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Complexity of Self-regular based Infeasible IPM

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Outline of Presentation

- Linear Optimization Problem
- Classical primal-Dual IMPs algorithm
- Newton Direction
- Self Regular Function
- Infeasible Self-Regular IPM
- Complexity Of the Algorithm
- Ongoing and Future Work

The Linear Optimization Problem

Consider the following LO problem:

$$\begin{aligned} \text{min} \quad & c^T x \\ \text{s.t} \quad & Ax = b, \\ & x \geq 0, \end{aligned}$$

The dual problem is:

$$(\text{LD}) \qquad \begin{aligned} \text{max} \quad & b^T y \\ \text{s.t} \quad & A^T y + \; s = c, \\ & s \geq 0, \end{aligned}$$

where $y \in \mathcal{R}^m$, $x, s \in \mathcal{R}^n$.

The LO Optimality Conditions

Optimality Conditions

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ Xs &= 0, \\ x \geq 0 \quad s &\geq 0, \end{aligned}$$

where $X = \text{diag}(x)$.

The Lo Central Path

The **primal-dual central path** is defined as the set of solutions $(x(\mu))$ and $(y(\mu), s(\mu))$ for $\mu > 0$ of the system

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ Xs &= \mu e, \\ x > 0 & \quad s > 0, \end{aligned}$$

where $e = [1, 1, \ldots, 1]^T$.

IPM condition:

$$Ax^0 = b, x^0 > 0 \quad A^T y + s = c, s > 0$$

Classical Primal-Dual Newton Method for LO

inputs

A proximity parameter τ;

an accuracy parameter $\epsilon > 0$;

(x^0, s^0) and $\mu^0 = 1$ such that $\Phi(x^0, s^0, \mu^0) \leq \tau$.

$x := x^0$; $s := s^0$; $\mu := \mu^0$;

while $n\mu \geq \epsilon$

$\mu := (1 - \theta)\mu$;

while $\Phi(x, s, \mu) \geq \tau$

Solve Newton system for $\Delta x, \Delta y, \Delta s$;

determine a step size α;

$x := x + \alpha \Delta x$;

$s := s + \alpha \Delta s$;

$y := y + \alpha \Delta y$.

end

end

Newton Direction

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta s \end{pmatrix} = \begin{pmatrix} -r_b \\ -r_c \\ \mu e - Xs \end{pmatrix},$$

$$r_b = Ax - b, \quad r_c = A^T y + s - c$$

$$v = \sqrt{\frac{xs}{\mu}}, \quad d_x = \frac{v\Delta x}{x}, \quad d_s = \frac{v\Delta s}{s}$$

$$\begin{pmatrix} \bar{A} & 0 & 0 \\ 0 & \bar{A}^T & I \\ I & 0 & I \end{pmatrix} \begin{pmatrix} d_x \\ \Delta y \\ d_s \end{pmatrix} = \begin{pmatrix} -r_b \\ -r_c \\ v^{-1} - v \end{pmatrix}, \quad \bar{A} = \frac{1}{\mu} A V^{-1} X$$

$$\Psi(V) = \frac{e^T v^2 - n}{2} - \sum_{i=1}^{n} \log(v_i)$$

Newton Direction

$$(AD^2A^T)\Delta y = AD^2 r_h - r_b,$$

$$\Delta x = D^2(A^T \Delta y - r_h),$$

$$\Delta s = x^{-1}(-\mu v \nabla \Psi(v) - s\Delta x),$$

$$x(\alpha) := x + \alpha \Delta x,$$

$$y(\alpha) := y + \alpha \Delta y,$$

$$s(\alpha) := s + \alpha \Delta s.$$

Complexity of small-update and large-update

small-update:

$$O(\sqrt{n}\log(\frac{n}{\epsilon})$$

large-update:

$$O(n\log(\frac{n}{\epsilon}))$$

Self-Regular Function

A function $\psi(t) \in \mathcal{C}^2 : (0, \infty) \to R$ is self-regular if it satisfies the following conditions:

SR.1 $\psi(t)$ is strictly convex with respect to $t > 0$ and vanishes at its global minimal point $t = 1$, i.e., $\psi(1) = \psi'(1) = 0$. Further, there exist positive constants $\nu_2 \geq \nu_1 > 0$ and $p \geq 1, \; q \geq 1$ such that

$$\nu_1(t^{p-1} + t^{-1-q}) \leq \psi''(t) \leq \nu_2(t^{p-1} + t^{-1-q}), \quad \forall t \in (0, \infty);$$

SR.2 For any $t_1, t_2 > 0$,

$$\psi(t_1^r t_2^{1-r}) \leq r\psi(t_1) + (1-r)\psi(t_2), \quad \forall r \in [0, 1].$$

$$\Upsilon_{p,q}(t) = \frac{t^{p+1} - 1}{p(p+1)} + \frac{t^{1-q} - 1}{q(q-1)} + \frac{p-q}{pq}(t-1), \quad p, q \geq 1,$$

Self-Regular Function

with $\nu_1 = \nu_2 = 1$. The second family is defined as

$$\Gamma_{p,q}(t) = \frac{t^{p+1} - 1}{p+1} + \frac{t^{1-q} - 1}{q-1}, \quad p \geq 1, \; q > 1,$$

with $\nu_1 = 1$ and $\nu_2 = q$.

$$\Psi(v) = \sum_{i=1}^{n} \psi(v_i), v := \sqrt{\frac{xs}{\mu}}$$

Best complexity for feasible IPMs :$O(\sqrt{n} \log(n) \log(\frac{n}{\epsilon}))$

$$\begin{pmatrix} \bar{A} & 0 & 0 \\ 0 & \bar{A}^T & I \\ I & 0 & I \end{pmatrix} \begin{pmatrix} d_x \\ \Delta y \\ d_s \end{pmatrix} = \begin{pmatrix} -r_b \\ -r_c \\ -\nabla \Psi(v) \end{pmatrix},$$

Properties of Proximity Function

$$\Phi(x,s,\mu) = \Psi(v) = \frac{e^T v^2 - n}{2} + \frac{e^T v^{-2} - n}{2}$$

$$\mu_g = \frac{x^T s}{n}, \quad \mu_h = \frac{n}{x^{-T} s^{-1}}, \quad \mu^* = \sqrt{\frac{x^T s}{x^{-T} s^{-1}}} = \sqrt{\mu_g \mu_h}$$

$$\mu_t = \frac{2\mu_g}{\tau + 1 + \sqrt{(\tau+1)^2 - \frac{4\mu_g}{\mu_h}}}, \quad \Phi(x,s,\mu_t) = \frac{(\tau-1)n}{2}$$

$$\mu_t \le \mu_h \le \mu* \le \mu_g$$

$$\Phi(x,s,\mu_g) = \Phi(x,s,\mu_h)$$

Properties of Proximity Function

$$\Phi(x, s, \mu_g) = \Phi(x, s, \mu^*) + \frac{\Phi(x, s, \mu^*)^2}{2n}$$

$$\frac{\mu_g}{\mu_h} = \frac{x^T s x^{-T} s^{-1}}{n^2} \leq \tau \iff \Phi(x, s, \frac{\mu_g}{\tau}) \leq \frac{(\tau - 1)n}{2}$$

Infeasible Neighborhood

$$\mathcal{N}(\tau,\beta)=\left\{(x,y,s)\mid\Psi(v)\le\frac{(\tau-1)n}{2},\ \|r_b\|\le\left\|r_b^0\right\|\frac{\mu_g}{\mu^0}\beta,\ \|r_c\|\le\left\|r_c^0\right\|\frac{\mu_g}{\mu^0}\beta\right\}$$

$$(x^0,y^0,s^0)=(\zeta e,0,\zeta e)$$

$$\sigma^2=\|d_x+d_s\|^2,\sigma_1^2=d_x^2+d_s^2=\sigma^2-2d_x^Td_s$$

$$C_3n^2\sigma^{\frac{2}{3}}\le\sigma_1^2\le C_3n^2\sigma^{\frac{2}{3}},$$

SR-Infeasible IPMs

Input:

Proximity parameters $\tau \geq 10$ and $\beta \geq 1$, Neighborhood $\mathcal{N}(\tau, \beta)$;

Initial point $(x^0, y^0, s^0) \in \mathcal{N}(\tau, \beta)$; such that $(x^0, s^0) > 0$

An accuracy parameter $\epsilon > 0$, and damping factor α.

begin

while max $\left\{(x^k)^T s^k, \left\|r_b^k\right\|, \left\|r_c^k\right\|\right\} \geq \epsilon$ **do**

Begin

If $\frac{\mu_g^k}{\mu_h^k} \geq \frac{\tau}{2}$ then $\mu := \mu_h^k$; otherwise

$\mu := \mu_t^k$.

Solve system for $\Delta x^k, \Delta y^k, \Delta s^k$.

Begin

Determine a step size α_k such that

$\Phi(x(\alpha_k), s(\alpha_k), \mu_t^k) \leq \Phi(x^k, s^k, \mu_t^k) - \frac{\alpha^*}{2}\Phi(x, s, \mu_t)$

and $(x(\alpha_k), y(\alpha_k), s(\alpha_k)) \in \mathcal{N}(\tau, \beta)$;

$x^{k+1} := x(\alpha_k)$; $y^{k+1} := y(\alpha_k)$; $s^{k+1} := s(\alpha_k)$;

$k = k + 1$.

end

end

Worst Case Complexity

$$\Phi(x(\alpha^*), s(\alpha^*), \mu_g(\alpha^*)) \leq \Phi(x^k, s^k, \mu_t^k)$$

$$\Phi(x(\alpha^*), s(\alpha^*), \mu_t^k) \leq \Phi(x^k, s^k, \mu_t^k) - \frac{\alpha^*}{2}\Phi(x^k, s^k, \mu_t^k)$$

$$\mu_t^{k+1} \leq (1 - \frac{\alpha^*}{4})\mu_t^k$$

$$O(n^2 \log(\frac{n}{\epsilon}))$$

Ongoing and future work

- Dynamic Large-update SR IPMs Developed for a big class of SR functions.
- Implementation of SR Proximity Based IPMs has been Done by X. Zhu and guoqing Zhang.
- Developing new algorithm for pure primal or pure dual problem based on SR proximity.