Parallel Distributed Shortest Paths

Ned Nedialkov

McMaster University
Canada

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Outline

For more details see A. Grama, G. Karypis, V. Kumar, A. Gupta, Introduction to Parallel Computing

Shortest paths

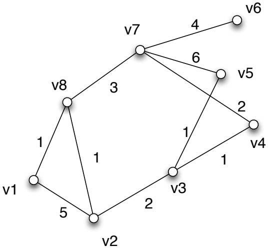
- $G = (V, E, w)$ weighted graph
- G can be directed or undirected
- V is a set of vertices
- E is a set of edges
- $w : E \rightarrow R^+$ is a weight function
- Given $s \in V$, find the shortest paths from s to all vertices in V
- Dijkstra's algoritm
  - Finds all shortest distances from s
  - Greedy algorithm: always choses the closest vertex
  - Original algorithm $O(|V|^2)$
  - Using a min-priority queue implemented by a Fibonacci heap $O(|E| + |V| \log |V|)$, Fredman & Tarjan 1984

Dijkstra's algoritm

Algorithm (Dijkstra)

Input: $G = (V, E, w)$, source s
Output: d an array of shortest distances
Compute:
% initialization
$V_T = \{s\}, d[s] = 0$
for all $v \in (V - V_T)$ **do**
  if $(s, v) \in E$ then $d[v] = w(s, v)$
  else $d[v] = \infty$
% find distances
while $V_T \neq V$ **do**
  find u such that $d[u] = \min\{d[v] \mid v \in V - V_T\}$
  $V_T = V_T \cup \{u\}$
  for all $v \in V - V_T$ do
    $d[v] = \min\{d[v], d[u] + w(u, v)\}$

Example



$s = v_1,\ V_T = \{v_1\}$
After the for loop
$d = (0, 5, \infty, \infty, \infty, \infty, \infty, 1)$

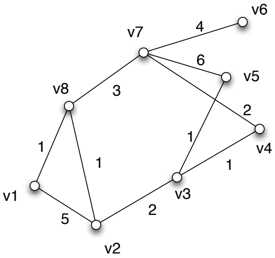
Iterations of the while loop

1. $u = v_8,\ V_T = \{v_1, v_8\}$
   $V - V_T = \{v_2, v_3, v_4, v_5, v_6, v_7\}$
   $d = (0, 2, \infty, \infty, \infty, \infty, 4, 1)$
2. $u = v_2,\ V_T = \{v_1, v_8, v_2\}$
   $V - V_T = \{v_3, v_4, v_5, v_6, v_7\}$
   $d = (0, 2, 4, \infty, \infty, \infty, 4, 1)$
3. $u = v_3,\ V_T = \{v_1, v_8, v_2, v_3\}$
   $V - V_T = \{v_4, v_5, v_6, v_7\}$
   $d = (0, 2, 4, 5, 5, \infty, 4, 1)$
4. $u = v_7,\ V_T = \{v_1, v_8, v_2, v_3, v_7\}$
   $V - V_T = \{v_4, v_5, v_6\}$
   $d = (0, 2, 4, 5, 5, 8, 4, 1)$

   The remaining iterations do not change d

Dense Graphs

Distribute the adjacency matrix using 1D block distribution, e.g.,



		v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
P_0	v_1		5						1
	v_2	5		2					1
P_1	v_3		2		1	1			
	v_4			1				2	
P_2	v_5			1				6	
	v_6							4	
P_3	v_7				2	6	4		3
	v_8	1	1					3	

Process P_i “owns" a subset V_i of V

P_i computes in the while loop

for all $v \in (V - V_T) \cap V_i$ do

$d[v] = \min\{d[v], d[u] + w(u, v)\}$

- Each P_i needs to know V_T
- P_i finds u such that $d[u] = \min\{d[v] \mid v \in (V - V_T) \cap V_i\}$
  That is, a minimum among the vertices it owns
- P_0 can do global reduction to find u such that $d[u] = \min\{d[v] \mid v \in (V - V_T)\}$
- P_0 broadcasts u
- Each P_i inserts into its V_T

- Let $|V| = n$
- Each process uses $O(n^2/p)$ storage
- Computations are in $O(n^2/p)$
- Reduction and broadcast are in $O(\log p)$
- There are n communication steps
- $T_p = O(n^2/p) + O(n \log p)$
- The speed up is

$$S = \frac{T_s}{T_p} = \frac{O(n^2)}{O(n^2/p) + O(n \log p)} = \frac{p}{1 + O(\frac{p \log p}{n})}$$

- The efficiency is

$$E = \frac{1}{1 + O(\frac{p \log p}{n})}$$

Sparse graphs: Johnson's algoritm

Algorithm (Johnson)

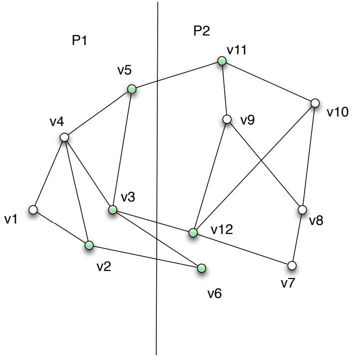
Input: $G = (V, E, w)$, source vertex s
Output: vector d with shortest distances to s
Compute:
% initialization
$Q = V$, s source vertex
for all $v \in Q$ **do** $d[v] = \infty$
$d[s] = 0$
% find distances
while $Q \neq \emptyset$ **do**
  u = extract vertex with smallest distance from Q
  for each $v \in \text{Adj}[u]$ **do**
    if $v \in Q$ and $d[u] + w(u, v) < d[v]$ then
      $d[v] = d[u] + w(u, v)$
      update $d[v]$ in Q

- Q is a priority queue
- Can be implemented as min-heap
- The top of the heap is the vertex with shortest distance
- Distance update (rebuilding the heap) is in $O(\log|V|)$
- For each vertex we scan its incident edges
- $|E|$ edges, $O(\log|V|)$ updates, $O(|E|\log|V|)$ running time
- How to parallelize?

Graph distribution

- Assume p processes
- Partition V into p sets, $V_1, V_2, \ldots, V_p$
- Process i
  - stores V_i
  - stores for each $u \in V_i$ all adjacent vertices to u, i.e., $v \in \mathrm{Adj}[u]$
    and the weights of the corresponding edges
- For $u \in V_i$, let $v \in \mathrm{Adj}[u]$
  - if $v \in V_i$, u is an internal vertex
  - if $v \in V_j$, u is a boundary vertex

Example

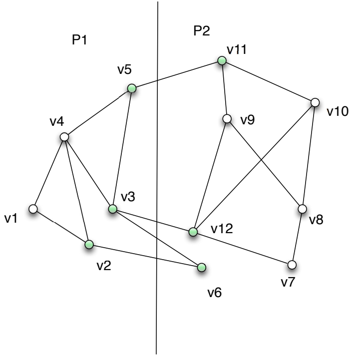


- Process P1 owns
  - internal vertices v_1, v_4
    boundary vertices v_2, v_3, v_5
  - stores vertices from P2: v_6, v_{11}, v_{12}
- Process P2 owns
  - internal vertices v_7, v_8, v_9, v_{10}
    boundary vertices v_6, v_{11},v_{12}
  - stores verices from P1 v_2, v_3, v_5
- Each process also stores the weights of the corresponding edges

Distributed implementation

- Assume p processes
- Partition V into subsets $V_1, \ldots, V_p$
- Process P_i owns V_i and stores adjacent vertices that are on other processes
- P_i maintains its own priority queue Q_i
- P_i keeps an array D, where $D[v]$ is the shortest distance to the source s
  - initially $D[v] = \infty$ for all $v \in V_i$
  - and $D[s] = 0$, where the source is
  - $D[v] = d[v]$, when v is extracted from Q_i
- Each process run JA on its Q_i

- Assume P_i extracts u from Q_i
- If $(u, v) \in E$ and $v \in P_j \neq P_i$, P_i may update $d[v]$ and then send it to P_j
- If v in Q_j, P_j updates Q_j with $d[v]$
- If $v \notin Q_j$, then P_j has already computed shorted distance for v, i.e. $D[v]$
  - if $D[v] \leq d[v] = d[u] + w(u, v)$, there is a longer path, nothing to update
  - if $D[v] > d[v] = d[u] + w(u, v)$, there is a shorter path: insert v back into Q_j



- Each process runs JA
- When u is extracted from Q_i
  if u is boundary, send $d[u]$ to all P_j's that contain a v adjacent to u
- e.g. when P1 updates the distance of v_{12}, send this distance to P2

- Assume s on P_0
- Initially only Q_0 is not empty
- P_0 starts working
- As distances of vertices on other processes become available, priority queues of other processes get populated
- Assume a grid graph, where s is at the bottom left corner
- The computation proceeds like a wave across the grid

2D-block mapping

- Let $|V| = n$ vertices
- Consider a grid of $\sqrt{p} \times \sqrt{p}$ processes
- Assign a block of $n/\sqrt{p} \times n/\sqrt{p}$ vertices to each process
- If s is in the left bottom corner, the wave moves diagonally up the grid of processes
- No more than $O(\sqrt{p})$ processes are busy at any time
- Assume the work of the sequential algorithm is $T_s = W$
- Ignoring communication cost, $T_p = W/\sqrt{p}$
- Speed up is $S = \sqrt{p}$
- Efficiency is $E = 1/\sqrt{p}$

1D-block mapping

- Subdivide the grid vertically in n/p stripes
- Assign each of them to a processor
- On average $p/2$ processes are busy
- $S = p/2$, $E = 1/2$