Overview
A group-theoretic method is presented for finding closed-form solutions for the normal mode frequencies for coils possessing at least a modicum of symmetry. The coil symmetry allows one to block-diagonalize Kirchhoff’s laws. One may then find the normal modes and their frequencies for each block independently. For many coils of interest in MR applications, the frequencies can therefore be expressed in closed form. Such solutions allow one to consider design questions about mode degeneracy of particular relevance to the parallel imaging community. Also, recent work has shown the value of having such solutions in developing coil ‘simulators.’ [1,2]

To demonstrate the utility of this method, an eight-runf three-ring birdcage was modeled and constructed. This coil has been successfully used as an eight channel SENSE coil. A theoretical approach based on group theory was used to calculate in closed-form the seventeen normal mode frequencies of the coil. The symmetry of the system is a key to finding analytical expressions for the normal mode frequencies. Good agreement was found between the frequency measurements and calculations. The method thus represents an effective approach to symmetric systems with larger degrees of freedom.

Theory
Finding the normal mode frequencies of the rf coil system involves solving the roots of $N^\text{th}$ order polynomials where $N$ in the number of independent loops in the system. For coils with some degree of symmetry, it is possible to use group theory to block-diagonalize Kirchhoff’s laws. With sufficient symmetry, the blocks may be small enough to find closed-form solutions for all the frequencies. The ranks of the blocks are given by the ranks of the irreducible representations of the group, and in practice have been determined to be at most three or four for coils with a moderate degree of symmetry. The block-diagonalization process is writing down a unitary transformation that transforms the system from a basis in which it is easy and convenient to write Kirchhoff’s laws to a basis in which it is easy to solve for the normal mode frequencies of the system.

Experimental Results for the 3-ring 8-axial-element Birdcage
A three-ring birdcage with eight axial elements has been constructed for SENSE applications. [3] It is illustrated in Fig. 1 and the goal was to use the extra degrees of freedom associated with the third ring to tune the dominant cage modes to near degeneracy. Ladder coils of this type were first suggested by Jevtic [4]. In the present paper, the goal is to examine the imaging capability of this coil and to compare its frequency spectrum to that obtained with a new theoretical approach.

The coil was constructed of copper tape with a cross section of 1 cm x 0.009 cm. The length of the eight axial elements in the main cage was 24.5 cm. The length of the eight axial elements in the additional cage was the much smaller value of 2 cm. The radius of the coil was 11.7 cm. The capacitors were chosen to be $C_1 = 4.7 \text{ pf}$, $C_2 = 56 \text{ pf}$, $C_3 = 45.8 \text{ pf}$, and $C_4 = 56 \text{ pf}$. The normal mode frequencies were measured via a two-loop free-space measurement of the unloaded coil. The spread in the normal mode frequencies in the ‘main’-cage modes was measured to be 980 kHz centered about 65 MHz. The addition of preamplifier decoupling resulted in isolation that was greater than 18 dB.

The images in Fig. 2 were all experimentally obtained using the coil as an eight-channel receive coil on a phantom. Fig. 3a was reconstructed via a sum of squares method and as expected the image is bright near the coils and darker in the center. Fig. 3b was reconstructed using the CLEAR method for uniformity correction with improved SNR and as expected the intensity of the image is uniform. Fig. 3c is a SENSE reconstruction with reduction factor (R=2), also showing the good uniformity and as expected the image is bright near the coils and darker in the center. Fig. 3b was reconstructed using the CLEAR method for uniformity correction with improved SNR and as expected the intensity of the image is uniform. Fig. 3c is a SENSE reconstruction with reduction factor (R=2), also showing the good uniformity.

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Theoretical Results
The numerical values of the frequencies can be calculated using the measured capacitances and the computed inductances. The inductive couplings were found via numerical integration. The seven main-cage theoretical modes are centered at 66.01 MHz, with a larger spread of 1.5 MHz. In view of capacitance tolerance values of 5% and other uncertainties, this calculation is in quite good agreement with experiment.

Design Tool
The symmetry of the coil may dictate certain modes having degenerate frequencies; for instance, the birdcage symmetry leads to a minimum of $N/2-1$ pairs of degenerate frequencies. (This degeneracy is the key for using the birdcage in quadrature receive mode.) Using the tools available from group theory, one may be able to ask that instead of pairs there should be, say, triplets of degenerate modes. This condition would then dictate the rank of the irreducible representations of the group to three. One now decides which group satisfies this condition, and the symmetry of the coil, and hence its design, is thus determined.

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References