Curvature Based Image Registration

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Abstract

A fully automated, non-rigid image registration algorithm is presented. The deformation field is found by minimizing a suitable measure subject to a curvature based constraint. It is a well-known fact that non-rigid image registration techniques may converge poorly if the initial position is not sufficiently near to the solution. A common approach to address this problem is to perform a time consuming rigid pre-registration step. In this paper we show that the new *curvature registration* not only produces accurate and smooth solutions but also allows for an automatic rigid alignment. Thus, in contrast to other popular registration schemes, the new method no longer requires a pre-registration step. Furthermore, we present an implementation of the new scheme based on the numerical solution of the underlying Euler-Lagrange equations. The real discrete cosine transform is the backbone of our implementation and leads to a stable and fast $\mathcal{O}(N \log N)$ algorithm, where N denotes the number of voxels. Finally, we report on some numerical test runs.

Keywords Image processing, image registration, variational methods, elastic registration, non-rigid registration, curvature, pre-registration, Euler-Lagrange equations

1 Introduction

Image registration is a challenging problem in digital image processing. The problem arises in many areas of applications, for example geophysics, medicine, and robotics. For an overview, we refer to [6, 12, 14, 11] and references therein.

The problem arises when images of one object are taken for example at different times, from different perspectives, and/or different image devices. The basic goal is to combine information and to integrate useful data obtained from the separate images. However, due to spatial distortion of the

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object under consideration, for example introduced by motion, the same object in different images is not directly comparable. The idea of image registration is to geometrically transform images in order to compensate for these distortions.

In particular in medicine, registration has a vast range of applications. It is essential in order to study the evolution of a pathology of a patient, or to take full advantage of the complementary information coming from multimodal imagery, only to name two examples.

A wide variety of image registration techniques have been designed to handle specific applications. Here, we focus on non-rigid, non-parametric, and fully automatic schemes which make use of a deformable image model, see, e.g., [5, 2, 1, 7, 3]. Most of these schemes may be viewed as a procedure which minimizes a suitable distance measure subject to a regularization term or some interpolation restrictions; see [8] for an overview.

There are several problems with fully automatic registration approaches. One of which is the problem that the technique is sensitive to initial positioning of the images to be matched, as the deformation process is always done iteratively. To produce a good initial position, typically a rigid or an affine-linear pre-registration step, also known as *global matching*, is performed. If the initial rigid alignment is not sufficiently near the solution, the non-rigid matching procedure may converge poorly.

Here, we propose a novel non-parametric fully automatic registration technique which relies on a curvature based penalizing term. This approach not only provides smooth solutions but also allows for an automatic rigid alignment. Thus, in contrast to other popular registration schemes, the pre-registration step becomes redundant.

Our derivation is based on a variational formulation. The wanted minimizer is characterized by the *Euler-Lagrange equations* which turn out to be a system of non-linear partial differential equations. In this paper we will carry out the derivation in detail with a special emphasis on the boundary conditions involved. Finally, we devise a fast and stable implementation for a finite difference approximation of the underlying partial differential equation and present some numerical examples.

2 Approach

Given are two images which are referred to as *reference* R and *template* T. For a particular spatial point $\mathbf{x} \in \Omega \subset \mathbb{R}^d$, the value $T(\mathbf{x})$ is the intensity at $\mathbf{x}$. Without loss of generality we assume $\Omega = [0,1]^d$, where $d \in \mathbb{N}$ denotes the spatial dimension of the images. The registration algorithm described in this paper is applicable to images with any number of dimensions d.

The purpose of the registration is to determine a transformation of T onto R. Ideally, one wants to determine a displacement field $\mathbf{u} : \Omega \to \Omega$

such that $T(\mathbf{x}-\mathbf{u}(\mathbf{x})) = R(\mathbf{x})$. The question is how to find such a mapping $\mathbf{u} = (u_1, \ldots, u_d)$. A typical approach is the minimization of a suitable distance measure $\mathcal{D}$. For example, the so-called sum of squared differences measure is

$$\mathcal{D}^{\mathrm{SSD}}[R,T;\mathbf{u}] := \frac{1}{2}\int_\Omega (T(\mathbf{x}-\mathbf{u}(\mathbf{x})) - R(\mathbf{x}))^2 \, d\mathbf{x}, \tag{1}$$

and other choices are discussed in [8]. Since the minimization of $\mathcal{D}$ is in general an ill-posed problem, an additional regularizing term $\mathcal{S}$ has to be introduced. This regularizing term may also serve to rule out discontinuous and/or suboptimal solutions or to privilege more likely solutions. The problem now reads: Find a mapping $\mathbf{u}$ which minimizes the joint criterion

$$\mathcal{J}[\mathbf{u}] := \alpha\mathcal{S}[\mathbf{u}] + \mathcal{D}[R,T;\mathbf{u}].$$

In this note, we investigate the novel smoothing term

$$\mathcal{S}^{\mathrm{curv}}[\mathbf{u}] := \frac{1}{2}\sum_{\ell=1}^{d}\int_\Omega (\Delta u_\ell)^2 \, d\mathbf{x}. \tag{2}$$

The reason for this particular choice is twofold. The integral might be viewed as an approximation to the curvature of the ℓth component of the displacement field and therefore does penalize oscillations. Most interestingly, $\mathcal{S}^{\mathrm{curv}}$ has a non-trivial kernel containing affine linear transformations, i.e.,

$$\mathcal{S}^{\mathrm{curv}}[\mathbf{C}\mathbf{x}+\mathbf{b}] = 0, \quad \mathbf{C} \in \mathbb{R}^{d\times d}, \ \mathbf{b} \in \mathbb{R}^d.$$

Thus, in contrast to many other non-linear registration techniques, the new scheme does not penalize affine linear transformations and consequently does not require an additional affine linear pre-registration step to be successful.

We now work out necessary conditions for a minimizer $\mathbf{u}$ of the functional $\mathcal{J}$. To this end we compute the Gâteaux derivatives of $\mathcal{D}^{\mathrm{SSD}}$ and $\mathcal{S}^{\mathrm{curv}}$, respectively. The proof is straightforward, see also [10].

Lemma 1

1. *Let $\mathcal{D}^{\mathrm{SSD}}$ be defined by (1), $R \in L_2(\mathbb{R}^d)$, $T \in C^2(\mathbb{R}^d)$, and $\mathbf{u} \in L_2(\Omega)^d$. For the perturbation $\mathbf{v} \in L_2(\Omega)^d$, the Gâteaux derivative of $\mathcal{D}^{\mathrm{SSD}}$ is given by*

$$d\mathcal{D}^{\mathrm{SSD}}[T,R;\mathbf{u};\mathbf{v}] = \int_\Omega \langle \mathbf{f}(\mathbf{x},\mathbf{u}(\mathbf{x})), \mathbf{v}(\mathbf{x})\rangle_{\mathbb{R}^d} \, d\mathbf{x},$$

where $\mathbf{f}(\mathbf{x},\mathbf{u}(\mathbf{x})) = -(T(\mathbf{x}-\mathbf{u}(\mathbf{x})) - R(\mathbf{x}))\ \nabla T(\mathbf{x}-\mathbf{u}(\mathbf{x}))$.

2. *Let $\mathcal{S}^{\mathrm{curv}}$ be defined by (2) for $\mathbf{u} \in C^4(\mathbb{R}^d)^d$. The for the perturbation $\mathbf{v} \in C^4(\mathbb{R}^d)^d$ the Gâteaux derivative of $\mathcal{S}^{\mathrm{curv}}$ is given by*

$$
\begin{aligned}
d\mathcal{S}^{\mathrm{curv}}[\mathbf{u};\mathbf{v}] = & \int_\Omega \langle \Delta^2 \mathbf{u}, \mathbf{v}\rangle_{\mathbb{R}^d} d\mathbf{x} \\
& + \sum_{\ell=1}^d \int_{\partial\Omega} \Delta u_\ell \langle \nabla v_\ell, \mathbf{n}\rangle_{\mathbb{R}^d} d\mathbf{x} - \sum_{\ell=1}^d \int_{\partial\Omega} v_\ell \langle \nabla \Delta u_\ell, \mathbf{n}\rangle_{\mathbb{R}^d} d\mathbf{x}.
\end{aligned}
$$

Remarks

1. Without loss of generality we assumed $T(\mathbf{x}) = R(\mathbf{x}) = 0$ for $\mathbf{x} \notin \Omega$.

2. The above lemma may be phrased for less smooth functions $\mathbf{u}$ and $\mathbf{v}$ as well. However, as we intend to solve the resulting PDE by means of a finite difference scheme, the considered smoothness conditions are what we want.

3. The necessary optimality condition for $\mathcal{S}^{\mathrm{curv}}$ gives the following natural boundary conditions

$$\Delta u_\ell = \langle \nabla \Delta u_\ell, \mathbf{n}\rangle_{\mathbb{R}^d} = 0 \quad \text{on} \quad \partial\Omega, \quad \ell = 1, \ldots, d. \tag{3}$$

4. For our finite difference based numerical scheme, we substitute the natural boundary conditions (3) by the explicit boundary conditions

$$\nabla u_\ell = \nabla \Delta u_\ell = 0 \qquad \partial\Omega, \quad \ell = 1, \ldots, d. \tag{4}$$

   In this case, we find that when $\mathcal{S}^{\mathrm{curv}}[\mathbf{u}]$ is restricted to $\mathbf{u}$ in the subspace of $C^4(\mathbb{R}^d)^d$ in which (4) holds, then for perturbations $\mathbf{v}$ in the same space,

$$d\mathcal{S}^{\mathrm{curv}}[\mathbf{u};\mathbf{v}] = \int_\Omega \langle \Delta^2 \mathbf{u}, \mathbf{v}\rangle_{\mathbb{R}^d} d\mathbf{x}.$$

   It is this choice which results in a highly structured, fast solvable matrix problem (see [8]).

5. We like to point out that there is a close relation between our new curvature based approach and the computation of so-called *thin-plate-splines*, cf., e.g., [13]. These functions are defined as minimizer of the following smoothing term

$$\mathcal{S}^{\mathrm{TPS}}[\mathbf{u}] = \frac{1}{2} \sum_{\ell=1}^d \int_{\mathbb{R}^d} \sum_{|\kappa|=q} c_\kappa \left(D^\kappa u_\ell\right)^2 d\mathbf{x},$$

   subject to some interpolatory constraints. Interestingly, the Gâteaux derivatives of $\mathcal{S}^{\mathrm{TPS}}$ and $\mathcal{S}^{\mathrm{curv}}$ share the same main part but differ slightly in their boundary integrals.

6. Thin-plate-splines are designed to minimize the bending energy of $\mathbf{u}$. This together with the above mentioned connection offers a nice interpretation of the curvature based approach.

3 Numerical approach

In accordance with the calculus of variations and Lemma 1, a function $\mathbf{u}$ which minimizes the joint functional $\mathcal{J}$ for the particular choice $\mathcal{S} = \mathcal{S}^{\text{curv}}$ has to satisfy the Euler-Lagrange equation

$$\mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) + \alpha\Delta^2\mathbf{u}(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in \Omega, \tag{5}$$

subject to the boundary conditions. The so-called force field $\mathbf{f}$ is the Gâteaux derivative of the distance measure $\mathcal{D}$. The above fourth-order non-linear PDE is known as the bipotential or biharmonic equation and is well studied; see, e.g., [9]. It describes the displacement of a plate subject to the load $\mathbf{f}$.

A popular approach to solve this PDE is to introduce an artificial time t and to compute the steady state solution of the time dependent PDE. Here, we employ the following semi-implicit iterative scheme,

$$\partial_t\mathbf{u}^{k+1}(\mathbf{x}, t) - \alpha\Delta^2\mathbf{u}^{k+1}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, \mathbf{u}^k), \quad k = 1, 2, \ldots, \tag{6}$$

where $\mathbf{u}^0$ is some initial deformation, typically $\mathbf{u}^0 = 0$.

To solve (6) numerically, we apply a finite difference discretization adapted to the particular simple geometry of our domain Ω. This approach results in a system of linear equations $\mathcal{A}\vec{U}^{k+1} = \vec{B}^k$, where $\vec{U}^k$ and $\vec{B}^k$ represent grid values of u and $b = u + \tau f$, respectively, and τ denotes a time step. Consequently, the main work in the overall scheme is the repeated solution of this linear system. Since the coupling with respect to the components of u_ℓ is only via the forces $\mathbf{f}$ the system decouples into

$$(I_N + \tau\alpha A^2)\vec{U}_\ell^{k+1} = \vec{U}_\ell^k + \tau\vec{F}_\ell^k, \quad \ell = 1, \ldots, d. \tag{7}$$

Here, N denotes the number of pixels or voxels and I_N the identity-matrix. Finally, the boundary conditions (4) ensure that the linear systems in (7) can be solved in a fast and stable way by discrete cosine transformations, which lead to an $\mathcal{O}(N \log N)$ implementation; for details, see [8].

4 Experiments

To illustrate the performance of our new approach we present the registration of X-rays of human hands. The upper left picture displays the template whereas the reference is depicted in the bottom right corner. The six pictures in between show some intermediate registration results. Note that apart from the fact that the template hand is much thinner as opposed to the reference it is also rotated to the right. It is this rotation which would cause serious trouble for other registration techniques like the elastic or fluid registration scheme. As is apparent from the picture sequence, the curvature based approach does produce a visually correct registration.

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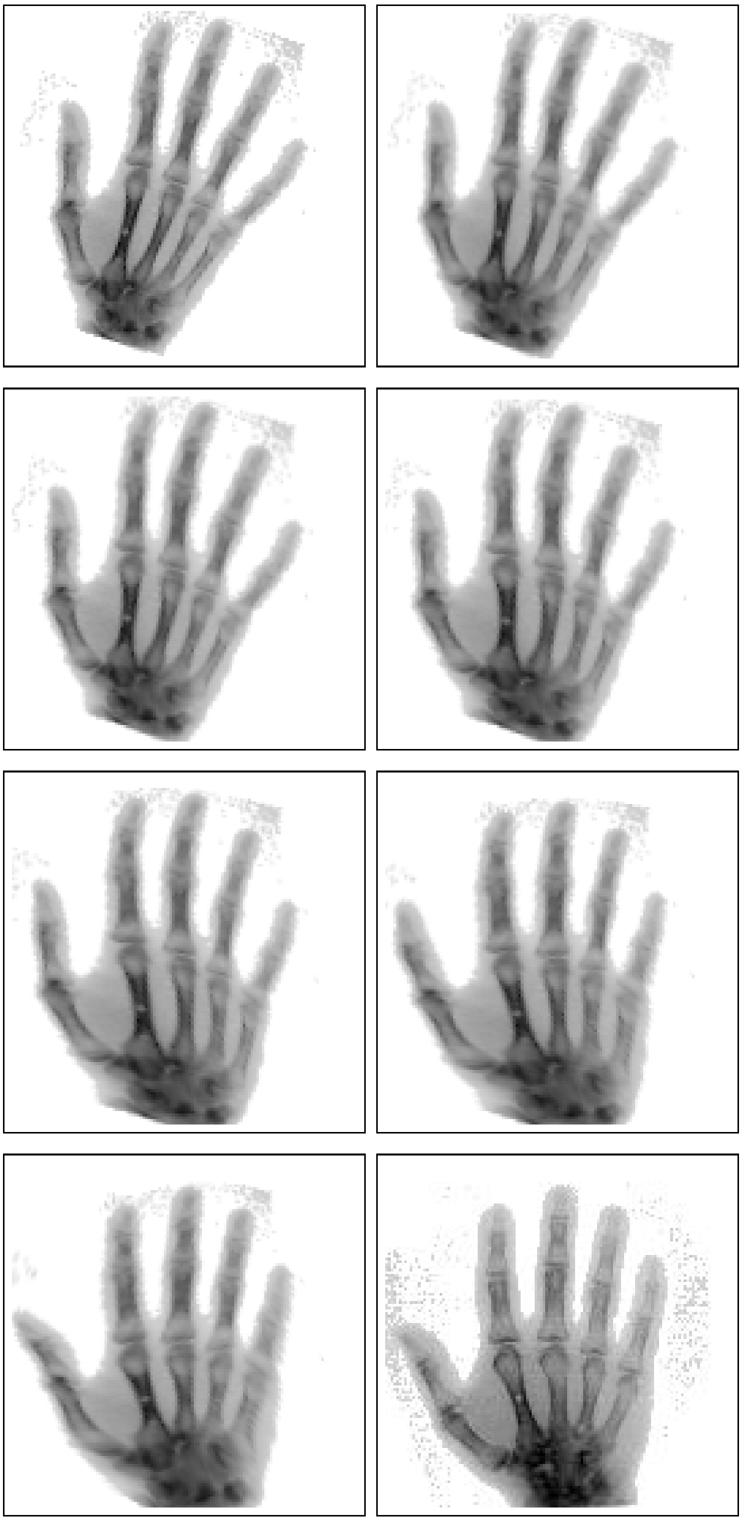
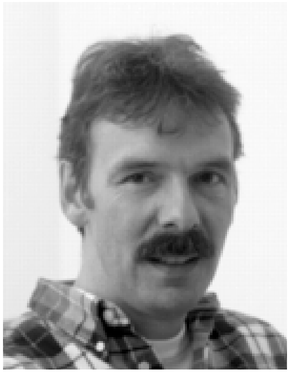


Fig. 1: Registration of X-rays of human hands. TOP: Template on the left. BOTTOM: Reference on the right. The six pictures in between show some intermediate registration results.



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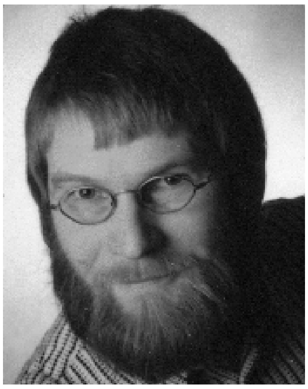

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