

Name _____

Student Number _____

Instructor: S. Qiao

CAS708/CES700 Final Examination

Duration of examination: two hours

1. (6 marks) Rewrite the following row version of an upper triangular solver (backward substitution) into a column version.

```
% Given an upper triangular matrix U and a right hand side b,
% this program solves the upper triangular system U*x = b and
% overwrites b with the solution x.
```

```
n = length(b);
for j=n:-1:2
    b(j) = b(j)/U(j,j);
    b(j-1) = b(j-1) - U(j-1,j:n)*b(j:n);
end
b(1) = b(1)/U(1,1);
```

Solution

```
n = length(b);
for j=n:-1:2
    b(j) = b(j)/U(j,j);
    b(1:j-1) = b(1:j-1) - b(j)*U(1:j-1,j);
end
b(1) = b(1)/U(1,1);
```

2. (6 marks) Given an n -by- n nonsingular matrix A and a second n -by- n matrix B , what is an efficient algorithm for computing $A^{-1}B$?

Solution Let $B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n]$.

```
PA = L*U;           % LU decomposition
for i=1:n           % for each column bi of B
    solve L*y = P*bi; % solve triangular systems
    solve U*ci = y;
end
```

Then $C = [\mathbf{c}_1 \ \mathbf{c}_2 \ \cdots \ \mathbf{c}_n] = A^{-1}B$.

3. (6 marks) Given the matrix

$$A = \begin{bmatrix} 1.001 & 0.01 & 0 \\ 0.01 & 0.999 & 0.01 \\ 0 & 0.01 & 0.05 \end{bmatrix}.$$

(a) Can you estimate the ranges of the eigenvalues of A ? For each eigenvalue, explain why or why not.

Solution Since A is a symmetric perturbation of the diagonal matrix $D = \text{diag}(1.001, 0.999, 0.05)$, the condition numbers for the eigenvalues are 1, and the size of the perturbation matrix is about 0.01. Thus the ranges of the eigenvalues of A are:

$$|\lambda_1 - 1.001|, |\lambda_2 - 0.999|, |\lambda_3 - 0.05| \leq 0.01.$$

(b) Can you estimate the eigenvectors of A ? For each eigenvector, explain why or why not.

Solution Since λ_3 is well separated from λ_1 and λ_2 , we expect its corresponding eigenvector x_3 is close to $[0 \ 0 \ 1]^T$. Since λ_1 and λ_2 are clustered, their corresponding eigenvectors are sensitive to perturbation. However, both are close to $[\times \ \times \ 0]^T$.

4. (10 marks) Given

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

(a) Find its Jordan canonical form.

Solution The matrix A has multiple eigenvalue $\lambda_1 = \lambda_2 = 0$ and only one right eigenvector $[1 \ 1]^T$, thus it has the Jordan block

$$J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The first column of S in the Jordan canonical form $A = SJS^{-1}$ is the right eigenvector. The second column of S can be determined by the equation $AS = SJ$.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}.$$

(b) Find its real Schur canonical form.

Solution The first column of the orthogonal Q in the Schur form $A = QTQ^T$ is the normalized right eigenvector $[1/\sqrt{2} \ 1/\sqrt{2}]^T$. Thus

$$Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Since A has multiple eigenvalue $\lambda_1 = \lambda_2 = 0$,

$$T = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix},$$

where t can be determined by the equation $AQ = QT$.

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T.$$

5. (15 marks) Solve the least squares problem:

$$\min_x \|Ax - b\|_2 \quad \text{where} \quad A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

using the following three methods.

(a) The normal equations and Cholesky factorization.

Solution

$$A^T A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} \end{bmatrix}^T = LL^T \quad \text{and} \quad A^T b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Solving $Ly = A^T b$, we get $y = [0 \quad 2\sqrt{2}/\sqrt{3}]$. Solving $L^T x = y$, we have

$$x = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

(b) The QR decomposition.

Solution

$$G_{12} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{bmatrix} : \quad A, b \rightarrow \begin{bmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \sqrt{2} \\ -1 \end{bmatrix}$$

$$G_{32} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} : \quad \rightarrow \begin{bmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \frac{2\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Solving the upper triangular system, we get

$$x = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

(c) The SVD:

$$A = U \Sigma V^T \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{\sqrt{2}}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T.$$

Solution

$$U^T b = \begin{bmatrix} -\frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma^{-1}(U^T b) = \begin{bmatrix} -\frac{\sqrt{2}}{3} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad x = V(\Sigma^{-1}U^T b) = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

6. (2 marks) What is the 2-norm condition number for solving the normal equations in the above question?

Solution: $\text{cond}_2(A^T A) = (\text{cond}_2(A))^2 = (\sqrt{3}/1)^2 = 3$.

7. (9 marks) Transform the second order initial value problem:

$$y'' = -3y' - y, \quad y(0) = 0 \quad \text{and} \quad y'(0) = 3,$$

into a system of first order equations.

Solution Let $u_1 = y$ and $u_2 = y'$, then

$$\mathbf{u}' = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} u_2 \\ -3u_2 - u_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \mathbf{u}$$

with initial conditions

$$\mathbf{u}(0) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

(a) Carry out one step of (forward) Euler's method with step size $h = 0.1$.

Solution

$$\mathbf{u}(0.1) = \mathbf{u}(0) + 0.1 \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \mathbf{u}(0) = \begin{bmatrix} 0.3 \\ 2.1 \end{bmatrix}.$$

(b) Carry out one step of backward Euler's method with step size $h = 0.1$.

Solution

$$\mathbf{u}(0.1) = \mathbf{u}(0) + 0.1 \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \mathbf{u}(0.1).$$

Solving for $\mathbf{u}(0.1)$, we get

$$\mathbf{u}(0.1) = \begin{bmatrix} 0.229 \\ 2.29 \end{bmatrix}.$$

END!