Interpolation

Problem setting: Given $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$, $x_1 < x_2 < \cdots < x_n$, construct a function f:

$$f(x_i) = y_i, \quad i = 1, 2, ..., n.$$

For the simplicity of discussion, sometimes we assume $i = 0, 1, ..., n$.
Desirable properties of f:

- smooth: analytic and $|f''(x)|$ not too large
- simple: polynomial of minimum degree

1 Polynomial Interpolation

Advantages: easy to evaluate and differentiate
Weierstrass Approximation Theorem:

> If f is any continuous function on the finite closed interval [a,b], then for every $\epsilon > 0$ there exists a polynomial $p_n(x)$ of degree $n = n(\epsilon)$ such that
>
> $$\max_{x \in [a,b]} |f(x) - p_n(x)| < \epsilon.$$

Although the proof of Weierstrass theorem is constructive, this method is impractical because the degree of the polynomial is often too high.

2 Global Polynomial Interpolation

Use one interpolating polynomial for the entire interval.

A naive approach

Given $(x_0, y_0), ..., (x_n, y_n)$, we construct the linear system:

$$\begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

Solve for the coefficients a_i of the polynomial

$$p_n(y)(x) = a_0 + a_1 x + \cdots + a_n x^n$$

- Unique solution, since x_i are distinct. The Vandermond matrix is nonsingular.
- Problems: the coefficient matrix is ill-conditioned; the calculation of the powers can easily overflow or underflow.

What is the condition number of the coefficient matrix constructed by $x_i = 1900 + 10i$, $i = 0, 1, ..., 7$?

A more practical way: Lagrange polynomials

Basis of polynomials: $\{l_j(x)\}$ $(j = 0, 1, ..., n)$ of degree n such that

$$l_j(x_i) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Construct

$$l_j(x) = \prod_{i \neq j} \frac{x - x_i}{x_j - x_i},$$

thus

$$p_n(y)(x) = \sum_{j=0}^{n} l_j(x) y_j$$

is the interpolating polynomial.

- The Lagrange polynomials are easy to construct, but expensive to evaluate.
- The naive approach is hard to compute, but easy to evaluate. The most efficient and stable way to evaluate a polynomial $p_n(x) = a_n x^n + \cdots + a_1 x + a_0$ is the following algorithm.

Algorithm. Evaluate the polynomial $a_n x^n + \cdots + a_1 x + a_0$ at $x = t$.

```
p = a_n;
for i = n-1 to 0
        p = p*t + a_i;
end
```

The Newton Form

This is an interpolation method which is not so difficult to construct and not so hard to evaluate. In this method, we use the basis

$$1,\ x - x_0,\ (x - x_0)(x - x_1),\ ...,\ (x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

and write the interpolating polynomial as a linear combination of the basis:

$$p_n(x) = c_0 + c_1(x - x_0) + \cdots + c_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$

Divided differences

How do we determine the coefficients c_k? The coefficients c_k form the solution of the following lower triangular system

$$\begin{bmatrix} 1 & & & \\ 1 & x_1 - x_0 & & \\ \vdots & \vdots & \ddots & \\ 1 & x_n - x_0 & \dots & (x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1}) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

In theory, this system can be solved in $O(n^2)$ operations. Unfortunately, the computation of the products in the coefficient matrix can easily overflow or underflow. Now, we derive another way of calculating c_k.

- Since this is a lower triangular system, c_k is the leading coefficient of the polynomial interpolating at $(x_0, y_0), ..., (x_k, y_k)$. In other words, adding a new pair (x_k, y_k) does not affect the previously computed $c_0, c_1, ..., c_{k-1}$.

Let $f[x_j, x_{j+1}, ..., x_k]$ $(k \geq j)$ denote the leading coefficient of the $(k - j)$-degree polynomial interplating at $(x_j, y_j), ..., (x_k, y_k)$. In particular,

$$f[x_k] = y_k, \quad \text{for} \quad k = 0, ..., n$$

and

$$f[x_0, x_1, ..., x_k] = c_k.$$

We can prove a two-term recursion:

$$f[x_j, ..., x_k] = \frac{f[x_{j+1}, ..., x_k] - f[x_j, ..., x_{k-1}]}{x_k - x_j}, \quad \text{for} \quad k > j.$$

Define

$p(x)$: the polynomial interpolating at $(x_j, y_j), ..., (x_k, y_k)$
$q(x)$: the polynomial interpolating at $(x_j, y_j), ..., (x_{k-1}, y_{k-1})$
$r(x)$: the polynomial interpolating at $(x_{j+1}, y_{j+1}), ..., (x_k, y_k)$

then

$$p(x) = q(x) + \frac{x - x_j}{x_k - x_j}(r(x) - q(x));$$

since both sides have the same value at $x = x_j, x_{j+1}, ..., x_{k-1}, x_k$. Comparing the leading coefficients on the both sides, we have

$$f[x_j, ..., x_k] = \frac{f[x_{j+1}, ..., x_k] - f[x_j, ..., x_{k-1}]}{x_k - x_j}.$$

This equality suggests a triangular table for computing c_k

$$\begin{array}{llll} y_0 = f[x_0] & & & \\ y_1 = f[x_1] & f[x_0, x_1] & & \\ y_2 = f[x_2] & f[x_1, x_2] & f[x_0, x_1, x_2] & \\ y_3 = f[x_3] & f[x_2, x_3] & f[x_1, x_2, x_3] & f[x_0, x_1, x_2, x_3] \end{array}$$

and the following algorithm, assuming that y_i are stored in a vector $\mathbf{c}$.

```
for j=1 to n
   for i=n to j
      c(i) = (c(i) - c(i-1))/(x(i) - x(i-j));
   end
end
```

Evaluation

Evaluating a Newton form is similar to evaluating a polynomial:

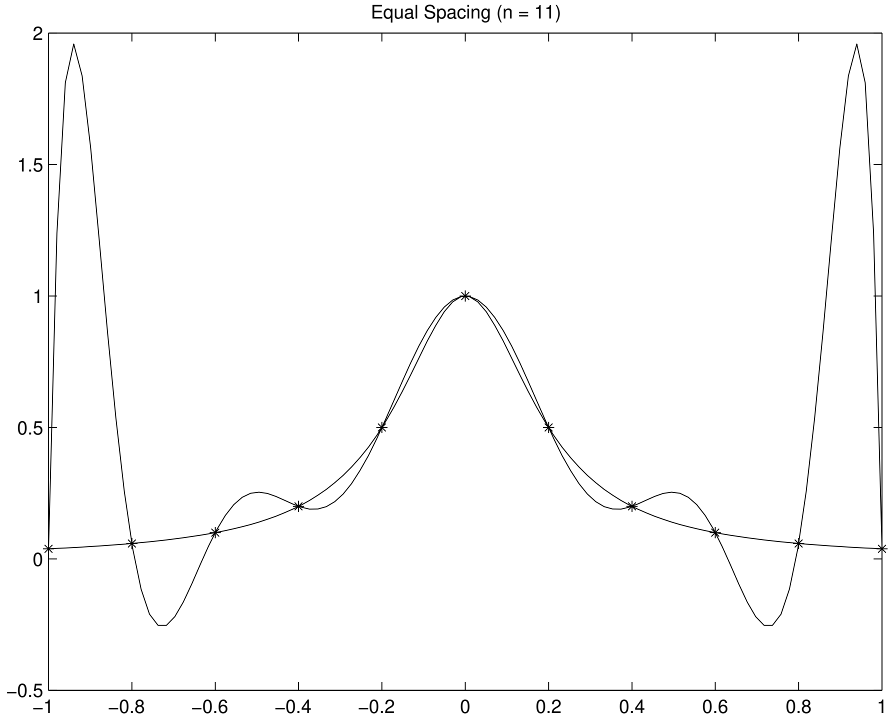
$p = c_n$;
for $i = n - 1$ to 0
    $p = p(t - x_i) + c_i$;
end

The dangers of global polynomial interpolation

An example. Runge's function (continuous derivatives of all order)

$$y(x) = \frac{1}{1 + 25x^2} \qquad \text{on} \quad [-1, 1]$$

equally spaces $x_0 = -1$, $x_1, \cdots, x_n = 1$.



A possible solution: Change the positions of the knots x_i, Chebyshev points instead of equal spacing. But, it does not always work.

- It is often best not to use global polynomial interpolation.

3 Piecewise Polynomial Interpolation

Partition

$$\alpha = x_1 < x_2 < \cdots < x_n = \beta,$$

interpolate on each $[x_i, x_{i+1}]$ with a low degree polynomial.

Linear

Linear polynomial on each subinterval:

$$L_i(z) = a_i + b_i(z - x_i), \quad z \in [x_i, x_{i+1}]$$

$$a_i = y_i \quad b_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

Algorithm. Piecewise linear interpolation. Given vectors **x** and **y** with interpolating points, this function returns the piecewise linear interpolation coefficients in the vectors **a** and **b**.

```
function [a,b] = pwL(x,y)
n = length(x);
a = y(1:n-1);
b = diff(y)./diff(x);
```

Evaluation

Given the piecewise interpolation $L(z)$ represented by the coefficient vectors a, b, how do we evaluate this function at $z \in [\alpha, \beta]$?

First, we locate $[x_i, x_{i+1}]$ such that $z \in [x_i, x_{i+1}]$. Then, we evaluate $L(z)$ using $L_i(z)$.

- Methods: linear search, binary search.
- Observation: If $[x_i, x_{i+1}]$ is associated with the current z, then it is likely that this subinterval will be the one for the next value.
- Idea: Use the previous subinterval as a guess. If not, do binary search.

Algorithm. Given the vector x of breakpoints and a scalar z between x_1 and x_n, this function locates i so that $x_i \le z \le x_{i+1}$. The optional g is a guess.

```
function i = Locate(x,z,g)
  if nargin==3    % try the guess
     if (x(g)<=z) & (z<=x(g+1))
        i = g;
        return    % quick return
     end
  end

  n = length(x);
  if z==x(n)
     i = n-1;     % quick return
  else            % binary search
     left = 1; right = n;
```

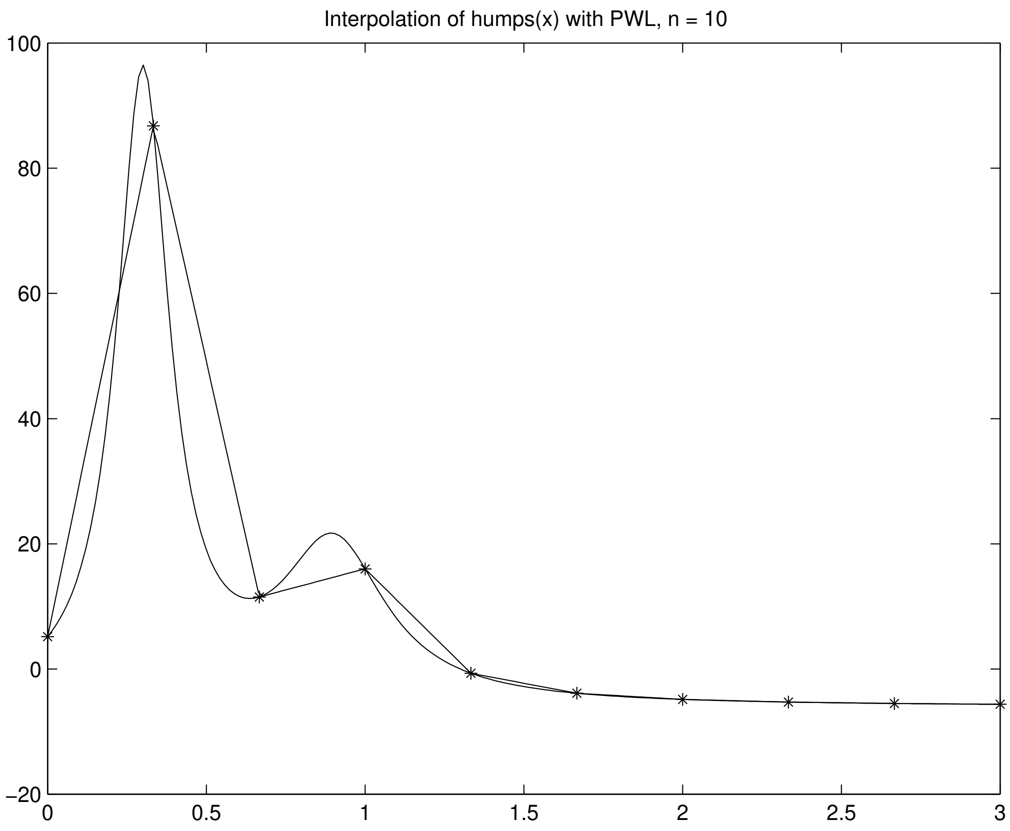
```
    while right > left+1
       mid = floor((left + right)/2);
       if z < x(mid)
          right = mid;
       else
          left = mid;
       end
    end
    i = left;
  end
```

Algorithm. Given a piecewise linear interpolation coefficients in a and b and its breakpoints in x, this function returns the values of the interpolation evaluated at the points in z.

```
function v = pwLEval(a,b,x,z)
  m = length(z);
  v = zeros(m,1);
  g = 1;
  for j=1:m
     i = Locate(x,z(j),g);
     v(j) = a(i) + b(i)*(z(j) - x(i));
     g = i;
  end
```

Example.

$$y = \frac{1}{(x-0.3)^2 + 0.01} + \frac{1}{(x-0.9)^2 + 0.04} - 6$$



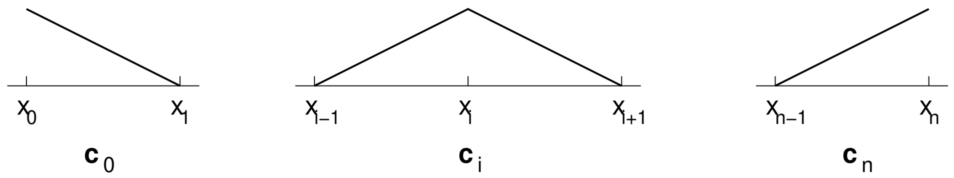
Hat Functions

The hat functions defined by

$$c_i(z) = \begin{cases} (z - x_{i-1})/h_{i-1} & \text{if } z \in [x_{i-1}, x_i], \\ (x_{i+1} - z)/h_i & \text{if } z \in [x_i, x_{i+1}], \\ 0 & \text{elsewhere,} \end{cases}$$

where the hat function $c_0(z)$ omits the first of the above cases; $c_n(z)$, the second, form a basis for piecewise linear polynomial interpolation. A piecewise linear polynomial interpolation $L(z)$ can be expressed as a linear combination of the hat functions

$$L(z) = \sum_{i=0}^{n} y_i c_i(z).$$



4 Natural Cubic Spline

Given $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$, find $s(x)$:

- in each subinterval $[x_i, x_{i+1}]$, $s(x)$ is cubic. There are $n - 1$ intervals, four coeficients each interval, total of $4(n - 1)$ coeficients.
- $s(x_i) = y_i$, $i = 1, ..., n$. For each interval $[x_i, x_{i+1}]$, $s(x_i) = y_i$ and $s(x_{i+1}) = y_{i+1}$. There are $2(n - 1)$ constraints.
- $s'(x)$ and $s''(x)$ are continuous at $x_2, x_3, ..., x_{n-1}$, total of $2(n - 2)$ constraints.
- $s''(x_1) = s''(x_n) = 0$, two constraints.

Constructing $s(x)$

In the subinterval $[x_i, x_{i+1}]$, let

$$h_i = x_{i+1} - x_i, \quad w = (x - x_i)/h_i, \quad \bar{w} = 1 - w.$$

We have introduced a new variable w to replace x. Notice that w: $0 \to 1$, $\bar{w}$: $1 \to 0$, as x: $x_i \to x_{i+1}$

Construct

$$s(x) = wy_{i+1} + \bar{w}y_i + h_i^2[(w^3 - w)\sigma_{i+1} + (\bar{w}^3 - \bar{w})\sigma_i]$$

where σ_i, total of n, to be determined. The first two terms are constructed so that $s(x)$ interpolates (x_i, y_i) and (x_{i+1}, y_{i+1}) and the last two terms are constructed so that $s''(x)$ interpolates $(x_i, 6\sigma_i)$ and $(x_{i+1}, 6\sigma_{i+1})$. Specifically, using $w' = 1/h_i$ and $\bar{w}' = -1/h_i$, we can verify

1. $s(x_i) = y_i$, $s(x_{i+1}) = y_{i+1}$, independent of σ, which leads to the continuity of $s(x)$.
2. $s''(x) = 6w\sigma_{i+1} + 6\bar{w}\sigma_i$.

Clearly $s''(x_i) = 6\sigma_i$, meaning $s''(x)$ is continuous. It then remains to make $s'(x)$ continuous.

Consider

$$s'(x) = \frac{y_{i+1} - y_i}{h_i} + h_i[(3w^2 - 1)\sigma_{i+1} - (3\bar{w}^2 - 1)\sigma_i]$$

Let $\Delta_i = (y_{i+1} - y_i)/h_i$. On $[x_i, x_{i+1}]$,

$$s'_+(x_i) = \Delta_i + h_i(-\sigma_{i+1} - 2\sigma_i).$$

On $[x_{i-1}, x_i]$,

$$s'_-(x_i) = \Delta_{i-1} + h_{i-1}(2\sigma_i + \sigma_{i-1}).$$

Making $s'(x)$ continuous

$$s'_+(x_i) = s'_-(x_i), \quad i = 2, 3, ..., n-1$$

we get $n-2$ equations

$$h_{i-1}\sigma_{i-1} + 2(h_{i-1} + h_i)\sigma_i + h_i\sigma_{i+1} = \Delta_i - \Delta_{i-1}$$

for $i = 2, 3, ..., n-1$. Set $\sigma_1 = \sigma_n = 0$.

Matrix form
Diagonal: $[2(h_1 + h_2), \cdots, 2(h_{n-2} + h_{n-1})]$
supper/subdiagonal: $[h_2, \cdots, h_{n-2}]$
unknowns: $[\sigma_2, \cdots, \sigma_{n-1}]^{\mathrm{T}}$
right-hand side: $[\Delta_2 - \Delta_1, \cdots, \Delta_{n-1} - \Delta_{n-2}]^{\mathrm{T}}$

- The matrix is symmetric
- The matrix is tridiagonal
- The matrix is diagonally dominant when $x_1 < x_2 < \cdots < x_n$
- Can apply Gaussian elimination without pivoting

Evaluating $s(x)$
If $s(x)$ is evaluated many times, arrange $s(x)$ so that

$$s(x) = y_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

for $x_i \le x \le x_{i+1}$
and store b_i, c_i, d_i (instead of σ_i)

$$b_i = \frac{y_{i+1} - y_i}{h_i} - h_i(\sigma_{i+1} + 2\sigma_i), \quad c_i = 3\sigma_i, \quad d_i = \frac{\sigma_{i+1} - \sigma_i}{h_i},$$

for $i = 1, 2, ..., n-1$

Algorithm. Given a vector $\mathbf{x}$ with breakpoints and vector $\mathbf{y}$ with function values, this algorithm computes the coefficients $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ of natural spline interpolation.

1. Compute h_i and Δ_i;
2. Form the tridiagonal matrix (two arrays) and the right hand side;
3. Solve for σ_i;
4. Compute the coefficients $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$.

Summary

- Global polynomial interpolation: General ideas and methods
- Piecewise polynomial interpolation: Construction of piecewise polynomial (linear and cubic), evaluation of a piecewise function.
- Basis functions: Lagrange polynomials and hat functions.

Software packages

IMSL csint, csdec, csher, csval

MATLAB polyfit, spline, ppval

NAG e01aef, e01baf, e01bef, e02bbf, e01bff

References

George E. Forsythe, Michael A. Malcolm, and Cleve B. Moler, *Computer Methods for Mathematical Computations*, Prentice-Hall, Inc., 1977.

Charles F. Van Loan, *Introduction to Scientific Computing*, Prentic-Hall, 1997.