Recursive Doubling

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Outline

The problem

Let $x_1 = b_1$. Consider computing

$$x_i = a_i x_{i-1} + b_i, \qquad i = 2, 3, \dots; \quad a_i, b_i \text{ are given}$$

- This appears inherently serial
- It can be done in parallel through *recursive doubling*
- It seems this method originates in
  Peter M. Kogge and Harold S. Stone. 1973. A Parallel Algorithm for the Efficient Solution of a General Class of Recurrence Equations. IEEE Trans. Comput. 22, 786-793
- See `http://www.acsel-lab.com/Projects/fast_adder/references/papers/Kogge-Stone-73.pdf`

- We present an example of recursive doubling used in the above reference
- It contains more general recurrence relations and algorithm

Example

$$
\begin{aligned}
x_1 &= b_1 \\
x_2 &= a_2 x_1 + b_2 = a_2 b_1 + b_2 = \boxed{\text{q(2,1)}} \\
x_3 &= a_3 x_2 + b_3 = (a_3 a_2) x_1 + (a_3 b_2 + b_3) = \boxed{(a_3 a_2) x_1 + q(3,2)} \\
x_4 &= a_4 x_3 + b_4 = (a_4 a_3) x_2 + (a_4 b_3 + b_4) = \boxed{(a_4 a_3) x_2 + q(4,3)} \\
x_5 &= a_5 x_4 + b_5 = (a_5 a_4) x_3 + (a_5 b_4 + b_5) \\
&= \boxed{(a_5 a_4)(a_3 a_2) x_1 + (a_5 a_4) q(3,2) + q(5,4)} = \boxed{(a_5 a_4)(a_3 a_2) x_1 + q(5,2)} \\
x_6 &= a_6 x_5 + b_6 = (a_6 a_5) x_4 + a_6 b_5 + b_6 \\
&= \boxed{(a_6 a_5)(a_4 a_3) x_2 + (a_6 a_5) q(4,3) + q(6,5)} = \boxed{(a_6 a_5)(a_4 a_3) x_2 + q(6,3)} \\
x_7 &= a_7 x_6 + b_7 = a_7 (a_6 x_5 + b_6) + b_7 = a_7 a_6 x_5 + a_7 b_6 + b_7 \\
&= \boxed{(a_7 a_6)(a_5 a_4) x_3 + (a_7 a_6) q(5,4) + q(7,6)} = \boxed{(a_7 a_6)(a_5 a_4) x_3 + q(7,4)} \\
x_8 &= a_8 x_7 + b_8 = (a_8 a_7) x_6 + a_8 b_7 + b_8 \\
&= \boxed{(a_8 a_7)(a_6 a_5) x_4 + (a_8 a_7) q(6,5) + q(8,7)} = \boxed{(a_8 a_7)(a_6 a_5) x_4 + q(8,5)}
\end{aligned}
$$

Computation

- We evaluate the x_i in stages
- We assume b_i is stored on process i, $i = 1, \ldots, 8$, and a_i is stored on process i, $i = 2, \ldots, 8$

Stage 1

PE	stores, sends	computes	
1	$b_1 \rightarrow 2$		
2	$(a_2, b_2) \rightarrow 3$		$x_2 = q(2,1) = a_2 b_1 + b_2$
3	$(a_3, b_3) \rightarrow 4$	$p(3,2) = a_3 a_2$	$q(3,2) = a_3 b_2 + b_3$
4	$(a_4, b_4) \rightarrow 5$	$p(4,3) = a_4 a_3$	$q(4,3) = a_4 b_3 + b_4$
5	$(a_5, b_5) \rightarrow 6$	$p(5,4) = a_5 a_4$	$q(5,4) = a_5 b_4 + b_5$
6	$(a_6, b_6) \rightarrow 7$	$p(6,5) = a_6 a_5$	$q(6,5) = a_6 b_5 + b_6$
7	$(a_7, b_7) \rightarrow 8$	$p(7,6) = a_7 a_6$	$q(7,6) = a_7 b_6 + b_7$
8	(a_8, b_8)	$p(8,7) = a_8 a_7$	$q(8,7) = a_8 b_7 + b_8$

Stage 2

PE	stores, sends	computes
1	$x_1 \to 3$	
2	$x_2 \to 4$	
3	$p(3,2), q(3,2) \to 5$	$x_3 = p(3,2)x_1 + q(3,2)$
4	$p(4,3), q(4,3) \to 6$	$x_4 = p(4,3)x_2 + q(4,3)$
5	$p(5,4), q(5,4) \to 7$	$p(5,2) = p(5,4)q(3,2) + q(5,4)$
6	$p(6,5), q(6,5) \to 8$	$p(6,3) = p(6,5)q(4,3) + q(6,5)$
7	$p(7,6), q(7,6)$	$p(7,4) = p(7,6)q(5,4) + q(7,6)$
8	$p(8,7), q(8,7)$	$p(8,5) = p(8,7)q(6,5) + q(8,7)$

Stage 3

PE	stores, sends		computes
1		$x_1 \to 5$	
2		$x_2 \to 6$	
3	$p(3,2) \to 5$	$x_3 \to 7$	
4	$p(4,3) \to 6$	$x_4 \to 8$	
5	$p(5,4) \to 7$	$q(5,2)$	$x_5 = p(5,4)p(3,2)x_1 + q(5,2)$
6	$p(6,5) \to 8$	$q(6,3)$	$x_6 = p(6,5)p(4,3)x_2 + q(6,3)$
7	$p(7,6)$	$q(7,4)$	$x_7 = p(7,6)p(5,4)x_3 + q(7,4)$
8	$p(8,7)$	$q(8,5)$	$x_8 = p(8,7)p(6,5)x_4 + q(8,5)$

Comments

In the above,

$$p(m,n) = \prod_{i=n}^{m} a_i, \quad p(m,n) = 1 \text{ if } n > m$$

$$q(m,n) = \sum_{j=n}^{m} \left(\prod_{r=j+1}^{m} a_r \right) b_j = \sum_{j=n}^{m} p(m, j+1) b_j$$

Then

$$q(2i,1) = x_{2i} = \left(\prod_{r=i+1}^{2i} a_r \right) q(i,1) + q(2i, i+1)$$

$q(i, 1)$ and $q(2i, i+1)$ have the same number of additions and multiplications since

$$q(i, 1) = \sum_{j=1}^{i} \left(\prod_{r=j+1}^{i} a_r \right) b_j,$$

$$q(2i, i+1) = \sum_{j=i+1}^{2i} \left(\prod_{r=j+1}^{2i} a_r \right) b_j$$