal{C}(X)$, the set of all congruences for α, as every $\omega \in \mathcal{F}$ has property $\omega \leq \omega \cdot \alpha$ and is thus a congruence.

As $\mathcal{C}(X)$ (the set of all congruences for α) is a complete sublattice of $\mathcal{E}(X)$, we have $\omega_o = \sup(\mathcal{F}) \in \mathcal{C}(X)$, as required.

Observers Proof (2)

Subclaim 2: $\omega_o \leq \ker\gamma$

Proof: Let $x, x' \in X$. Assume $x(\sup(\mathcal{F}))x'$ **(1)**

Will now show implies $x(\ker\gamma)x'$

From definition of sup, **(1)** implies: **(2)**
$(\exists\, \text{integer}\, k \geq 1)(\exists F_1, \ldots, F_k \in \mathcal{F})(\exists x_0, x_1, \ldots, x_k \in X)$
$x_0 = x \;\&\; x_k = x' \;\&\; (\forall i) 1 \leq i \leq k \Rightarrow x_i \equiv x_{i-1} (\bmod\, F_i)$

From definition of $\mathcal{F}$, $(\forall F \in \mathcal{F})\, F \leq \ker\gamma$

$\Rightarrow (\forall i) 1 \leq i \leq k \Rightarrow x_i \equiv x_{i-1} (\bmod\, \ker\gamma)$ **(3)**

$\Rightarrow x(\ker\gamma)x'$ by the transitive property of equivalence relations.
Subclaim 2 complete.

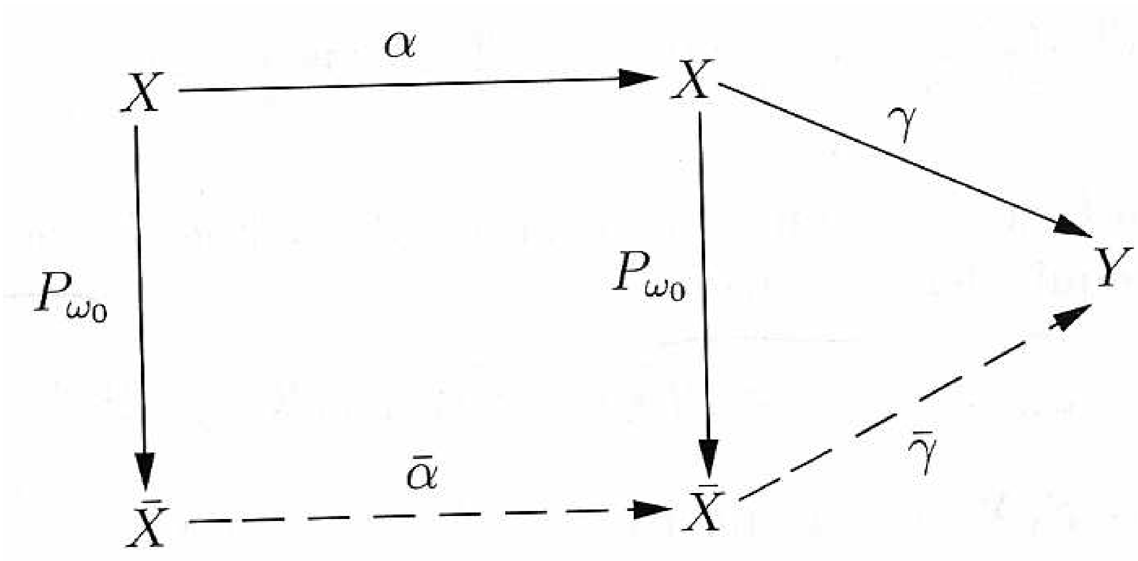
By **subclaims 1** and **2**, we have that ω_o is a congruence for α and that $\omega_o \leq \ker\gamma$, as required.

QED

Observer Diagram

If ω_o is an observer for (X, α, γ), then it will induce the maps $\overline{\alpha}$ and $\overline{\gamma}$, such that the diagram below will commute.

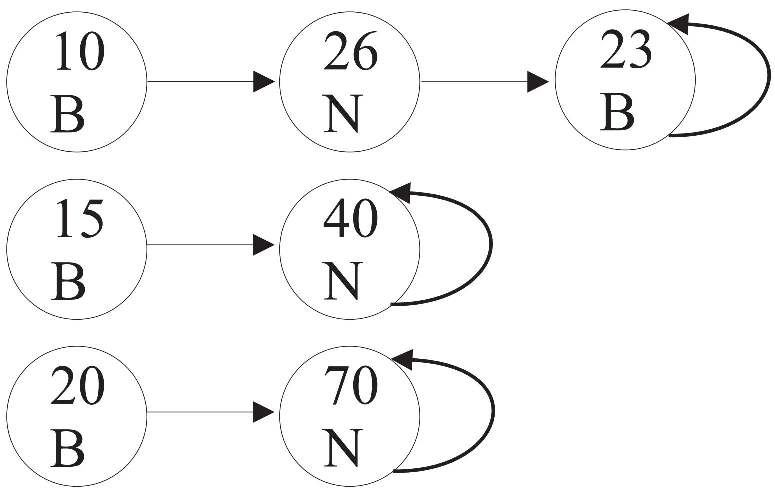
Think of γ as the output map for the system.



We can thus take the dynamic system $(\overline{X}, \overline{\alpha}, \overline{\gamma})$ as a consistent aggregated (high level) model of (X, α, γ).

Observer Example

Assume we have the dynamic system (X, α, γ) shown in the figure and table below:



x	10	15	20	23	26	40	70
$\alpha(x)$	26	40	70	23	23	40	70
$\gamma(x)$	B	B	B	B	N	N	N

Define $\gamma : X \rightarrow Y := \{B, N, D\}$ as follows:

$$\begin{aligned} \gamma(x) &= B \text{ if } x < 25, \quad \gamma(x) = N \text{ if } 25 \leq x \leq 100, \\ \gamma(x) &= D \text{ if } x > 100 \end{aligned}$$

Observer Example (2)

We will use the function $\Psi : \mathcal{E}(X) \rightarrow \mathcal{E}(X)$ to calculate our observer, ω_o:

$$\Psi(\omega) := (\ker \gamma) \wedge (\omega \cdot \alpha), \quad \omega \in \mathcal{E}(X)$$

It can be shown that, if we start with $\omega = \ker \gamma$, Ψ is monotone and that ω_o is the greatest fixed point of Ψ.

Our *refinement algorithm* will be:

1) Set $\omega' = \ker \gamma$

2) $\omega = \Psi(\omega') := (\ker \gamma) \wedge (\omega' \cdot \alpha)$

3) if $\omega \neq \omega'$ then $\omega' = \omega$; goto step 2

4) $\omega_o = \omega$; stop

Observer Example (3)

From table: $\ker\gamma = \{\{10, 15, 20, 23\}, \{26, 40, 70\}\}$

i	$\omega' = \ker\gamma$ calculate: $\omega' \cdot \alpha = \{\{10, 15, 20, 40, 70\}, \{23, 26\}\}$ calculate: $\omega = (\ker\gamma) \wedge (\omega' \cdot \alpha) = \{\{10, 15, 20\}, \{23\}, \{26\}, \{40, 70\}\}$ $\omega \neq \omega'$ so $\omega' = \omega$
ii	calculate: $\omega' \cdot \alpha = \{\{10\}, \{15, 20, 40, 70\}, \{23, 26\}\}$ calculate: $\omega = \{\{10\}, \{23\}, \{26\}, \{15, 20\}, \{40, 70\}\}$ $\omega \neq \omega'$ so $\omega' = \omega$
iii	calculate: $\omega' \cdot \alpha = \{\{10\}, \{15, 20, 40, 70\}, \{23, 26\}\}$ calculate: $\omega = \{\{10\}, \{23\}, \{26\}, \{15, 20\}, \{40, 70\}\}$ $\omega = \omega'$ so $\omega_o = \omega$; stop

Observer Example (4)

Our observer is thus:

$$\omega_o = \{\{10\}, \{23\}, \{26\}, \{15, 20\}, \{40, 70\}\}$$

Define: $\overline{X} := \{a, c, b, d, e\}$ to label cells:

$$\{10\}, \{23\}, \{26\}, \{15, 20\}, \{40, 70\}$$

We can now define our aggregated system $(\overline{X}, \overline{\alpha}, \overline{\gamma})$ as follows:

$\overline{x}$	a	b	c	d	e
$\overline{\alpha}(\overline{x})$	b	c	c	e	e
$\overline{\gamma}(\overline{x})$	B	N	B	B	N