Assignment 1

Due. January 29, Wednesday, 9:00.

1. What is the number of positive normal floating-point numbers in a floating-point system with base β, precision t, maximal exponent $e_{\max}$, and minimal exponent $e_{\min}$?

2. Assuming a small floating-point system, where $\beta = 2$, $t = 3$, $e_{\min} = -2$, and $e_{\max} = 3$.
   (a) List the floating-point numbers $x \in [1, 3/2)$.
   (b) List the floating-point numbers $x \in [3/2, 2)$.
   (c) List the floating-point numbers $y \in (1/2, 2/3]$.
   (d) List the floating-point numbers $y \in (2/3, 1]$.
   (e) Find two different floating-point numbers x_1 and x_2 for which the computed reciprocals $\mathrm{fl}(1/x_1)$ and $\mathrm{fl}(1/x_2)$ are the same, assuming the nearest rounding.
   (f) Does it follow that there exist floating-point numbers x for which
   $$\mathrm{fl}(1/(1/x))$$
   is not exactly x, assuming the nearest rounding?

3. What is the number of double precision floating-point numbers lying in the interval $[10^3, 2^{10})$? What is the number of 15 decimal digit floating-point numbers lying in the same interval? Does it follow that there must exist two distinct double precision numbers x_1 and x_2 which are converted into a same 15 decimal digit floating-point number?
   Does it follow that a 15 decimal digit floating-point number can be uniquely converted back to the original double precision floating-point number?
   If your answer is no, how many decimal digits are necessary for unique conversion?

4. In 250 B.C.E. the Greek mathematician Archimedes estimated the number π as follows. He looked at a circle with diameter 1, and hence circumference π. Inside the circle he inscribed a square. The perimeter of the square is smaller than the circumference of the circle, and so it is a lower bound for π. Archimedes then considered an inscribed octagon, 16-gon, etc., each time doubling the number of sides of the inscribed polygon, and producing ever better estimates for π. Using 96-sided inscribed and circumscribed polygons, he was able to show that $223/71 < \pi < 22/7$. There is a recursive formula for these estimates. Let p_n be the perimeter of the inscribed polygon with 2^n sides. Then $p_2 = 2\sqrt{2}$. In general,
   $$p_{n+1} = 2^n \sqrt{2(1 - \sqrt{1 - (p_n/2^n)^2})}$$
   Compute p_n for $n = 3, 4, ..., 60$. Try to explain your results.
   Kahan suggested a revision:
   $$p_{n+1} = 2^n \sqrt{r_{n+1}}$$
   where r_{n+1} can be computed iteratively
   $$r_{n+1} = \frac{r_n}{2 + \sqrt{4 - r_n}} \qquad r_3 = \frac{2}{2 + \sqrt{2}}.$$
   Use this revision to calculate r_n and p_n for $n = 3, 4, ..., 60$. Try to explain your results.