McMaster University

Advanced Optimization Laboratory

Title:

Sensitivity Analysis in Convex Quadratic Optimization:
Simultaneous Perturbation of the Objective
and Right-Hand-Side Vectors

Authors:

Alireza Ghaffari Hadigheh, Oleksandr Romanko, Tamás Terlaky

AdvOl-Report No. 2003/6

December 2003, Hamilton, Ontario, Canada

SENSITIVITY ANALYSIS IN CONVEX QUADRATIC OPTIMIZATION: SIMULTANEOUS PERTURBATION OF THE OBJECTIVE AND RIGHT-HAND-SIDE VECTORS

ALIREZA GHAFFARI HADIGHEH

Department of Applied Mathematics, Mathematical Sciences Faculty, University of Tabriz, Tabriz, Iran, a-r-ghaffari@tabrizu.ac.ir, hadigheha@optlab.mcmaster.ca

OLEKSANDR ROMANKO and TAMÁS TERLAKY

Advanced Optimization Laboratory, Department of Computing and Software, McMaster University, Hamilton, Ontario, Canada, romanko@mcmaster.ca • terlaky@mcmaster.ca

In this paper we study the behavior of Convex Quadratic Optimization problems when variation occurs simultaneously in the right-hand side vector of the constraints and in the coefficient vector of the linear term in the objective function. It is proven that the optimal value function is piecewise-quadratic. The concepts of transition point and invariancy interval are generalized to the case of simultaneous perturbation. Criteria of convexity, concavity or linearity of the optimal value function on invariancy intervals are derived. Furthermore, differentiability of the optimal value function is studied, and linear optimization problems are given to calculate the left and right derivatives. An algorithm, that is capable to compute the transition points and optimal partitions on all invariancy intervals, is outlined. We specialize the method to Linear Optimization problems.

1 Introduction

In this paper we are concerned with the sensitivity analysis of perturbed Convex Quadratic Optimization (CQO) problems where the coefficient vector of the linear term of the objective function and the right-hand side (RHS) vector of the constraints are varied simultaneously. This type of sensitivity analysis is often referred to as parametric programming. Research on the topic was triggered when a variant of parametric CQO problems was considered by Markowitz (1956). He developed the critical line method to determine the optimal value function of his parametric problem and applied it to mean-variance portfolio analysis. The basic result for parametric quadratic programming obtained by Markowitz is that the optimal value function (efficient frontier in financial terminology) is piecewise quadratic and can be obtained by computing successive corner portfolios while in between these corner portfolios the optimal solutions vary linearly. Non-degeneracy was assumed and a variant of the simplex method was used for computations.

Difficulties that may occur in parametric analysis when the problem is degenerate are studied extensively in the Linear Optimization (LO) literature. In case of degeneracy the optimal basis need not to be unique and multiple optimal solutions may exist. While simplex methods were used to perform the computations in earlier studies (see e.g., Murty (1983) for a comprehensive survey), recently research on parametric analysis was revisited from the point of view of interior-point methods (IPMs). For degenerate LO problems, the availability of strictly complementary solutions produced by IPMs allows to overcome many difficulties associated with the use of bases. Alder and Monteiro (1992) pioneered the use of IPMs in parametric analysis for LO (see also Jansen et al. (1997)). Berkelaar, Roos and Terlaky (1997) emphasized shortcomings of using optimal bases in parametric LO showing by an example that different optimal bases computed by different LO packages give different optimality intervals.

Naturally, results obtained for parametric LO were extended to the CQO. Berkelaar et al. (1997) showed that the optimal partition approach can be generalized to the quadratic case by introducing tripartition of variables instead of bipartition. They performed the sensitivity analysis for the cases when perturbation occurs either in the coefficient vector of the linear term of the objective value function or in the RHS of the constraints. In this paper we show that the results obtained in Berkelaar, Roos and Terlaky (1997) and Berkelaar et al. (1997) can be generalized further to accommodate simultaneous perturbation of the data even in the presence of degeneracy. The theoretical results allow us to present a universal computational algorithm for the parametric analysis of LO/CQO problems.

The paper is organized as follows. In Section 2, CQO problem is introduced and some elementary concepts are reviewed. Some simple properties of the optimal value function are summarized in Section 3. Section 4 is devoted to deriving more properties of the optimal value function. It is shown that the optimal value function is continuous and piecewise quadratic, and an explicit formula is presented to identify it on the subintervals. Criteria for convexity, concavity or linearity of the optimal value function on these subintervals are derived. We investigate the first and second order derivatives of the optimal value function as well. Auxiliary LO problems can be used to compute the left and right derivatives. It is shown that the optimal partition on the neighboring intervals can be identified by solving an auxiliary self-dual CQO problem. The results are summarized in a computational algorithm. Specialization of the method to LO problems is described in Section 5. For illustration, the results are tested on a simple problem in Section 6. We conclude the paper with some remarks and we sketch further research directions.

2 Preliminaries

A primal CQO problem is defined as:

$$(QP) \qquad \min \{ c^T x + \frac{1}{2} x^T Q x \; : \; Ax = b, \; x \geq 0 \} ,$$

where $Q \in \mathbb{R}^{n \times n}$ is a symmetric positive semidefinite matrix, $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ are fixed data and $x \in \mathbb{R}^n$ is an unknown vector.

The Wolfe-Dual of (QP) is given by

$$(QD) \qquad \max \{ b^T y - \frac{1}{2} u^T Q u \; : \; A^T y + s - Qu = c, \; s \geq 0 \} ,$$

where $s, u \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are unknown vectors. The feasible regions of (QP) and (QD) are denoted by

$$\mathcal{QP} = \{x : Ax = b, x \geq 0\},$$

$$\mathcal{QD} = \left\{(u, y, s) : A^T y + s - Qu = c,\ \ s, u \geq 0\right\},$$

and their associated optimal solutions sets are $\mathcal{QP}^*$ and $\mathcal{QD}^*$, respectively. It is well known that for any optimal solution of (QP) and (QD) we have $Qx = Qu$ and $s^T x = 0$, see e.g., Dorn (1960). For the duality gap $s^T x$ being zero is equivalent to $s_i x_i = 0$ for all $i \in \{1, 2, \ldots, n\}$. This property of the nonnegative variables x and s is called the *complementarity property.* It is obvious that there are optimal solutions with $x = u$. Since we are only interested in the solutions where $x = u$, therefore, u will be replaced by x in the dual problem. It is easy to show, see e.g., Berkelaar et al. (1997) and Dorn (1960), that for two optimal solutions (x^*, y^*, s^*) and $(\tilde{x}, \tilde{y}, \tilde{s})$ of (QP) and (QD) it holds that $Qx^* = Q\tilde{x}$, $c^T x^* = c^T \tilde{x}$ and $b^T y^* = b^T \tilde{y}$ and consequently,

$$\tilde{x}^T s^* = \tilde{s}^T x^* = 0. \tag{1}$$

The *optimal partition* of the index set $\{1, 2, \ldots, n\}$ is defined as

$$\mathcal{B} = \{\ i : x_i > 0 \text{ for an optimal solution } x \in \mathcal{QP}^*\},$$

$$\mathcal{N} = \{\ i : s_i > 0 \text{ for an optimal solution } (x, y, s) \in \mathcal{QD}^*\},$$

$$\mathcal{T} = \{1, 2, \ldots, n\} \setminus (\mathcal{B} \cup \mathcal{N}),$$

and denoted by $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$. Berkelaar et al. (1997) and Berkelaar, Roos and Terlaky (1997) showed that this partition is unique. The *support set* of a vector v is defined as $\sigma(v) = \{i : v_i > 0\}$ and is used extensively in this paper. An optimal solution (x, y, s) is called *maximally complementary* if it possesses the following properties:

$$x_i > 0 \text{ if and only if } i \in \mathcal{B},$$
$$s_i > 0 \text{ if and only if } i \in \mathcal{N}.$$

For any maximally complementary solution (x, y, s) the relations $\sigma(x) = \mathcal{B}$ and $\sigma(s) = \mathcal{N}$ hold. The existence of a maximally complementary solution is a direct consequence of the convexity of the optimal sets $\mathcal{QP}^*$ and $\mathcal{QD}^*$. It is known that IPMs find a maximally complementary solution in the limit, see e.g., McLinden (1980) and Güler and Ye (1993).

The general perturbed CQO problem is

$$(QP_{\lambda_b,\lambda_c}) \qquad \min\left\{ (c+\lambda_c\triangle c)^T x + \frac{1}{2}x^T Qx \,:\, Ax = b+\lambda_b\triangle b,\ x\geq 0 \right\},$$

where $\triangle b \in \mathbb{R}^m$ and $\triangle c \in \mathbb{R}^n$ are nonzero perturbation vectors, and λ_b and λ_c are real parameters. The optimal value function $\phi(\lambda_b, \lambda_c)$ denotes the optimal value of $(QP_{\lambda_b,\lambda_c})$ as the function of the parameters λ_b and λ_c. As we already mentioned, Berkelaar, Roos and Terlaky (1997) and Berkelaar et al. (1997) were the first to analyze parametric CQO by using the optimal partition approach when variation occurs either in the RHS or the linear term of the objective value function data, i.e., either $\triangle c$ or $\triangle b$ is zero. In these cases the domain of the optimal value function $\phi(\lambda_b, 0)$ (or $\phi(0, \lambda_c)$) is a closed interval of the real line and the function is piecewise convex (concave) quadratic on its domain. The authors presented an explicit formula for the optimal value function on these subintervals and introduced the concept of transition points that separate them. They proved that the optimal partition is invariant on the subintervals which are characterized by consecutive transition points. The authors also studied the behavior of the first and second order derivatives of the optimal value function and proved that the transition points coincide with the points where the first or second order derivatives do not exist. It was proven that by solving auxiliary self-dual CQO problems, one can identify the optimal partitions on the neighboring subintervals.

3 The Optimal Value Function in Simultaneous Perturbation Sensitivity Analysis

In this section, we introduce explicitly the perturbed CQO problem when perturbation occurs in the RHS data and the linear term of the objective value function of (QP), simultaneously. In the problem $(QP_{\lambda_b,\lambda_c})$ that was mentioned in the pervious section, λ_b and λ_c are independent parameters. In this paper we only concerned with the case when they coincide, i.e., $\lambda_b = \lambda_c = \lambda$. Consequently, the perturbation takes the form λh, where $h = (\triangle b^T, \triangle c^T)^T \in \mathbb{R}^{m+n}$ is a nonzero perturbing direction and $\lambda \in \mathbb{R}$ is a parameter. Thus, we define the following primal and dual perturbed problems corresponding to (QP) and (QD), respectively:

$$(QP_\lambda) \qquad \min\left\{ (c+\lambda\triangle c)^T x + \frac{1}{2}x^T Qx \,:\, Ax = b+\lambda\triangle b,\ x\geq 0 \right\},$$

$$(QD_\lambda) \qquad \max\{\, (b+\lambda\triangle b)^T y - \frac{1}{2}x^TQx \;:\; A^Ty + s - Qx = c+\lambda\triangle c,\ s\geq 0\,\}.$$

Let $\mathcal{QP}_\lambda$ and $\mathcal{QD}_\lambda$ denote the feasible sets of the problems (QP_λ) and (QD_λ), respectively. Their optimal solution sets are analogously denoted by $\mathcal{QP}^*_\lambda$ and $\mathcal{QD}^*_\lambda$. The optimal value function of (QP_λ) and (QD_λ) is

$$\phi(\lambda) = (c+\lambda\triangle c)^T x^*(\lambda) + \frac{1}{2}x^*(\lambda)^T Q x^*(\lambda) = (b+\lambda\triangle b)^T y^*(\lambda) - \frac{1}{2}x^*(\lambda)^T Q x^*(\lambda),$$

where $x^*(\lambda) \in \mathcal{QP}^*_\lambda$ and $(x^*(\lambda), y^*(\lambda), s^*(\lambda)) \in \mathcal{QD}^*_\lambda$. Further, we define

$$\begin{aligned} \phi(\lambda) &= +\infty \qquad \text{if } \mathcal{QP}_\lambda = \emptyset,\\ \phi(\lambda) &= -\infty \qquad \text{if } \mathcal{QP}_\lambda \neq \emptyset \text{ and } (QP_\lambda) \text{ is unbounded.} \end{aligned}$$

Let us denote the domain of $\phi(\lambda)$ by

$$\Lambda = \{\lambda : \mathcal{QP}_\lambda \neq \emptyset \text{ and } \mathcal{QD}_\lambda \neq \emptyset\}.$$

Since it is assumed that (QP) and (QD) have optimal solutions, it follows that $\Lambda \neq \emptyset$. We can easily prove the following property of Λ.

Lemma 3.1 $\Lambda \subseteq \mathbb{R}$ *is a closed interval.*

Proof: Let $\lambda \notin \Lambda$. There are two cases: the primal problem (QP_λ) is feasible but unbounded or it is infeasible. We only prove the second case, the first one can be proved analogously. If the primal problem (QP_λ) is infeasible then by the Farkas Lemma (see e.g., Murty 1983 or Roos, Terlaky and Vial 1997) there is a vector y such that $A^Ty \leq 0$ and $(b+\lambda\triangle b)^T y > 0$. Fixing y and considering λ as a variable, the set $S(y) = \left\{\lambda : (b+\lambda\triangle b)^T y > 0\right\}$ is an open half-line in λ, thus the given vector y is a certificate of infeasibility of (QP_λ) for an open interval. Thus, the union $\bigcup_y S(y)$, where y is a Farkas certificate for the infeasibility of (QP_λ) for some $\lambda \in \mathbb{R}$, is open. Consequently, the domain of the optimal value function is closed. We also need to show that this closed set is connected. Let $\lambda_1, \lambda_2 \in \Lambda$ be two arbitrary numbers. Let $(x(\lambda_1), y(\lambda_1), s(\lambda_1)) \in \mathcal{QP}^*_{\lambda_1} \times \mathcal{QD}^*_{\lambda_1}$. Similarly, $(x(\lambda_2), y(\lambda_2), s(\lambda_2)) \in \mathcal{QP}^*_{\lambda_2} \times \mathcal{QD}^*_{\lambda_2}$. For any $\lambda \in (\lambda_1, \lambda_2)$ and $\theta = \frac{\lambda_2-\lambda}{\lambda_2-\lambda_1}$ we have

$$\lambda = \theta\lambda_1 + (1-\theta)\lambda_2.$$

Let us define

$$x(\lambda) = \theta x(\lambda_1) + (1-\theta)x(\lambda_2),$$

$$y(\lambda) = \theta y(\lambda_1) + (1-\theta)y(\lambda_2),$$

$$s(\lambda) = \theta s(\lambda_1) + (1-\theta)s(\lambda_2).$$

It is easy to check that $(x(\lambda), y(\lambda), s(\lambda)) \in \mathcal{QP}^*_\lambda \times \mathcal{QD}^*_\lambda$. This implies that the set Λ is connected and thus Λ is a closed interval. The proof is complete. □

4 Properties of the Optimal Value Function

In this section we investigate the properties of the optimal value function. These are generalizations of the corresponding properties that have been proven in Berkelaar et al. (1997) for the case when $\triangle c = 0$ or $\triangle b = 0$.

4.1 Basic Properties

For $\lambda^* \in \Lambda$, let $\pi = \pi(\lambda^*)$ denote the optimal partition and let (x^*, y^*, s^*) be a maximally complementary solution at λ^*. We use the following notation that generalizes the notation introduced in Berkelaar et al. (1997):

$$\mathcal{O}(\pi) = \{\lambda \in \Lambda : \pi(\lambda) = \pi\};$$

$$\mathcal{S}_\lambda(\pi) = \{(x,y,s) : x \in \mathcal{QP}_\lambda,\ (x,y,s) \in \mathcal{QD}_\lambda,\ x_\mathcal{B} > 0,\ x_{\mathcal{N}\cup\mathcal{T}} = 0,\ s_\mathcal{N} > 0,\ s_{\mathcal{B}\cup\mathcal{T}} = 0\};$$

$$\overline{\mathcal{S}}_\lambda(\pi) = \{(x,y,s) : x \in \mathcal{QP}_\lambda,\ (x,y,s) \in \mathcal{QD}_\lambda,\ x_\mathcal{B} \geq 0,\ x_{\mathcal{N}\cup\mathcal{T}} = 0,\ s_\mathcal{N} \geq 0,\ s_{\mathcal{B}\cup\mathcal{T}} = 0\};$$

$$\Lambda(\pi) = \{\lambda \in \Lambda : \mathcal{S}_\lambda(\pi) \neq \emptyset\};$$

$$\overline{\Lambda}(\pi) = \left\{\lambda \in \Lambda : \overline{\mathcal{S}}_\lambda(\pi) \neq \emptyset\right\};$$

$$D_\pi = \left\{(\triangle x, \triangle y, \triangle s) : A\triangle x = \triangle b, A^T\triangle y + \triangle s - Q\triangle x = \triangle c, \triangle x_{\mathcal{N}\cup\mathcal{T}} = 0, \triangle s_{\mathcal{B}\cup\mathcal{T}} = 0\right\}.$$

The following theorem resembles Theorem 3.1 from Berkelaar et al. (1997) and presents the basic relations between the open interval where the optimal partition is invariant and its closure.

Theorem 4.1 *Let $\pi = \pi(\lambda^*) = (\mathcal{B}, \mathcal{N}, \mathcal{T})$ for some λ^* denote the optimal partition and (x^*, y^*, s^*) denote an associated maximally complementary solution at λ^*. Then,* *(i)* $\Lambda(\pi)=\{\lambda^*\}$ *if and only if* $D_\pi=\emptyset$*;*

(ii) $\Lambda(\pi)$ *is an open interval if and only if* $D_\pi\neq\emptyset$*;*

(iii) $\mathcal{O}(\pi)=\Lambda(\pi)$ *and* $cl\,\mathcal{O}(\pi)=cl\,\Lambda(\pi)=\overline{\Lambda}(\pi)$*;*

(iv) $\overline{\mathcal{S}}_\lambda(\pi)=\{(x,y,s):x\in\mathcal{QP}^*_\lambda,(x,y,s)\in\mathcal{QD}^*_\lambda\}$ *for all* $\lambda\in\overline{\Lambda}(\pi)$*.*

Proof: First let us recall the characteristics of a maximally complementary solution. Any maximally complementary solution (x^*,y^*,s^*) associated with a given λ^* satisfies $Ax^*=b+\lambda^*\triangle b$, $A^Ty^*+s^*-Qx^*=c+\lambda^*\triangle c$, $x^*_{\mathcal{B}}>0$, $x^*_{\mathcal{N}\cup\mathcal{T}}=0$, $s^*_{\mathcal{N}}>0$ and $s^*_{\mathcal{B}\cup\mathcal{T}}=0$. Let $(\triangle x,\triangle y,\triangle s)\in D_\pi$, and define

$$\overline{x}=x^*+(\overline{\lambda}+\lambda^*)\triangle x, \tag{2}$$

$$\overline{y}=y^*+(\overline{\lambda}+\lambda^*)\triangle y, \tag{3}$$

$$\overline{s}=s^*+(\overline{\lambda}+\lambda^*)\triangle s. \tag{4}$$

If $\overline{\lambda}$ is in an ϵ-neighborhood of λ^* for enough small ϵ, then

$$\begin{aligned} A\overline{x}&=b+\overline{\lambda}\triangle b,\\ A^T\overline{y}+\overline{s}-Q\overline{x}&=c+\overline{\lambda}\triangle c,\\ \overline{x}_{\mathcal{N}\cup\mathcal{T}}&=0,\\ \overline{s}_{\mathcal{B}\cup\mathcal{T}}&=0,\\ \overline{x}_{\mathcal{B}}>0,\quad \overline{s}_{\mathcal{N}}&>0. \end{aligned} \tag{5}$$

(i) $[\Rightarrow]$: Let $\Lambda(\pi)=\{\lambda^*\}$, and assume to the contrary that D_π is not empty. Then, there exists $(\triangle x,\triangle y,\triangle s)$ such that $A\triangle x=\triangle b$ and $A^T\triangle y+\triangle s-Q\triangle x=\triangle c$ with $\triangle x_{\mathcal{N}\cup\mathcal{T}}=0$ and $\triangle s_{\mathcal{B}\cup\mathcal{T}}=0$. Let (x^*,y^*,s^*) be a maximally complementary solution associated with λ^*, i.e., $Ax^*=b+\lambda^*\triangle b$, $A^Ty^*+s^*-Qx^*=c+\lambda^*\triangle c$, $x^*_{\mathcal{N}\cup\mathcal{T}}=0$, $s^*_{\mathcal{B}\cup\mathcal{T}}=0$, $x^*_{\mathcal{B}}>0$ and $s^*_{\mathcal{N}}>0$. Let $(\overline{x},\overline{y},\overline{s})$ be defined by (2)–(4). From (5), one can conclude $\overline{\lambda}\in\Lambda(\pi)$ that contradicts to the assumption $\Lambda(\pi)=\{\lambda^*\}$.

(i) $[\Leftarrow]$: Let $D_\pi=\emptyset$, and suppose to the contrary that $\overline{\lambda},\lambda^*\in\Lambda(\pi)$, with $\overline{\lambda}\neq\lambda^*$ and $(\overline{x},\overline{y},\overline{s})$ is a maximally complementary solution at $\overline{\lambda}$. Thus, from (2)–(4) we can compute $(\triangle x,\triangle y,\triangle s)$ and conclude that $(\triangle x,\triangle y,\triangle s)\in D_\pi$. This contradicts to the fact that $D_\pi=\emptyset$ and thus $\Lambda(\pi)=\{\lambda^*\}$.

(ii) $[\Rightarrow]$: Let $\lambda^*\in\Lambda(\pi)$. Then, there is a maximally complementary solution (x^*,y^*,s^*) at

λ^*. Moreover, since $\Lambda(\pi)$ is an open interval, there exists a $\overline{\lambda}$ in an ϵ-neighborhood of λ^* with $\overline{\lambda} \neq \lambda^*$ and $\overline{\lambda} \in \Lambda(\pi)$. Let $(\overline{x}, \overline{y}, \overline{s})$ denote a maximally complementary solution at $\overline{\lambda}$. From (2)–(4), we can compute $(\triangle x, \triangle y, \triangle s)$ and conclude that $(\triangle x, \triangle y, \triangle s) \in D_\pi \neq \emptyset$.

(ii) $[\Leftarrow]$: Suppose that D_π is non-empty. Then, there exists $(\triangle x, \triangle y, \triangle s)$ such that $A\triangle x = \triangle b$, $A^T\triangle y + \triangle s - Q\triangle x = \triangle c$, $\triangle x_{\mathcal{N}\cup\mathcal{T}} = 0$ and $\triangle s_{\mathcal{B}\cup\mathcal{T}} = 0$. On the other hand, a maximally complementary solution (x^*, y^*, s^*) at λ^* exists such that $Ax^* = b + \lambda^*\triangle b$, $A^T y^* + s^* - Qx^* = c + \lambda^*\triangle c$, $x^*_{\mathcal{N}\cup\mathcal{T}} = 0, s^*_{\mathcal{B}\cup\mathcal{T}} = 0$, $x^*_{\mathcal{B}} > 0$ and $s^*_{\mathcal{N}} > 0$. Consider $(\overline{x}, \overline{y}, \overline{s})$ as defined in (2)–(4). For any $\overline{\lambda} \in \mathbb{R}$, $(\overline{x}, \overline{y}, \overline{s})$ satisfies

$$A\overline{x} = b + \overline{\lambda}\triangle b, \qquad A^T\overline{y} + \overline{s} - Q\overline{x} = c + \overline{\lambda}\triangle c,$$

and

$$\overline{x}^T\overline{s} = (\overline{\lambda} - \lambda^*)(\triangle x^T s^* + \triangle s^T x^*).$$

From the definitions of π and D_π, one can conclude that $\overline{x}^T\overline{s} = 0$. Thus $(\overline{x}, \overline{y}, \overline{s})$ is a pair of primal-dual optimal solutions of $(QP_{\overline{\lambda}})$ and $(QD_{\overline{\lambda}})$ as long as $\overline{x} \geq 0$ and $\overline{s} \geq 0$, that gives a closed interval around λ^*. Furthermore, for an open interval Λ, $\overline{x}_{\mathcal{B}} > 0$ and $\overline{s}_{\mathcal{N}} > 0$. Let $\lambda' < \lambda^* < \overline{\lambda}$, where $\lambda', \overline{\lambda} \in \Lambda$. If (x', y', s') and $(\overline{x}, \overline{y}, \overline{s})$ are defined by (2)–(4), then $x'_{\mathcal{B}}, \overline{x}_{\mathcal{B}} > 0$, $x'_{\mathcal{B}\cup\mathcal{T}} = \overline{x}_{\mathcal{B}\cup\mathcal{T}} = 0$, $s'_{\mathcal{N}}, \overline{s}_{\mathcal{N}} > 0$, $s'_{\mathcal{N}\cup\mathcal{T}} = \overline{s}_{\mathcal{N}\cup\mathcal{T}} = 0$. To prove that $\overline{\lambda} \in \Lambda(\pi)$, we need to show that $(\overline{x}, \overline{y}, \overline{s})$ is not only optimal for $(QP_{\overline{\lambda}})$ and $(QD_{\overline{\lambda}})$, but also maximally complementary.

Let us assume that the optimal partition $\overline{\pi} = (\overline{\mathcal{B}}, \overline{\mathcal{N}}, \overline{\mathcal{T}})$ at $\overline{\lambda}$ is not identical to π, i.e., there is a solution $(x(\overline{\lambda}), y(\overline{\lambda}), s(\overline{\lambda}))$ such that

$$x_{\mathcal{B}}(\overline{\lambda}) > 0, \; s_{\mathcal{N}}(\overline{\lambda}) > 0, \text{ and } x_{\mathcal{T}}(\overline{\lambda}) + s_{\mathcal{T}}(\overline{\lambda}) \neq 0. \tag{6}$$

Let us define

$$\begin{aligned}
\tilde{x} &= \frac{\overline{\lambda} - \lambda^*}{\overline{\lambda} - \lambda'} x(\overline{\lambda}) + \frac{\lambda^* - \lambda'}{\overline{\lambda} - \lambda'} x', \\
\tilde{y} &= \frac{\overline{\lambda} - \lambda^*}{\overline{\lambda} - \lambda'} y(\overline{\lambda}) + \frac{\lambda^* - \lambda'}{\overline{\lambda} - \lambda'} y', \\
\tilde{s} &= \frac{\overline{\lambda} - \lambda^*}{\overline{\lambda} - \lambda'} s(\overline{\lambda}) + \frac{\lambda^* - \lambda'}{\overline{\lambda} - \lambda'} s'.
\end{aligned}$$

By definition $(\tilde{x}, \tilde{y}, \tilde{s})$ is optimal for λ^*, while by (6) it has a positive $\tilde{x}_i + \tilde{s}_i$ coordinate in $\mathcal{T}$, contradicting to the definition of the optimal partition π at λ^*.

We still need to show that $\Lambda(\pi)$ is a connected interval. The proof follows the same reasoning as the proof of Lemma 3.1 and is omitted.

(iii) Let $\lambda \in \mathcal{O}(\pi)$, then by definition $\pi(\lambda) = \pi$, and hence for $\lambda \in \Lambda$ there is a maximally complementary solution (x, y, s) which satisfies $Ax = b + \lambda \triangle b$, $A^T y + s - Qx = c + \lambda \triangle c$, $x_{\mathcal{N} \cup \mathcal{T}} = 0$, $s_{\mathcal{B} \cup \mathcal{T}} = 0$, $x_{\mathcal{B}} > 0$ and $s_{\mathcal{N}} > 0$, from which we conclude that $\lambda \in \Lambda(\pi)$. Analogously, one can prove that if $\lambda \in \Lambda(\pi)$ then $\lambda \in \mathcal{O}(\pi)$. Consequently, $\mathcal{O}(\pi) = \Lambda(\pi)$. Because $\Lambda(\pi)$ is a polyhedral set, it follows that $\mathrm{cl}\,\mathcal{S}_\lambda(\pi) = \overline{\mathcal{S}}_\lambda(\pi)$. Since $\Lambda(\pi)$ is an open interval, we conclude that $\mathrm{cl}\,\mathcal{O}(\pi) = \mathrm{cl}\,\Lambda(\pi) = \overline{\Lambda}(\pi)$.

(iv) This result follows from Lemma 2.3 of Berkelaar et al. (1997). □

The following two corollaries are direct consequences of Theorem 4.1.

Corollary 4.2 *Let $\lambda_2 > \lambda_1$ be such that $\pi(\lambda_1) = \pi(\lambda_2)$. Then, $\pi(\lambda)$ is constant for all $\lambda \in [\lambda_1, \lambda_2]$.*

Corollary 4.3 *Let $(x^{(1)}, y^{(1)}, s^{(1)})$ and $(x^{(2)}, y^{(2)}, s^{(2)})$ be maximally complementary solutions of (QP_{λ_1}), (QD_{λ_1}) and (QP_{λ_2}), (QD_{λ_2}), respectively. Then, for any $\lambda \in [\lambda_1, \lambda_2]$*

$$\begin{aligned} x(\lambda) &= \frac{\lambda_2 - \lambda}{\lambda_2 - \lambda_1} x^{(1)} + \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1} x^{(2)}, \\ y(\lambda) &= \frac{\lambda_2 - \lambda}{\lambda_2 - \lambda_1} y^{(1)} + \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1} y^{(2)}, \\ s(\lambda) &= \frac{\lambda_2 - \lambda}{\lambda_2 - \lambda_1} s^{(1)} + \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1} s^{(2)} \end{aligned}$$

is a maximally complementary solution of (QP_λ) and (QD_λ) if and only if $\lambda_1, \lambda_2 \in \Lambda(\pi)$.

The next theorem shows how to determine the endpoints of the interval $\overline{\Lambda}(\pi)$. It is a direct consequence of Theorem 4.1 as well.

Theorem 4.4 *Let $\lambda^* \in \Lambda$ and let (x^*, y^*, s^*) be a maximally complementary solution of (QP_{λ^*}) and (QD_{λ^*}) with optimal partition $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$. Then the left and right extreme points of the closed interval $\overline{\Lambda}(\pi) = [\lambda_\ell, \lambda_u]$ that contains λ^* can be obtained by minimizing and maximizing λ over $\overline{\mathcal{S}}_\lambda(\pi)$, respectively, i.e., by solving*

$$\begin{aligned} \lambda_\ell = \min_{\lambda,x,y,s} \{\, \lambda : \; & Ax - \lambda \triangle b = b, \; x_{\mathcal{B}} \geq 0, \; x_{\mathcal{N} \cup \mathcal{T}} = 0, \\ & A^T y + s - Qx - \lambda \triangle c = c, \; s_{\mathcal{N}} \geq 0, \; s_{\mathcal{B} \cup \mathcal{T}} = 0 \,\}, \end{aligned} \tag{7}$$

and

$$\lambda_u = \max_{\lambda,x,y,s} \{ \lambda : Ax - \lambda \triangle b = b,\ x_{\mathcal{B}} \geq 0,\ x_{\mathcal{N}\cup\mathcal{T}} = 0, \qquad (8)$$
$$A^T y + s - Qx - \lambda \triangle c = c,\ s_{\mathcal{N}} \geq 0,\ s_{\mathcal{B}\cup\mathcal{T}} = 0 \}.$$

Proof: We will prove the theorem for λ_u only. The proof for λ_ℓ goes analogously. Problem (8) is feasible since problems (QP_λ) and (QD_λ) are feasible for the given $\lambda = \lambda^*$. We continue by considering two cases:

(i) Problem (8) is unbounded. Then for every $\lambda \geq \lambda^*$ there exists a feasible solution (x, y, s) for (QP_λ) and (QD_λ). Further, from the complementarity property $x^T s = 0$ we conclude that (x, y, s) is also optimal for (QP_λ) and (QD_λ). Theorem 4.1 imply that λ and λ^* belong to the same interval $\Lambda(\pi)$. Since this holds for any $\lambda \geq \lambda^*$, the right boundary of $\Lambda(\pi)$ is $+\infty$.

(ii) Problem (8) has an optimal solution $(\tilde{\lambda}, \tilde{x}, \tilde{y}, \tilde{s})$. Similarly to (i), $(\tilde{x}, \tilde{y}, \tilde{s})$ is feasible for $(QP_{\tilde{\lambda}})$ and $(QD_{\tilde{\lambda}})$. From the complementarity property $\tilde{x}^T \tilde{s} = 0$ we conclude that $(\tilde{x}, \tilde{y}, \tilde{s})$ is optimal for $(QP_{\tilde{\lambda}})$ and $(QD_{\tilde{\lambda}})$, and then Theorem 4.1 imply that $\tilde{\lambda}$ and λ^* belong to the interval $\Lambda(\pi)$ and so $\tilde{\lambda} \leq \lambda_u$.

Since for every $\lambda \in \Lambda(\pi)$ problem (8) has a feasible solution and for any $\tilde{\lambda} \geq \lambda_u$ the optimal partition is different, the proof is completed. □

The open interval $\Lambda(\pi)$ is referred to as *invariancy interval* because the optimal partition is invariant on it. The points λ_ℓ and λ_u, that separate neighboring invariancy intervals, are called *transition points*. The following theorem shows that the optimal value function is quadratic on an invariancy interval $(\lambda_\ell, \lambda_u)$, where λ_ℓ and λ_u are obtained by solving (7) and (8). It presents an explicit formula for the optimal value function as well as simple criteria to determine convexity, concavity or linearity of the optimal value function on a specific invariancy interval.

Theorem 4.5 *Let λ_ℓ and λ_u be obtained by solving (7) and (8), respectively. The optimal value function $\phi(\lambda)$ is quadratic on $\mathcal{O}(\pi) = (\lambda_\ell, \lambda_u)$.*

Proof: If $\lambda_\ell = \lambda_u$ the statement is trivial, so we may assume that $\lambda_\ell < \lambda_u$. Let $\lambda_\ell < \lambda_1 < \lambda < \lambda_2 < \lambda_u$ are given and let $(x^{(1)}, y^{(1)}, s^{(1)})$, $(x^{(2)}, y^{(2)}, s^{(2)})$ and $(x(\lambda), y(\lambda), s(\lambda))$ be pairs of primal-dual optimal solutions corresponding to λ_1, λ_2 and λ, respectively. Thus, there is a

$\theta = \frac{\lambda-\lambda_1}{\lambda_2-\lambda_1} \in (0,1)$ such that

$$\begin{aligned}
\lambda &= \lambda_1 + \theta\triangle\lambda, \\
x(\lambda) &= x^{(1)} + \theta\triangle x = x^{(1)} + \frac{\lambda-\lambda_1}{\lambda_2-\lambda_1}\triangle x, \\
y(\lambda) &= y^{(1)} + \theta\triangle y = y^{(1)} + \frac{\lambda-\lambda_1}{\lambda_2-\lambda_1}\triangle y, \\
s(\lambda) &= s^{(1)} + \theta\triangle s = s^{(1)} + \frac{\lambda-\lambda_1}{\lambda_2-\lambda_1}\triangle s,
\end{aligned}$$

where $\triangle\lambda = \lambda_2 - \lambda_1, \triangle x = x^{(2)} - x^{(1)}, \triangle y = y^{(2)} - y^{(1)}$ and $\triangle s = s^{(2)} - s^{(1)}$. We also have

$$A\triangle x = \triangle\lambda\triangle b, \tag{9}$$

$$A^T\triangle y + \triangle s - Q\triangle x = \triangle\lambda\triangle c. \tag{10}$$

The optimal value function at λ is given by

$$\begin{aligned}
\phi(\lambda) &= (b+\lambda\triangle b)^T y(\lambda) - \tfrac{1}{2}x(\lambda)^T Q x(\lambda) \\
&= (b+(\lambda_1+\theta\triangle\lambda)\triangle b)^T(y^{(1)}+\theta\triangle y) - \tfrac{1}{2}(x^{(1)}+\theta\triangle x)^T Q(x^{(1)}+\theta\triangle x) \\
&= (b+\lambda_1\triangle b)^T y^{(1)} + \theta(\triangle\lambda\triangle b^T y^{(1)} + (b+\lambda_1\triangle b)^T\triangle y) + \theta^2\triangle\lambda\triangle b^T\triangle y \\
&\quad -\tfrac{1}{2}{x^{(1)}}^T Q x^{(1)} - \theta {x^{(1)}}^T Q\triangle x - \tfrac{1}{2}\theta^2\triangle x^T Q\triangle x.
\end{aligned} \tag{11}$$

From equations (9) and (10), one gets

$$\triangle x^T Q\triangle x = \triangle\lambda(\triangle b^T\triangle y - \triangle c^T\triangle x), \tag{12}$$

$${x^{(1)}}^T Q\triangle x = (b+\lambda_1\triangle b)^T\triangle y - \triangle\lambda\triangle c^T x^{(1)}. \tag{13}$$

Substituting (12) and (13) into (11) we obtain

$$\phi(\lambda) = \phi(\lambda_1+\theta\triangle\lambda) = \phi(\lambda_1) + \theta\triangle\lambda(\triangle b^T y^{(1)} + \triangle c^T x^{(1)}) + \tfrac{1}{2}\theta^2\triangle\lambda(\triangle c^T\triangle x + \triangle b^T\triangle y). \tag{14}$$

Using the notation

$$\gamma_1 = \triangle b^T y^{(1)} + \triangle c^T x^{(1)}, \tag{15}$$

$$\gamma_2 = \triangle b^T y^{(2)} + \triangle c^T x^{(2)}, \tag{16}$$

$$\gamma = \frac{\gamma_2-\gamma_1}{\lambda_2-\lambda_1} = \frac{\triangle c^T\triangle x + \triangle b^T\triangle y}{\lambda_2-\lambda_1}, \tag{17}$$

one can rewrite (14) as

$$\phi(\lambda) = (\phi(\lambda_1) - \lambda_1\gamma_1 + \frac{1}{2}\lambda_1^2\gamma) + (\gamma_1 - \lambda_1\gamma)\lambda + \frac{1}{2}\gamma\lambda^2. \tag{18}$$

Because λ_1 and λ_2 are two arbitrary elements from the interval $(\lambda_\ell, \lambda_u)$, the claim of the theorem follows directly from (18). The proof is complete. $\square$

It should be mentioned that the sign of $\triangle c^T \triangle x + \triangle b^T \triangle y$ in (14) is independent of λ_1 and λ_2, because both λ_1 and λ_2 are two arbitrary numbers in $(\lambda_\ell, \lambda_u)$. The following corollary is a straightforward consequence of (18).

Corollary 4.6 *For two arbitrary $\lambda_1 < \lambda_2 \in (\lambda_\ell, \lambda_u)$, let $(x^{(1)}, y^{(1)}, s^{(1)})$ and $(x^{(2)}, y^{(2)}, s^{(2)})$ be pairs of primal-dual optimal solutions corresponding to λ_1 and λ_2, respectively. Moreover, let $\triangle x = x^{(2)} - x^{(1)}$ and $\triangle y = y^{(2)} - y^{(1)}$. Then, the optimal value function $\phi(\lambda)$ is quadratic on $\mathcal{O}(\pi) = (\lambda_\ell, \lambda_u)$ and it is*

(i) strictly convex if $\triangle c^T \triangle x + \triangle b^T \triangle y > 0$;

(ii) linear if $\triangle c^T \triangle x + \triangle b^T \triangle y = 0$;

(iii) strictly concave if $\triangle c^T \triangle x + \triangle b^T \triangle y < 0$.

Remark 4.7 *For $\triangle c = 0$ equation (18) reduces to the one presented in Theorem 3.5 in Berkelaar et al. (1997).*

Remark 4.8 *Note that π represents either an optimal partition at a transition point, when $\lambda_\ell = \lambda_u$, or on the interval between two consequent transition points λ_ℓ and λ_u. Thus $\Lambda = \bigcup_\pi \Lambda(\pi) = \bigcup_\pi \overline{\Lambda}(\pi)$, where π runs throughout all possible partitions.*

Corollary 4.9 *The optimal value function $\phi(\lambda)$ is continuous and piecewise quadratic on Λ.*

Proof: The fact that the optimal value function is piecewise quadratic follows directly from Theorem 4.5. We need only to prove the continuity of $\phi(\lambda)$. Continuity at interior points of any invariancy interval follows from (18). Let λ^* be the left transition point of the given invariancy interval (i.e., $\lambda^* = \lambda_\ell$) and $(x(\lambda^*), y(\lambda^*), s(\lambda^*))$ be a pair of primal-dual optimal solutions at λ^*. We need to prove that $\lim_{\lambda \downarrow \lambda^*} \phi(\lambda) = \phi(\lambda^*)$. For any λ close enough to λ^*, there is a $\overline{\lambda}$ such that for $\lambda^* < \lambda < \overline{\lambda}$ with $\pi(\lambda) = \pi(\overline{\lambda})$. Let $(x(\overline{\lambda}), y(\overline{\lambda}), s(\overline{\lambda}))$ be a maximally complementary optimal solution at $\overline{\lambda}$ and $\theta = \frac{\lambda^*-\lambda}{\lambda^*-\overline{\lambda}} \in (0,1)$. We define

$$x(\lambda) = \theta x(\overline{\lambda}) + (1-\theta)x(\lambda^*),$$

$$y(\lambda) = \theta y(\overline{\lambda}) + (1-\theta)y(\lambda^*),$$

$$s(\lambda) = \theta s(\overline{\lambda}) + (1-\theta)s(\lambda^*),$$

that shows the convergence of the subsequence $(x(\lambda), y(\lambda), s(\lambda))$ to $(x(\lambda^*), y(\lambda^*), s(\lambda^*))$ when θ goes to zero. As in our case $\lambda^* = \lambda_\ell$, thus it follows from (7) that $x(\lambda)^T s(\lambda) = \theta(1-\theta)(x(\overline{\lambda})^T s(\lambda^*) + s(\overline{\lambda})^T x(\lambda^*)) = 0$. It means that the subsequence $(x(\lambda), y(\lambda), s(\lambda))$ is complementary.

We know that $x(\lambda) = x(\lambda^*) + \theta(x(\overline{\lambda}) - x(\lambda^*))$ and

$$\begin{aligned}\phi(\lambda) = (c+\lambda\triangle c)^T x(\lambda) + \tfrac{1}{2}x(\lambda)^T Q x(\lambda) &= (c+\lambda\triangle c)^T x(\lambda^*) + \tfrac{1}{2}x(\lambda^*)^T Q x(\lambda^*) \\ &+ \theta(c+\lambda\triangle c)^T(x(\overline{\lambda}) - x(\lambda^*)) + \theta(x(\overline{\lambda}) - x(\lambda^*))^T Q x(\lambda^*) \\ &+ \tfrac{1}{2}\theta^2(x(\overline{\lambda}) - x(\lambda^*))^T Q (x(\overline{\lambda}) - x(\lambda^*)),\end{aligned} \tag{19}$$

with $\theta = \frac{\lambda^*-\lambda}{\lambda^*-\overline{\lambda}} \in (0,1)$. When $\lambda \downarrow \lambda^*$ (i.e., $\theta \downarrow 0$), we have $\phi(\lambda) \to \phi(\lambda^*)$ that proves left continuity of the optimal value function $\phi(\lambda)$ at $\lambda^* = \lambda_\ell$. Analogously, one can prove right continuity of the optimal value function considering $\lambda^* = \lambda_u$ by using problem (8) that completes the proof. □

4.2 Derivatives, Invariancy Intervals, and Transition Points

In this subsection, the first and second order derivatives of the optimal value function are studied. We also investigate the relationship between the invariancy intervals and neighboring transition points where these derivatives may not exist.

Two auxiliary LO problems were presented in Theorem 4.4 to identify transition points and consequently to determine invariancy intervals. The following theorem allows us to compute the left and right first order derivatives of the optimal value function by solving another two auxiliary LO problems.

Theorem 4.10 *For a given $\lambda \in \Lambda$, let (x^*, y^*, s^*) be a pair of primal-dual optimal solutions of (QP_λ) and (QD_λ). Then, the left and right derivatives of the optimal value function $\phi(\lambda)$ at λ*

satisfy

$$\phi'_-(\lambda) = \min_{x,y,s}\{\triangle b^T y : (x,y,s) \in \mathcal{QD}^*_\lambda\} + \max_x\{\triangle c^T x : x \in \mathcal{QP}^*_\lambda\}, \quad (20)$$

$$\phi'_+(\lambda) = \max_{x,y,s}\{\triangle b^T y : (x,y,s) \in \mathcal{QD}^*_\lambda\} + \min_x\{\triangle c^T x : x \in \mathcal{QP}^*_\lambda\}. \quad (21)$$

Proof: Let ε be a sufficiently small real number. For *any* optimal solution $x(\lambda) \in \mathcal{QP}^*_\lambda$ and $(x(\lambda), y(\lambda), s(\lambda)) \in \mathcal{QD}^*_\lambda$ we have

$$\begin{aligned}\phi(\lambda+\varepsilon) &= (b + (\lambda+\varepsilon)\triangle b)^T y(\lambda+\varepsilon) - \tfrac{1}{2}x(\lambda+\varepsilon)^T Q x(\lambda+\varepsilon)\\ &= (b+\lambda\triangle b)^T y(\lambda) - \tfrac{1}{2}x(\lambda)^T Q x(\lambda) + \varepsilon(\triangle b^T y(\lambda) + \triangle c^T x(\lambda+\varepsilon))\\ &\quad + x(\lambda+\varepsilon)^T s(\lambda) + \tfrac{1}{2}(x(\lambda+\varepsilon) - x(\lambda))^T Q (x(\lambda+\varepsilon) - x(\lambda)) \qquad (22)\\ &\geq (b+\lambda\triangle b)^T y(\lambda) - \tfrac{1}{2}x(\lambda)^T Q x(\lambda) + \varepsilon(\triangle b^T y(\lambda) + \triangle c^T x(\lambda+\varepsilon))\\ &= \phi(\lambda) + \varepsilon(\triangle b^T y(\lambda) + \triangle c^T x(\lambda+\varepsilon)),\end{aligned}$$

where the constraints of (QD_λ) and $(QP_{\lambda+\varepsilon})$ were used. Analogously, the constraints of (QP_λ) and $(QD_{\lambda+\varepsilon})$ imply

$$\phi(\lambda+\varepsilon) \leq \phi(\lambda) + \varepsilon(\triangle b^T y(\lambda+\varepsilon) + \triangle c^T x(\lambda)). \quad (23)$$

We prove only (21). The proof of (20) goes analogously. Using (22) and (23) for a positive ε we derive the following inequality

$$\triangle b^T y(\lambda) + \triangle c^T x(\lambda+\varepsilon) \leq \frac{\phi(\lambda+\varepsilon) - \phi(\lambda)}{\varepsilon} \leq \triangle b^T y(\lambda+\varepsilon) + \triangle c^T x(\lambda). \quad (24)$$

For any ε small enough, there is a $\overline{\lambda}$ such that $\lambda < \lambda+\varepsilon < \overline{\lambda}$ and $\lambda+\varepsilon$ and $\overline{\lambda}$ are in the same invariancy interval. Let $(x(\overline{\lambda}), y(\overline{\lambda}), s(\overline{\lambda}))$ be an optimal solution at $\overline{\lambda}$ and for $\theta = \frac{\lambda-\overline{\lambda}}{\lambda+\varepsilon-\overline{\lambda}} > 1$, we define

$$\tilde{x}(\theta) = \theta x(\lambda+\varepsilon) + (1-\theta)x(\overline{\lambda}),$$

$$\tilde{y}(\theta) = \theta y(\lambda+\varepsilon) + (1-\theta)y(\overline{\lambda}),$$

$$\tilde{s}(\theta) = \theta s(\lambda+\varepsilon) + (1-\theta)s(\overline{\lambda}),$$

which shows that when $\varepsilon \downarrow 0$ ($\theta \downarrow 1$) the subsequence $(x(\lambda+\varepsilon), y(\lambda+\varepsilon), s(\lambda+\varepsilon))$ converges to $(\tilde{x}(1), \tilde{y}(1), \tilde{s}(1)) = (\tilde{x}, \tilde{y}, \tilde{s})$. Since $\lambda+\varepsilon$ and $\overline{\lambda}$ are in the same invariancy interval, thus $\tilde{x}^T\tilde{s} = \theta(1-\theta)(x(\lambda+\varepsilon)^T s(\overline{\lambda}) + s(\lambda+\varepsilon)^T x(\overline{\lambda})) = 0$ shows that $\tilde{x}$ is a primal optimal solution of (QP_λ) and $(\tilde{x}, \tilde{y}, \tilde{s})$ is a dual optimal solution of (QD_λ). Letting $\varepsilon \downarrow 0$ we get

$$\lim_{\varepsilon\downarrow 0} \triangle c^T x(\lambda+\varepsilon) = \triangle c^T \tilde{x} \quad \text{and} \quad \lim_{\varepsilon\downarrow 0} \triangle b^T y(\lambda+\varepsilon) = \triangle b^T \tilde{y}. \quad (25)$$

From (24) and (25) one can easily obtain the inequality

$$\triangle b^T y(\lambda) + \min_{\tilde{x}(\lambda)\in\mathcal{QP}^*_\lambda} \triangle c^T \tilde{x}(\lambda) \leq \phi'_+(\lambda) \leq \max_{\tilde{y}(\lambda)\in\mathcal{QD}^*_\lambda} \triangle b^T \tilde{y}(\lambda) + \triangle c^T x(\lambda). \tag{26}$$

Since $x(\lambda)$ is *any* optimal solution of (QP_λ) and $(x(\lambda), y(\lambda), s(\lambda))$ is *any* optimal solution of (QD_λ), from (26) we conclude that

$$\max_{\tilde{y}(\lambda)\in\mathcal{QD}^*_\lambda} \triangle b^T \tilde{y}(\lambda) + \min_{\tilde{x}(\lambda)\in\mathcal{QP}^*_\lambda} \triangle c^T \tilde{x}(\lambda) \leq \phi'_+(\lambda) \leq \max_{\tilde{y}(\lambda)\in\mathcal{QD}^*_\lambda} \triangle b^T \tilde{y}(\lambda) + \min_{\tilde{x}(\lambda)\in\mathcal{QP}^*_\lambda} \triangle c^T \tilde{x}(\lambda).$$

Then

$$\phi'_+(\lambda) = \max_{\tilde{y}(\lambda)\in\mathcal{QD}^*_\lambda} \triangle b^T \tilde{y}(\lambda) + \min_{\tilde{x}(\lambda)\in\mathcal{QP}^*_\lambda} \triangle c^T \tilde{x}(\lambda),$$

completing the proof. □

Remark 4.11 *It is worthwhile to make some remarks about Theorem 4.10. We consider problem (21) only. Similar results hold for problem (20). Let*

$$\mathcal{QPD} = \{(x, y, s) : Ax = b + \lambda\triangle b,\ x \geq 0,\ x^T s^* = 0,$$
$$A^T y + s - Qx = c + \lambda\triangle c,\ s \geq 0,\ s^T x^* = 0\}.$$

First, in the definition of the set $\mathcal{QPD}$ *the constraints* $x \geq 0,\ x^T s^* = 0$ *and* $s \geq 0,\ s^T x^* = 0$ *are equivalent to* $x_\mathcal{B} \geq 0,\ x_{\mathcal{N}\cup\mathcal{T}} = 0$ *and* $s_\mathcal{N} \geq 0,\ s_{\mathcal{B}\cup\mathcal{T}} = 0$*, where* $(\mathcal{B}, \mathcal{N}, \mathcal{T})$ *is the optimal partition in the transition point* λ*. Second, let us consider the first and the second subproblems of (21). Observe that the optimal solutions produced by each subproblem are both optimal for* (QP_λ) *and* (QD_λ) *and so the vector* Qx*, appearing in the constraints, is always identical for both subproblems (see, e.g., Dorn 1960). This means that we can maximize the first subproblem over the dual optimal set* $\mathcal{QD}^*_\lambda$ *only and minimize the second subproblem over the primal optimal set* $\mathcal{QP}^*_\lambda$ *only. In other words, instead of solving two subproblems in (21) separately, we can solve the problem*

$$\min_{x,y,s}\{\triangle c^T x - \triangle b^T y : (x, y, s) \in \mathcal{QPD}\} \tag{27}$$

that produces the same optimal solution $(\hat{x}, \hat{y}, \hat{s})$ *as a solution of problem (21). Then the right derivative* $\phi'_+(\lambda)$ *can be computed by using the values* $(\hat{x}, \hat{y}, \hat{s})$ *as* $\phi'_+(\lambda) = \triangle b^T \hat{y} + \triangle c^T \hat{x}$*. Consequently, we refer to the optimal solutions of problems (21) and (27) interchangeably.*

The following theorem shows that if λ is not a transition point, then the optimal value function is differentiable and the derivative can be given explicitly.

Theorem 4.12 *If λ is not a transition point, then the optimal value function at λ is a differentiable quadratic function and its first order derivative is*

$$\phi'(\lambda) = \triangle b^T y(\lambda) + \triangle c^T x(\lambda).$$

Proof: The result can be established directly by differentiating the optimal value function given by (18). □

If we have non-degenerate primal and dual optimal solutions at transition point λ, the left and right first order derivatives coincide and can be computed by using the formula given in Theorem 4.12 as well. It follows from the fact that in such situation, the optimal solution of both the primal and the dual problems (QP_λ) and (QD_λ) are unique.

The next lemma shows an important property of strictly complementary solutions of (20) and (21) and will be used later on in the paper.

Lemma 4.13 *Let λ^* be a transition point of the optimal value function. Further, assume that the (open) invariancy interval to the right of λ^* contains $\overline{\lambda}$ with the optimal partition $\overline{\pi} = (\overline{\mathcal{B}}, \overline{\mathcal{N}}, \overline{\mathcal{T}})$. Let (x, y, s) be an optimal solution of (21) with $\lambda = \lambda^*$. Then, $\sigma(x) \subseteq \overline{\mathcal{B}}$ and $\sigma(s) \subseteq \overline{\mathcal{N}}$.*

Proof: Let $(\overline{x}, \overline{y}, \overline{s})$ be a maximally complementary solution at $\overline{\lambda}$ and let $(\lambda^*, \underline{x}, \underline{y}, \underline{s})$ be an optimal solution of (7) where the optimal partition is $\overline{\pi}$.

First, we want to prove that

$$\triangle c^T x = \triangle c^T \underline{x} \quad \text{and} \quad \triangle b^T y = \triangle b^T \underline{y}, \tag{28}$$

$$c^T x = c^T \underline{x} \quad \text{and} \quad b^T y = b^T \underline{y}. \tag{29}$$

For this purpose we use equation (14). In (14) let $\lambda_2 = \overline{\lambda}$, $x^{(2)} = \overline{x}$, $y^{(2)} = \overline{y}$. Continuity of the optimal value function, that is proved in Corollary 4.9, allows us to establish that equation (14) holds not only on invariancy intervals, but also at their endpoints, i.e., at the transition points.

Thus, we are allowed to consider the case when $\lambda_1 = \lambda^*$ and $(x^{(1)}, y^{(1)}, s^{(1)})$ is any optimal solution at the transition point λ^*.

Computing $\phi(\lambda)$ at the point $\overline{\lambda}$ (where $\theta = \frac{\lambda-\lambda_1}{\lambda_2-\lambda_1} = \frac{\overline{\lambda}-\lambda^*}{\overline{\lambda}-\lambda^*} = 1$) by (14) gives us

$$\begin{aligned}\phi(\overline{\lambda}) &= \phi(\lambda^*) + (\overline{\lambda} - \lambda^*)(\triangle b^T y^{(1)} + \triangle c^T x^{(1)}) + \frac{1}{2}(\overline{\lambda} - \lambda^*)[\triangle c^T(\overline{x} - x^{(1)}) + \triangle b^T(\overline{y} - y^{(1)})] \\ &= \phi(\lambda^*) + \frac{1}{2}(\overline{\lambda} - \lambda^*)[\triangle c^T(\overline{x} + x^{(1)}) + \triangle b^T(\overline{y} + y^{(1)})]. \end{aligned} \tag{30}$$

One can rearrange (30) as

$$\frac{\phi(\overline{\lambda}) - \phi(\lambda^*)}{\overline{\lambda} - \lambda^*} = \triangle c^T \left(\frac{\overline{x} + x^{(1)}}{2}\right) + \triangle b^T \left(\frac{\overline{y} + y^{(1)}}{2}\right).$$

Let $\overline{\lambda} \downarrow \lambda^*$, then we have

$$\phi'_+(\lambda^*) = \lim_{\overline{\lambda}\downarrow\lambda^*} \frac{\phi(\overline{\lambda}) - \phi(\lambda^*)}{\overline{\lambda} - \lambda^*} = \triangle c^T \left(\frac{\underline{x} + x^{(1)}}{2}\right) + \triangle b^T \left(\frac{\underline{y} + y^{(1)}}{2}\right). \tag{31}$$

Since $(x^{(1)}, y^{(1)}, s^{(1)})$ is an arbitrary optimal solution at λ^* and $\phi'_+(\lambda^*)$ is independent of the optimal solution choice at λ^*, one may choose $(x^{(1)}, y^{(1)}, s^{(1)}) = (x, y, s)$ and $(x^{(1)}, y^{(1)}, s^{(1)}) = (\underline{x}, y, s)$. From (31) we get

$$\phi'_+(\lambda^*) = \triangle c^T \left(\frac{\underline{x} + x}{2}\right) + \triangle b^T \left(\frac{\underline{y} + y}{2}\right) = \triangle c^T \left(\frac{\underline{x} + \underline{x}}{2}\right) + \triangle b^T \left(\frac{\underline{y} + y}{2}\right). \tag{32}$$

Equation (32) reduces to $\triangle c^T(\frac{\underline{x}+x}{2}) = \triangle c^T \underline{x}$ from which it follows that $\triangle c^T x = \triangle c^T \underline{x}$. Furthermore, let us consider $(x^{(1)}, y^{(1)}, s^{(1)}) = (x, y, s)$ and $(x^{(1)}, y^{(1)}, s^{(1)}) = (x, \underline{y}, \underline{s})$. From (31) we obtain $\triangle b^T y = \triangle b^T \underline{y}$.

Now, since both (x, y, s) and $(\underline{x}, \underline{y}, \underline{s})$ are optimal solutions in $\mathcal{QP}^*_{\lambda^*} \times \mathcal{QD}^*_{\lambda^*}$, it holds that $(c + \lambda^*\triangle c)^T x = (c+\lambda^*\triangle c)^T \underline{x}$ and $(b+\lambda^*\triangle b)^T y = (b+\lambda^*\triangle b)^T \underline{y}$ (see e.g., Dorn 1960). Consequently, it follows from (28) that $c^T x = c^T \underline{x}$ and $b^T y = b^T \underline{y}$.

As a result we can establish that

$$\begin{aligned} x^T \overline{s} &= x^T(c + \overline{\lambda}\triangle c + Q\overline{x} - A^T\overline{y}) = c^T x + \overline{\lambda}\triangle c^T x + x^T Q\overline{x} - (b + \lambda^*\triangle b)^T \overline{y} \\ &= c^T \underline{x} + \overline{\lambda}\triangle c^T \underline{x} + \underline{x}^T Q\overline{x} - (A\underline{x})^T \overline{y} = \underline{x}^T(c + \overline{\lambda}\triangle c + Q\overline{x} - A^T\overline{y}) = \underline{x}^T \overline{s} = 0, \end{aligned} \tag{33}$$

and

$$\begin{aligned} \overline{x}^T s &= \overline{x}^T(c + \lambda^*\triangle c + Qx - A^T y) = \overline{x}^T(c + \lambda^*\triangle c + Qx) - b^T y - \overline{\lambda}\triangle b^T y \\ &= \overline{x}^T(c + \lambda^*\triangle c + Q\underline{x}) - b^T \underline{y} - \overline{\lambda}\triangle b^T \underline{y} = \overline{x}^T(c + \lambda^*\triangle c + Q\underline{x} - A^T\underline{y}) = \overline{x}^T \underline{s} = 0. \end{aligned} \tag{34}$$

For $\theta \in (0,1)$ and $\tilde{\lambda} = (1-\theta)\lambda^* + \theta\overline{\lambda}$, let us consider

$$\begin{aligned}\tilde{x} &= (1-\theta)x + \theta\overline{x},\\ \tilde{y} &= (1-\theta)y + \theta\overline{y}, \qquad (35)\\ \tilde{s} &= (1-\theta)s + \theta\overline{s}.\end{aligned}$$

Utilizing equations (35) and the complementarity properties (33) and (34), we obtain that $\tilde{x}$ and $(\tilde{x}, \tilde{y}, \tilde{s})$ are feasible and complementary, and thus optimal solutions of $(QP_{\tilde{\lambda}})$ and $(QD_{\tilde{\lambda}})$, respectively. Noting that $(\overline{\mathcal{B}}, \overline{\mathcal{N}}, \overline{\mathcal{T}})$ is the optimal partition at $(\tilde{x}, \tilde{y}, \tilde{s})$, it follows from (35) that $x_{\overline{\mathcal{B}}} \geq 0,\ x_{\overline{\mathcal{N}}} = 0,\ x_{\overline{\mathcal{T}}} = 0$ and $s_{\overline{\mathcal{B}}} = 0,\ s_{\overline{\mathcal{N}}} \geq 0,\ s_{\overline{\mathcal{T}}} = 0$. Then we can conclude that $\sigma(x) \subseteq \overline{\mathcal{B}}$ and $\sigma(s) \subseteq \overline{\mathcal{N}}$. □

The next theorem utilizes two auxiliary linear optimization problems to calculate the left and right second order derivatives of $\phi(\lambda)$ and also gives a general result concerning the transition points of the optimal value function. Remark 4.11 can be used for finding optimal solutions of problems (20) and (21).

Theorem 4.14 *Let $\lambda \in \Lambda$, and x^* be an optimal solution of (QP_λ). Further, let (x^*, y^*, s^*) be an optimal solution of (QD_λ). Then, the left and right second order derivatives $\phi''_-(\lambda)$ and $\phi''_+(\lambda)$ are*

$$\begin{aligned}\phi''_-(\lambda) = \min_{\xi,\varrho,\mu,\eta,\rho,\delta} \{&\triangle c^T\xi : A\xi = \triangle b,\ \xi + \varrho + \mu x^* = 0,\ \varrho_{\sigma(s^-)} \geq 0,\ \varrho_{\sigma(x^-)} = 0,\\ &A^T\eta + \rho - Q\xi + \delta s^* = \triangle c,\ \rho_{\sigma(s^-)} \geq 0,\ \rho_{\sigma(x^-)} = 0\}\\ + \max_{\xi,\varrho,\mu,\eta,\rho,\delta} \{&\triangle b^T\eta : A\xi = \triangle b,\ \xi + \varrho + \mu x^* = 0,\ \varrho_{\sigma(s^-)} \geq 0,\ \varrho_{\sigma(x^-)} = 0,\\ &A^T\eta + \rho - Q\xi + \delta s^* = \triangle c,\ \rho_{\sigma(s^-)} \geq 0,\ \rho_{\sigma(x^-)} = 0\},\end{aligned}$$

where (x^-, y^-, s^-) is a strictly complementary optimal solution of (20)*, and*

$$\begin{aligned}\phi''_+(\lambda) = \max_{\xi,\varrho,\mu,\eta,\rho,\delta} \{&\triangle c^T\xi : A\xi = \triangle b,\ \xi + \varrho + \mu x^* = 0,\ \varrho_{\sigma(s^+)} \geq 0,\ \varrho_{\sigma(x^+)} = 0,\\ &A^T\eta + \rho - Q\xi + \delta s^* = \triangle c,\ \rho_{\sigma(s^+)} \geq 0,\ \rho_{\sigma(x^+)} = 0\}\\ + \min_{\xi,\varrho,\mu,\eta,\rho,\delta} \{&\triangle b^T\eta : A\xi = \triangle b,\ \xi + \varrho + \mu x^* = 0,\ \varrho_{\sigma(s^+)} \geq 0,\ \varrho_{\sigma(x^+)} = 0,\\ &A^T\eta + \rho - Q\xi + \delta s^* = \triangle c,\ \rho_{\sigma(s^+)} \geq 0,\ \rho_{\sigma(x^+)} = 0\},\end{aligned}$$

where (x^+, y^+, s^+) is a strictly complementary optimal solution of (21)*.*

Proof: The proof follows by using a similar pattern of reasoning as Theorem IV.61 in Roos, Terlaky and Vial (1997) for the linear problems (20) and (21). □

The following theorem, that summarizes the results we got up to now, is a direct consequence of Theorem 4.1 (equivalence of (i) and (ii)), the definition of a transition point (equivalence of (ii) and (iii)), and Corollary 4.3 and Lemma 4.13 (equivalence of (iii) and (iv)). The proof is identical to the proof of Theorem 3.10 in Berkelaar et al. (1997) and it also shows that in adjacent subintervals $\phi(\lambda)$ is defined by different quadratic functions.

Theorem 4.15 *The following statements are equivalent:*

(i) $D_\pi = \emptyset$*;*

(ii) $\Lambda(\pi) = \{\lambda^*\}$*;*

(iii) λ^* *is a transition point;*

(iv) ϕ' *or* ϕ'' *is discontinuous at* λ^**.*

By solving an auxiliary self-dual quadratic optimization problem one can obtain the optimal partition in the neighboring invariancy interval. The result is given by the next theorem.

Theorem 4.16 *Let* λ^* *be a transition point of the optimal value function. Let* (x^*, s^*) *be an optimal solution of (21) for* λ^**. Let us assume that the (open) invariancy interval to the right of* λ^* *contains* $\overline{\lambda}$ *with optimal partition* $\overline{\pi} = (\overline{\mathcal{B}}, \overline{\mathcal{N}}, \overline{\mathcal{T}})$*, and* $(\overline{x}, \overline{y}, \overline{s})$ *is a maximally complementary solution at* $\overline{\lambda}$*. Define* $T = \overline{\sigma}(x^*, s^*) = \{1, 2, \ldots, n\} \setminus (\sigma(x^*) \cup \sigma(s^*))$*. Consider the following self-dual quadratic problem*

$$\min_{\xi,\rho,\eta} \quad \{-\triangle b^T\eta + \triangle c^T\xi + \xi^T Q\xi : A\xi = \triangle b,\ A^T\eta + \rho - Q\xi = \triangle c, \\ \xi_{\sigma(s^*)} = 0,\ \rho_{\sigma(x^*)} = 0,\ \xi_{\overline{\sigma}(x^*,s^*)} \geq 0,\ \rho_{\overline{\sigma}(x^*,s^*)} \geq 0\}, \tag{36}$$

and let (ξ^*, η^*, ρ^*) *be a maximally complementary solution of (36). Then,* $\overline{\mathcal{B}} = \sigma(x^*) \cup \sigma(\xi^*)$*,* $\overline{\mathcal{N}} = \sigma(s^*) \cup \sigma(\rho^*)$ *and* $\overline{\mathcal{T}} = \{1, \ldots, n\} \setminus (\overline{\mathcal{B}} \cup \overline{\mathcal{N}})$*.*

Proof: For any feasible solution of (36) we have

$$-\triangle b^T\eta + \triangle c^T\xi + \xi^T Q\xi = \xi^T(Q\xi - A^T\eta + \triangle c) = \xi^T\rho = \xi_T^T\rho_T \geq 0.$$

The dual of (36) is

$$\max_{\delta,\xi,\gamma,\zeta}\{\ \triangle b^T\delta - \triangle c^T\zeta - \xi^T Q\xi :$$
$$A\zeta = \triangle b, A^T\delta + \gamma + Q\zeta - 2Q\xi = \triangle c,\ \gamma_{\sigma(s^*)} = 0, \zeta_{\sigma(x^*)} = 0, \gamma_T \geq 0, \zeta_T \geq 0\}.$$

For a feasible solution it holds

$$\triangle b^T\delta - \triangle c^T\zeta - \xi^T Q\xi = \xi^T A\zeta - \triangle c^T\zeta - \xi^T Q\xi = -\zeta^T\gamma - (\zeta-\xi)^T Q(\zeta-\xi) \leq 0.$$

So, the optimal value of (36) is zero. Let us assign

$$\xi = \zeta = \overline{x} - x^*,\ \rho = \gamma = \overline{s} - s^*,\ \eta = \delta = \overline{y} - y^*, \tag{37}$$

that satisfy the first two linear constraints of (36). Then, problem (36) is feasible and self-dual.

Using the fact that by Lemma 4.13 $\sigma(x^*) \subseteq \overline{\mathcal{B}}$ and $\sigma(s^*) \subseteq \overline{\mathcal{N}}$, it follows that

$$\xi_{\sigma(s^*)} = \overline{x}_{\sigma(s^*)} - x^*_{\sigma(s^*)} = 0, \quad \xi_T = \overline{x}_T - x^*_T = \overline{x}_T \geq 0$$

and

$$\rho_{\sigma(x^*)} = \overline{s}_{\sigma(x^*)} - s^*_{\sigma(x^*)} = 0, \quad \rho_T = \overline{s}_T - s^*_T = \overline{s}_T \geq 0.$$

From the proof of Lemma 4.13 we have $\overline{x}^T s^* = \overline{s}^T x^* = 0$, implying that (37) is an optimal solution. The fact that $(\overline{x}, \overline{y}, \overline{s})$ is maximally complementary shows that (37) is maximally complementary solution in (36) as well. For $\overline{x} = x^* + \xi$, we need to consider four cases to determine $\overline{\mathcal{B}}$:

1. $x^*_i > 0$ and $\xi_i > 0$;

2. $x^*_i > 0$ and $\xi_i = 0$;

3. $x^*_i > 0$ and $|\xi_i| < x^*_i$;

4. $x^*_i = 0$ and $\xi_i > 0$.

One can easily check that $\overline{\mathcal{B}} = \sigma(x^*) \cup \sigma(\xi^*)$. Analogous arguments hold for $\overline{\mathcal{N}}$, that completes the proof. □

4.3 Computational Algorithm

In this subsection we summarize the results in a computational algorithm. This algorithm is capable of finding the transition points; the right first order derivatives of the optimal value function at transition points; and optimal partitions at all transition points and invariancy intervals. Note that the algorithm computes all these quantities to the right from the given initial value λ^*. One can easily outline an analogous algorithm for the transition points to the left from λ^*. It is worthwhile to mention that all the subproblems used in this algorithm can be solved in polynomial time by IPMs.

Algorithm: Transition Points, First-Order Derivatives of the Optimal Value Function and Optimal Partitions at All Subintervals for CQO

Input:
  A nonzero direction of perturbation: $r = (\triangle b, \triangle c)$;
  a maximally complementary solution (x^*, y^*, s^*) of (QP_λ) and (QD_λ) for $\lambda = \lambda^*$;
  $\pi^0 = (\mathcal{B}^0, \mathcal{N}^0, \mathcal{T}^0)$, where $\mathcal{B}^0 = \sigma(x^*)$, $\mathcal{N}^0 = \sigma(s^*)$;
  $k := 0$; $x^0 := x^*$; $y^0 := y^*$; $s^0 := s^*$;
  ready:= false;
while not ready **do**
begin
  solve $\lambda_k = \max_{\lambda,x,y,s}\{\lambda : Ax - \lambda\triangle b = b,\ x_{\mathcal{B}^k} \geq 0,\ x_{\mathcal{N}^k \cup \mathcal{T}^k} = 0,$
  $A^T y + s - Qx - \lambda\triangle c = c,\ s_{\mathcal{N}^k} \geq 0,\ s_{\mathcal{B}^k \cup \mathcal{T}^k} = 0\}$;
  if this problem is unbounded: ready:= true; **else**
  let $(\lambda_k, x^k, y^k, s^k)$ be an optimal solution;
  begin
    Let $x^* := x^k$ and $s^* := s^k$;
    solve $\min_{x,y,s}\{\triangle c^T x - \triangle b^T y : (x, y, s) \in \mathcal{QPD}\}$
    if this problem is unbounded: ready:= true; **else**
    let (x^k, y^k, s^k) be an optimal solution;
    begin
      Let $x^* := x^k$ and $s^* := s^k$;
      solve $\min_{\xi,\rho,\eta}\{-\triangle b^T \eta + \triangle c^T \xi + \xi^T Q \xi : A\xi = \triangle b,$
      $A^T \eta + \rho - Q\xi = \triangle c,\ \xi_{\sigma(s^*)} = 0,$
      $\rho_{\sigma(x^*)} = 0,\ \xi_{\overline{\sigma}(x^*, s^*)} \geq 0,\ \rho_{\overline{\sigma}(x^*, s^*)} \geq 0\}$;
      $\mathcal{B}^{k+1} = \sigma(x^*) \cup \sigma(\xi^*)$, $\mathcal{N}^{k+1} = \sigma(s^*) \cup \sigma(\rho^*)$,
      $\mathcal{T}^{k+1} = \{1, \ldots, n\} \setminus (\mathcal{B}^{k+1} \cup \mathcal{N}^{k+1})$;
      $k := k + 1$;
    end
  end
end

---

5 Simultaneous Perturbation in Linear Optimization

The case, when perturbation occurs in the objective function vector c or the RHS vector b of an LO problem was extensively studied. A comprehensive survey can be found in the book of Roos, Terlaky and Vial (1997). Greenberg (2000) has investigated simultaneous perturbation of the objective and RHS vectors when the primal and dual LO problems are formulated in canonical form. He proved some properties of the optimal value function in that case and showed that the optimal value function is piecewise quadratic.

We start this section by emphasizing the differences in the optimal partitions of the optimal value function in LO and CQO problems and then proceed to specialize our results to the LO case. Let us define the simultaneous perturbation of an LO problem as

$$(LP_\lambda) \qquad \min \{ (c + \lambda \triangle c)^T x \ : \ Ax = b + \lambda \triangle b, \ x \geq 0 \} .$$

Its dual is

$$(LD_\lambda) \qquad \max \{ (b + \lambda \triangle b)^T y \ : \ A^T y + s = c + \lambda \triangle b, \ s \geq 0 \} .$$

The LO problem can be derived from the CQO problem by substituting the zero matrix for Q. As a result, vector x does not appear in the constraints of the dual problem, and the set $\mathcal{T}$ in the optimal partition is always empty.

The following theorem shows that to identify an invariancy interval, we don't need to solve problems (7) and (8) as they are formulated for the CQO case. Its proof is based on the fact that the constraints in these problems separate when $Q = 0$ and is omitted.

Theorem 5.1 *Let $\lambda^* \in \Lambda$ be given and let (x^*, y^*, s^*) be a strictly complementary optimal solution of (LP_{λ^*}) and (LD_{λ^*}) with the optimal partition $\pi = (\mathcal{B}, \mathcal{N})$. Then, the left and right extreme points of the interval $\overline{\Lambda}(\pi) = [\lambda_\ell, \lambda_u]$ that contains λ^* are $\lambda_\ell = \max \{\lambda_{P_\ell}, \lambda_{D_\ell}\}$ and $\lambda_u = \min \{\lambda_{P_u}, \lambda_{D_u}\}$, where*

$$\lambda_{P_\ell} = \min_{\lambda,x} \{\lambda : Ax - \lambda \triangle b = b, \ x_\mathcal{B} \geq 0, \ x_\mathcal{N} = 0\},$$

$$\lambda_{P_u} = \max_{\lambda,x} \{\lambda : Ax - \lambda \triangle b = b, \ x_\mathcal{B} \geq 0, \ x_\mathcal{N} = 0\},$$

$$\lambda_{D_\ell} = \min_{\lambda,y,s} \{\lambda : A^T y + s - \lambda \triangle c = c, \ s_\mathcal{N} \geq 0, \ s_\mathcal{B} = 0\},$$

$$\lambda_{D_u} = \max_{\lambda,y,s} \{\lambda : A^T y + s - \lambda \triangle c = c, \ s_\mathcal{N} \geq 0, \ s_\mathcal{B} = 0\}.$$

We also state the following lemma that does not hold for CQO problems.

Lemma 5.2 *Let λ_ℓ and λ_u be obtained from Theorem 5.1 and $(x^{(\ell)}, y^{(\ell)}, s^{(\ell)})$ and $(x^{(u)}, y^{(u)}, s^{(u)})$ be any strictly complementary solutions of (LP_λ) and (LD_λ) corresponding to λ_ℓ and λ_u, respectively. Then it holds that*

$$\triangle b^T \triangle y = \triangle c^T \triangle x,$$

where $\triangle y = y^{(u)} - y^{(\ell)}$ and $\triangle x = x^{(u)} - x^{(\ell)}$.

Proof: Subtracting the constraints of (LP_{λ_ℓ}) from (LP_{λ_u}) and the constraints of (LD_{λ_ℓ}) from (LD_{λ_u}) results in

$$A\triangle x = \triangle\lambda \triangle b, \tag{38}$$

$$A^T\triangle y + \triangle s = \triangle\lambda \triangle c, \tag{39}$$

where $\triangle\lambda = \lambda_u - \lambda_\ell$ and $\triangle s = s^{(u)} - s^{(\ell)}$. Premultiplying (38) by $\triangle y^T$ and (39) by $\triangle x^T$, the result follows from the fact that $\triangle x^T \triangle s = 0$, that completes the proof. □

Utilizing Lemma 5.2 and using the same notation as in (15)–(17), we can state the following theorem that gives explicit expressions for computing the objective value function. The theorem also gives the criteria to determine convexity, concavity and linearity of the objective value function on its subintervals.

Theorem 5.3 *Let $\lambda_1 < \lambda_2$ and $\pi(\lambda_1) = \pi(\lambda_2) = \pi$, let $(x^{(1)}, y^{(1)}, s^{(1)})$ and $(x^{(2)}, y^{(2)}, s^{(2)})$ be strictly complementary optimal solutions of problems (LP_λ) and (LD_λ) at λ_1 and λ_2, respectively. The following statements hold:*

(i) The optimal partition is invariant on (λ_1, λ_2).

(ii) The optimal value function is quadratic on this interval and is given by

$$\begin{aligned}\phi(\lambda) &= (\phi(\lambda_1) - \lambda_1\gamma_1 + \tfrac{1}{2}\lambda_1^2\gamma) + (\gamma_1 - \lambda_1\gamma)\lambda + \tfrac{1}{2}\gamma\lambda^2 \\ &= \phi(\lambda_1) + \theta\triangle\lambda(\triangle b^T y^{(1)} + \triangle c^T x^{(1)}) + \theta^2\triangle\lambda\triangle c^T\triangle x \\ &= \phi(\lambda_1) + \theta\triangle\lambda(\triangle b^T y^{(1)} + \triangle c^T x^{(1)}) + \theta^2\triangle\lambda\triangle b^T\triangle y\end{aligned}$$

(iii) On any subinterval, the objective value function is

- *strictly convex if $\triangle c^T\triangle x = \triangle b^T\triangle y > 0$,*

- *linear if* $\triangle c^T \triangle x = \triangle b^T \triangle y = 0$,
- *strictly concave if* $\triangle c^T \triangle x = \triangle b^T \triangle y < 0$.

Computation of derivatives can be done by solving smaller LO problems than the problems introduced in Theorem 4.10. The following theorem summarizes these results.

Theorem 5.4 *For a given* $\lambda \in \Lambda$*, let* (x^*, y^*, s^*) *be a pair of primal-dual optimal solutions of* (LP_λ) *and* (LD_λ)*. Then, the left and right first order derivatives of the optimal value function* $\phi(\lambda)$ *at* λ *are*

$$\begin{aligned}\phi'_-(\lambda) &= \min_{y,s}\{\triangle b^T y : A^T y + s = c + \lambda \triangle c,\ s \geq 0,\ s^T x^* = 0\} \\ &\quad + \max_{x}\{\triangle c^T x : Ax = b + \lambda \triangle b,\ x \geq 0,\ x^T s^* = 0\},\end{aligned}$$

$$\begin{aligned}\phi'_+(\lambda) &= \max_{y,s}\{\triangle b^T y : A^T y + s = c + \lambda \triangle c,\ s \geq 0,\ s^T x^* = 0\} \\ &\quad + \min_{x}\{\triangle c^T x : Ax = b + \lambda \triangle b,\ x \geq 0,\ x^T s^* = 0\}.\end{aligned}$$

Yildirim (2003) showed that results similar to Theorems 4.10 and 5.4 hold for parametric Convex Conic Optimization (CCO) problems.

6 Illustrative Example

Here we present some illustrative numerical results by using the algorithm outlined in Section 4.3. Computations can be performed by using any IPM solver for LO and CQO problems. Let us consider the following CQO problem with $x, c \in \mathbb{R}^5$, $b \in \mathbb{R}^3$, $Q \in \mathbb{R}^{5\times 5}$ being a positive semidefinite symmetric matrix, $A \in \mathbb{R}^{3\times 5}$ with $\text{rank}(A) = 3$. The problem's data are

$$Q = \begin{bmatrix} 4&2&0&0&0 \\ 2&5&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix},\quad c = \begin{bmatrix} -16 \\ -20 \\ 0 \\ 0 \\ 0 \end{bmatrix},\quad \triangle c = \begin{bmatrix} 7 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix},\quad A = \begin{bmatrix} 2&2&1&0&0 \\ 2&1&0&1&0 \\ 2&5&0&0&1 \end{bmatrix},\quad b = \begin{bmatrix} 11 \\ 8 \\ 20 \end{bmatrix},\quad \triangle b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

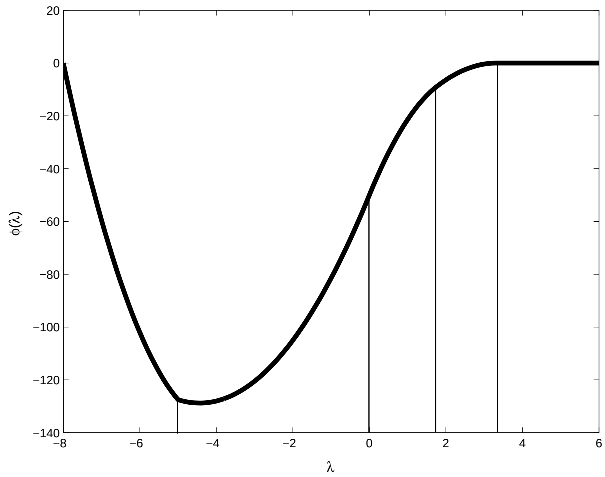

Figure 1: The Optimal Value Function

With this data the perturbed CQO instance is

$$
\begin{aligned}
\min\ & (-16+7\lambda)x_1 + (-20+6\lambda)x_2 + 2x_1^2 + 2x_1x_2 + \tfrac{5}{2}x_2^2 \\
\text{s.t.}\ & 2x_1 + 2x_2 + x_3 = 11+\lambda \\
& 2x_1 + x_2 + x_4 = 8+\lambda \\
& 2x_1 + 5x_2 + x_5 = 20+\lambda \\
& x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0.
\end{aligned} \tag{40}
$$

The results of our computations are presented in Table 1. The set Λ for the optimal value function $\phi(\lambda)$ is $[-8,+\infty)$. Figure 1 depicts the graph of $\phi(\lambda)$. Transition points and the optimal partitions at each transition point and on the invariancy intervals are identified by solving the problems in Theorems 4.4 and 4.16. The optimal value function on the invariancy intervals is computed by using formula (18). Convexity, concavity or linearity of the optimal value function can be determined by the sign of the quadratic term of the optimal value function (see Table 1). As shown in Figure 1, the optimal value function is convex on the first two invariancy intervals, concave on the third and fourth and linear on the last one. The first order derivative does not exists at transition point $\lambda = -5$.

7 Conclusions

In this paper we investigated the characteristics of the optimal value function of parametric convex quadratic optimization problems when variation occurs in both the RHS vector of the

	$\mathcal{B}$	$\mathcal{N}$	$\mathcal{T}$	$\phi(\lambda)$
$\lambda = -8.0$	{3,5}	{1,4}	{2}	
$-8.0 < \lambda < -5.0$	{2,3,5}	{1,4}	$\emptyset$	$68.0\lambda + 8.5\lambda^2$
$\lambda = -5.0$	{2}	{1,3,4,5}	$\emptyset$	
$-5.0 < \lambda < 0.0$	{1,2}	{3,4,5}	$\emptyset$	$-50.0 + 35.5\lambda + 4\lambda^2$
$\lambda = 0.0$	{1,2}	$\emptyset$	{3,4,5}	
$0.0 < \lambda < 1.739$	{1,2,3,4,5}	$\emptyset$	$\emptyset$	$-50.0 + 35.5\lambda - 6.9\lambda^2$
$\lambda = 1.739$	{2,3,4,5}	$\emptyset$	{1}	
$1.739 < \lambda < 3.333$	{2,3,4,5}	{1}	$\emptyset$	$-40.0 + 24.0\lambda - 3.6\lambda^2$
$\lambda = 3.333$	{3,4,5}	{1}	{2}	
$3.333 < \lambda < +\infty$	{3,4,5}	{1,2}	$\emptyset$	0

Table 1: Transition Points, Invariancy Intervals and Optimal Partitions

constraints and the coefficient vector of the objective function's linear term. The rate of variation, represented by the parameter λ, is identical for both perturbation vectors $\triangle b$ and $\triangle c$. We proved that the optimal value function is a continuous piecewise quadratic function on the closed set Λ. Criteria for convexity, concavity or linearity of the optimal value function were derived. Auxiliary linear problems are constructed to find its first and second order left and right derivatives. One of the main results is that the optimal partitions on the left or right neighboring intervals of a given transition point can be determined by solving an auxiliary self-dual quadratic problem. This means that we do not need to guess the length of the invariancy interval to the left or right from the current transition point and should not worry about "missing" short-length invariancy intervals. We already mentioned that all auxiliary problems can be solved in polynomial time. Finally, we outlined an algorithm to identify all invariancy intervals and draw the optimal value function. The algorithm is illustrated with a simple problem. In the special cases, when $\triangle c$ or $\triangle b$ is zero, our findings specialize to the results of Berkelaar et al. (1997) and Roos, Terlaky and Vial (1997). Simplification of some results to LO problems is given, that coincide with the findings of Greenberg (2000).

The most famous application of the CQO sensitivity analysis is the mean-variance portfolio optimization problem introduced by Markowitz (1956). The method presented in our paper allows to analyze not only the original Markowitz model, but also some of its extensions. One possible extension is when the investors's risk aversion parameter λ influences not only risk-return preferences, but also budget constraints.

It is worthwhile to mention that some encouraging results already exist for parametric Convex Conic Optimization (CCO). In CCO x is restricted to be in a closed, convex, solid and pointed cone. Yildirim (2003) extended the concept of the optimal partition to CCO problems. He proved that the optimal value function is quadratic and presented auxiliary conic problems for computing derivatives and boundaries of invariancy intervals.

As the content of the previous paragraphs suggests, our further research directions include generalizing the analysis of this paper to Second-Order Cone Optimization problems and exploring its applications to financial models.

Acknowledgements

The authors' research was partially supported by the NSERC Grant DG:5-48923, the Canadian Research Chair Program and MITACS. The first author was working on this paper while visiting the Advanced Optimization Laboratory, McMaster University, Hamilton, Canada.

References

Adler, I., and R. Monteiro. 1992. A geometric view of parametric linear programming. *Algorithmica* **8**, 161–176.

Berkelaar, A. B., B. Jansen, C. Roos, and T. Terlaky. 1997. *An interior point approach to parametric convex quadratic programming.* Working paper, Erasmus University Rotterdam, Rotterdam, The Netherlands.

Berkelaar, A. B., C. Roos, and T. Terlaky. 1997. *The optimal set and optimal partition approach to linear and quadratic programming.* Chapter 6 in Advances in Sensitivity Analysis and Parametric Programming, T. Gal and H. J. Greenberg, eds., Kluwer Academic Publishers, Boston, USA.

Boot, J. C. G. 1964. *Quadratic Programming: Algorithms, Anomalies and Applications.* North-Holland Publishing Company, Amsterdam, The Netherlands.

Dorn, W. S. 1960. Duality in quadratic programming. *Quarterly of Applied Mathematics* **18**, 155–162.

Greenberg, H. J. 2000. Simultaneous primal-dual right-hand-side sensitivity analysis from a strictly complementary solution of a linear program. *SIAM Journal of Optimization* **10**, 427–442.

Güler, O., and Y. Ye. 1993. Convergence behavior of interior-point algorithms. *Mathematical Programming* **60**, 215–228.

Jansen, B., J. J. de Jong, C. Roos, and T. Terlaky. 1997. Sensitivity analysis in linear programming: just be careful! *European Journal of Operational Research* **101**, 15–28.

Markowitz, H. M. 1956. The optimization of a quadratic function subject to linear constraints. *Naval Research Logistics Quarterly* **3**, 111–133.

McLinden, L. 1980. An analogue of Moreau's proximation theorem, with application to the nonlinear complementary problem. *Pacific Journal of Mathematics* **88**, 101–161.

Murty, K. G. 1983. *Linear Programming.* John Wiley & Sons, New York, USA.

Roos, C., T. Terlaky, and J.-Ph. Vial. 1997. *Theory and Algorithms for Linear Optimization: An Interior Point Approach*, John Wiley & Sons, Chichester, UK.

Yildirim, E. A. 2003. A unifying optimal partition approach to sensitivity analysis in conic optimization. To appear in *Journal of Optimization Theory and Applications.*