Measuring NMR Relaxation Time
Using an Exact Solution of the Bloch Equations

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Hahn Echo

- Sample $S(\omega, R_1, R_2)$
- Pulse Sequences (Operators) $f$
- Collecting data $d$

Objective: Using a series of delay times $\tau$, a series of data $d$ are collected, then we analyze the data $d$ to get the transverse relaxation rate $R_2$ which is one of properties of sample $S$. 
**Hahn Echo**

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- Pulse Sequences (Operators) $f$

Collecting data $d$  

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Hard Pulse

\[
\frac{\pi}{2} \gamma B_1 \times t_p \times 2\pi = \pi
\]

- high power
- short duration

- \( \gamma B_1 = 16500\text{Hz} \)
- \( \omega = 330\text{Hz} \)
- \( R_1 = 1s^{-1} \)
- \( R_2 = 18.8s^{-1} \)
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Hard Pulses are not always available

- Maximum power output by the amplifier is limited;
- Sample may boil if the power is too high;
- Some probes require low power;
- Sample may be a large molecule, which means the range of offset frequencies may be wide. For example,

\[ \Delta \omega = \gamma B_1 \]
Soft Pulse

\[ \frac{\pi}{2} \begin{array}{c} y \\ \tau \end{array} \quad \frac{\pi}{2} \begin{array}{c} \pm x \\ t_p \end{array} \]

\[ \gamma B_1 \times t_p \times 2\pi = \pi \]
- low power
- long duration
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Soft Pulse

\[ \frac{\pi}{2} y \tau \text{ } t_p \text{ } \tau \]

\[ \gamma B_1 \times t_p \times 2\pi = \pi \]

- low power
- long duration

\[ \gamma B_1 = 330Hz \]
\[ \omega = 330Hz \]
\[ R_1 = 1s^{-1} \]
\[ R_2 = 18.8s^{-1} \]
Mathematical View of Hahn Echo

\[
\frac{d M_x}{dt} = -\omega M_y + \gamma B_1 \sin \phi M_z - R_2 M_x \quad (1a)
\]

\[
\frac{d M_y}{dt} = \omega M_x - \gamma B_1 \cos \phi M_z - R_2 M_y \quad (1b)
\]

\[
\frac{d M_z}{dt} = -\gamma B_1 \sin \phi M_x + \gamma B_1 \cos \phi M_y - R_1 (M_z - 1) \quad (1c)
\]

\[
M(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (2)
\]

\[
\gamma B_1(t) = \begin{cases} 
0 & 0 \leq t \leq \tau \\
b1 & \tau \leq t \leq \tau + t_p \\
0 & \tau + t_p \leq t \leq 2\tau + t_p 
\end{cases} \quad (3)
\]

signal: \[\sqrt{M_x(2\tau + t_p)^2 + M_y(2\tau + t_p)^2}\]

Inverse Problem: Known \[\sqrt{M_x(2\tau_i + t_p)^2 + M_y(2\tau_i + t_p)^2}\], \(i \in [1..n]\), find out \(R_2\).
Fitting Problem

\[
\min \sum_{i=1}^{n} \left\| M_{\text{meas}}(i) - I_0 \sqrt{M_{x,i}^2 + M_{y,i}^2} \right\|^2
\]

Subject to

\[
\begin{align*}
\frac{d M_x}{d t} &= -\omega M_y + \gamma B_1 \sin \phi M_z - R_2 M_x \\
\frac{d M_y}{d t} &= \omega M_x - \gamma B_1 \cos \phi M_z - R_2 M_y \\
\frac{d M_z}{d t} &= -\gamma B_1 \sin \phi M_x + \gamma B_1 \cos \phi M_y - R_1 (M_z - 1)
\end{align*}
\]

with Eqs. (2), (3).

Variables: \( I_0 \) and \( R_2 \).
Bloch Equations

\[
\frac{d}{dt} \begin{pmatrix}
M_x \\
M_y \\
M_z \\
M_e
\end{pmatrix} = 
\begin{pmatrix}
-R_2 & -\omega & \gamma B_1 \sin \phi & 0 \\
\omega & -R_2 & -\gamma B_1 \cos \phi & 0 \\
-\gamma B_1 \sin \phi & \gamma B_1 \cos \phi & -R_1 & R_1 \\
0 & 0 & -R_1 & R_1
\end{pmatrix}
\begin{pmatrix}
M_x \\
M_y \\
M_z \\
M_e
\end{pmatrix}
\equiv A
\begin{pmatrix}
M_x \\
M_y \\
M_z \\
M_e
\end{pmatrix}
\]

Then the solution of the Hahn echo can be easily expressed as

\[
M(2\tau + t_p) = e^{A(\gamma B_1 = 0)\tau} \cdot e^{A(\gamma B_1 = b1)t_p} \cdot e^{A(\gamma B_1 = 0)\tau} \cdot M(0)
\]
\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} M_x \\ M_y \\ M_z \\ M_e \end{pmatrix} &= \begin{pmatrix} -R_2 & -\omega & \gamma B_1 \sin \phi & 0 \\ \omega & -R_2 & -\gamma B_1 \cos \phi & 0 \\ -\gamma B_1 \sin \phi & \gamma B_1 \cos \phi & -R_1 & R_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} M_x \\ M_y \\ M_z \\ M_e \end{pmatrix} \\
&\equiv A \begin{pmatrix} M_x \\ M_y \\ M_z \\ M_e \end{pmatrix}
\end{align*}
\]

Then the solution of the Hahn echo can be easily expressed as

\[
M(2\tau + t_p) = e^{A(\gamma B_1=0)\tau} \cdot e^{A(\gamma B_1=b1)t_p} \cdot e^{A(\gamma B_1=0)\tau} \cdot M(0)
\]
Reformed Fitting Problem

\[ \min \sum_{i=1}^{n} \left| M_{\text{meas}}(i) - I_0 \sqrt{M^2_{x,i} + M^2_{y,i}} \right|^2 \]

Subject to

\[ M(2\tau_i + t_p) = e^{A(\gamma B_1=0)i\tau_i} . e^{A(\gamma B_1=b1)t_p} . e^{A(\gamma B_1=0)i\tau_i} . M(0) \quad (i \in [1..n]) \]

Hard Pulse: \[ e^{2\tau_i R_2} \]

Soft Pulse: \[ ? \]

11M solved by MatrixExponential of Maple.
Reformed Fitting Problem

\[
\min \sum_{i=1}^{n} \left| M_{\text{meas}}(i) - I_0 \sqrt{M_{x,i}^2 + M_{y,i}^2} \right|^2
\]

Subject to

\[
M(2\tau_i + t_p) = e^{A(\gamma B_1=0)\tau_i} . e^{A(\gamma B_1=b1)t_p} . e^{A(\gamma B_1=0)\tau_i} . M(0) \quad (i \in [1..n])
\]

Hard Pulse: \( e^{2\tau_i R_2} \)

Soft Pulse: 11M solved by MatrixExponential of Maple.
Reformed Fitting Problem

\[
\min \sum_{i=1}^{n} \left| \left| M_{\text{meas}}(i) - I_0 \sqrt{M_{x,i}^2 + M_{y,i}^2} \right| \right|^2
\]

Subject to

\[
M(2\tau_i + t_p) = e^{A(\gamma B_1=0)\tau_i}.e^{A(\gamma B_1=b1)t_p}.e^{A(\gamma B_1=0)\tau_i}.M(0) \quad (i \in [1..n])
\]

Hard Pulse: \quad Soft Pulse:

\[
e^{2\tau_i R_2}
\]

11M solved by MatrixExponential of Maple.
Computing $\exp(At)$ via Lagrange Interpolation

**Theorem**

If $A$ is an $n \times n$ matrix with $n$ distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, then we have

$$\exp(At) = \sum_{k=1}^{n} \exp(t \lambda_k) L_k(A),$$

where the $L_k(A)$ are Lagrange interpolation coefficients given by

$$L_k(A) = \prod_{j=1, j \neq k}^{n} \frac{A - \lambda_j I}{\lambda_k - \lambda_j}$$

for $k = 1, 2, \cdots, n$.

The matrix $A$ has four distinct eigenvalues: one is a real number ($\lambda_1$), two are complex conjugates ($\lambda_2$, $\lambda_3$), one is zero ($\lambda_4$).

$$\lambda = [ \lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 ]^T$$

$$e^{At} = e^{\lambda_1 t} \frac{A(A - \lambda_2 I)(A - \lambda_3 I)}{\lambda_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + e^{\lambda_2 t} \frac{A(A - \lambda_1 I)(A - \lambda_3 I)}{\lambda_2(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} +$$

$$e^{\lambda_3 t} \frac{A(A - \lambda_1 I)(A - \lambda_2 I)}{\lambda_3(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} + \frac{(A - \lambda_1 I)(A - \lambda_2 I)(A - \lambda_3 I)}{-\lambda_1 \lambda_2 \lambda_3}$$

$$\equiv e^{\lambda_1 t}L_1(A) + e^{\lambda_2 t}L_2(A) + e^{\lambda_3 t}L_3(A) + L_4(A)$$

- process experimental data
- design new experiments
Fitting Problem with Exact Solution of the Bloch Equations

\[
\min \sum_{i=1}^{n} \left| \left| M_{\text{meas}}(i) - I_0 \sqrt{M_{x,i}^2 + M_{y,i}^2} \right| \right|^2
\]

Subject to

\[
\begin{align*}
\lambda_j &= \cdots \quad (j = 1, 2, 3) \\
&\vdots \\
M_{x,i} &= \cdots \\
M_{y,i} &= \cdots
\end{align*}
\]

Fitting Problem is Solvable?
Approximate Solutions of the Bloch Equations

Approximate Eigenvalues

\[ \lambda \approx \begin{bmatrix} -R_2 \\ -R_2 + i \sqrt{b_1^2 + \omega^2} \\ -R_2 - i \sqrt{b_1^2 + \omega^2} \\ 0 \end{bmatrix} \]

Split-Operators

\[ e^{(X+Y)t} \approx e^{(t/2)X}e^{(t)Y}e^{(t/2)X} \]

\[ M_x(t) = c_1 e^{\lambda_1 t} + (c_2 \sin(Im(\lambda_2) t) + c_3 \cos(Im(\lambda_2) t))e^{Re(\lambda_2) t} + c_4 \]
Approximate Solutions of the Bloch Equations

Approximate Eigenvalues

\[
\lambda \approx \begin{bmatrix}
-R_2 \\
-R_2 + i \sqrt{b_1^2 + \omega^2} \\
-R_2 - i \sqrt{b_1^2 + \omega^2} \\
0
\end{bmatrix}
\]

Split-Operators

\[
e^{(X+Y)t} \approx e^{(t/2)X}e(t)Ye^{(t/2)X}
\]

\[
M_x(t) = c_1 e^{\lambda_1 t} + (c_2 \sin(Im(\lambda_2) t) + c_3 \cos(Im(\lambda_2) t))e^{Re(\lambda_2) t} + c_4
\]
Results

(a) 

(b) 

(c) 

(d)
Results

(a) 

(b) 

(c) 

(d)
Field Inhomogeneity

\[ \min \sum_{i=1}^{n} \left| M_{\text{meas}}(i) - I_0 \sqrt{\left( \sum_k \sum_j M_{x,i,j,k} \right)^2 + \left( \sum_k \sum_j M_{y,i,j,k} \right)^2} \right|^2 \]

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Conclusion

- The relaxation time is fitted by the exact solution of the Bloch equations.
- Ordinary differential constraints are pre-solved by the Lagrange interpolation method which is able to give the symbolic gradient and Hessian.
- The equivalent constrained optimization problem has been developed to dramatically reduce the size of the objective function and eliminate the computation of redundant terms.
- More complicated fitting problems can be constructed with more precise description of experiments to improve the accuracy of the measurements.
- The method can be applied to any arbitrary single spin-$\frac{1}{2}$ experiments and it is possible to extend to coupled-spin systems.

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Thanks!