Assignment 4

Due. In class Nov. 30 (Friday)

1. Apply two iterations of the implicit QR method using the Wilkinson shift to the matrix

$$T = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

and show the updated T.

Solution
Iteration 1:
Shift: 1
The first rotation:

$$\begin{bmatrix} 0.7071 & 0.7071 & 0 \\ -0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the second rotation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

The updated T

$$\begin{bmatrix} 3 & 0.7071 & 0 \\ 0.7071 & 2 & -0.7071 \\ 0 & -0.7071 & 1 \end{bmatrix}.$$

Iteration 2:
Shift: 0.6340
The first rotation:

$$\begin{bmatrix} 0.9581 & 0.2863 & 0 \\ -0.2863 & 0.9581 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the second rotation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8426 & 0.5385 \\ 0 & 0.5385 & 0.8426 \end{bmatrix}.$$

The updated T

$$\begin{bmatrix} 3.3060 & 0.3760 & 0 \\ 0.3760 & 2.1076 & -0.0304 \\ 0 & -0.0304 & 0.5864 \end{bmatrix}.$$

2. This question is the continuation of Question 3 in Assignment 2.

Since the sound speed varies with depth, sound rays will travel in curved paths. The effect is continuous version of the familiar refraction of light waves caused by the air-water interface in a fish bowl. The basic equation is a continuous version of Snell's law. Let x denote the horizontal (radial) distance in feet from a source of sound and $z(x)$ denote the depth of a particular ray at distance x. Let $\theta = \theta(x)$ denote the angle between a horizontal line and the tagent to the ray at x, i.e.,

$$\tan\theta = \frac{dz}{dx}.$$

Snell's law can be written

$$\frac{\cos\theta}{c(z)} = \text{constant}.$$

These two equations together yield a first-order ordinary differential equation which appears to define z as a function of x. However, it turns out that this equation fails to have unique solutions at those points where the ray becomes horizontal. To eliminate the unwanted solutions, we differentiate both equations with respect to x and combine the results to obtain a second order equation

$$\frac{d^2z}{dx^2} = -\frac{c'(z)}{A^2c(z)^3},$$

where A is the constant occuring in Snell's law. The constant and the initial conditions can be conveniently expressed in terms of z_0, the depth of the source, and θ_0, the ray angle at the source. We find

$$\begin{aligned} z(0) &= z_0, \\ \frac{dz}{dx}(0) &= \tan\theta_0, \\ A^2 &= \left(\frac{\cos\theta_0}{c(z_0)}\right)^2. \end{aligned}$$

This brings to the second part of the problem.

(b) Use `ode45` to find the ray with $z_0 = 2000$ feet and $\theta_0 = 5.4$ degrees. Trace the ray over a distance of 24 nautical miles, plotting its depth. Assume 1 nautical mile is 6076 feet. You should find that the depth at 24 miles is close to 3000 feet.

Now suppose that a sound source at a depth of 2000 feet transmits to a receiver 24 miles away at a depth of 3000 feet. The above calculation shows that one of the rays from the source to the receiver leaves the source at an angle close to 5.4 degrees. How many other rays are there? Let $x_f = 24$ nautical miles. As θ_0 varies, $z(x_f)$ also varies. We are interested in finding values of θ_0 for which $z(x_f) = 3000$.

(c) Write a function `zxf(theta)`, which traces the ray with initial angle `theta` and returns the value $z(x_f) - 3000$. Plot the values of this function for `theta` in the range -10 to 10 degrees. Since this function is fairly expensive to evaluate, the increment used in this plot will depend on the amount of computer time you have available and the efficiency of your program.

(d) Use `fzero` with starting values obtained from part (c) to find rays which pass through (or as near as possible to) the receiver.

Solution Setting $y_1 = z$ and $y_2 = z'$, we have the following system of first order ODEs

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_2 \\ -\frac{c'(y_1)}{A^2c(y_1)^3} \end{bmatrix},$$

with the initial condition

$$y_1(0) = z_0 \quad \text{and} \quad y_2(0) = \tan\theta_0.$$

The above system can be solved by `ode45`.

```
% depth data, ft
z = [0.0; 500; 1000; 1500; 2000; 2500; 3000; 3500; 4000;
    5000; 6000; 7000; 8000; 9000; 10000; 11000; 12000];
global Z
Z = z;
% speed data, ft/sec
cz = [5042; 4995; 4948; 4887; 4868; 4863; 4865; 4869; 4875;
      4875; 4887; 4905; 4918; 4933; 4949; 4973; 4991];
global CZ
CZ = cz;

% constants
odetol = 0.001;   % tolerance in ode45, part (b)
thetastep = 0.1;  % step size for angle, part(c)

%% (b) %%
% use RKF45 to trace the depth the ray starting with depth 2000 feet
% and angle 5.4 degrees over a distance of 24 nautical miles.
xInit = 0.0;             % source position, from the source
xFinal = 6076*24;        % 24 nautical miles, in ft, from source
zInit = 2000.0;          % initial depth
thetaInit = 5.4*pi/180;  % initial angle
% constant A
[czInit, dczInit] = seval(zInit,z,cz,b,c,d);
                         % initial speed using spline
A = cos(thetaInit)/czInit;
Asq = A*A;
global ASQ
ASQ = Asq;
%
% solve the differential equation
xspan = [xInit; xFinal];
dzInit = tan(thetaInit);
yInit = [zInit; dzInit]; % initial condition
[x,zx] = ode45(@deqn, xspan, yInit, odetol);
% print table
indx(1) = 1;
k = 2;
for i=1:23
    while ((x(k) >= (i-1)*6076) & (x(k) < i*6076))
        k = k + 1;
    end
    indx(i+1) = k;
```

```
end
indx(25) = length(x);
Table = [x(indx)/6076, zx(indx)’];
format short
Table,
% figure
%plot(x, zx(:,1))
%xlabel(’distance from source (ft.)’), ylabel(’depth (ft.)’),

%% (c) %%
% depth, as a function of the angle theta, at 24 miles from
% the source. The dept is biased by 3000 ft.
theta = [-10:thetastep:10];
d = zxf(theta);
% figure
%plot(theta, d)
%xlabel(’angle (degrees)’), ylabel(’depth - 3000 (ft.)’),

%% (d) %%
% find rays which pass through the receiver
% From (c), the approximations are
t0 = [-8.3, -4.3, -3.2, -2.8, 3.7, 5.4, 7.2];
for k=1:length(t0)
    ray(k) = fzero(@zxf, t0(k), fzerotol);
end
% print
ray,

%-------------------------------------------------------------------
% subfunctions
%-------------------------------------------------------------------

%%%%%%%%%%%%%
% zxf       %
%%%%%%%%%%%%%
function d = zxf(theta)
%
% input
%    theta    vector of angles, in degrees
% output
%    d        depths to 3000 ft at 24 miles from the source

global Z
global CZ
global B
global C
global D
```

```
% use RKF45 to find the depth
xInit = 0.0;
xFinal = 6076*24;        % 24 nautical miles, in ft
zInit = 2000.0;          % initial depth
[czInit, dczInit] = seval(zInit,Z,CZ,B,C,D);

for k = 1:length(theta);
   thetaInit = theta(k)*pi/180;
   A = cos(thetaInit)/czInit;
   Asq = A*A;
   global ASQ
   ASQ = Asq;
%
   xspan = [xInit xFinal];
   dzInit = tan(thetaInit);
   yInit = [zInit; dzInit]; % initial condition
   [x,zx] = ode45(@deqn, xspan, yInit);
%
   d(k) = zx(length(x)) - 3000;
end

%%%%%%%%%%%%%%
% deqn       %
%%%%%%%%%%%%%%
function yprime = deqn(x,y)
% yprime = deqn(x,y)
%
% This function defines the system of differential equations
% corresponding to the second order ordinary differential equation

% inputs
%    x    distance from the source, scalar variable
%    y    a 2-vector, y(1) is the depth, y(2) is the derivative:
%
%                y(1) = z   y(2) = dz/dx
%
% output
%    yprime  derivative of y w.r.t. x

global Z
global CZ
global B
global C
global D
global ASQ

% variable declaration
yprime = zeros(size(y));
```

```
% initialization
yprime(1) = y(2);
% find derivative of c(z) using spline
[cz,dcz] = seval(y(1),Z,CZ,B,C,D);
% second derivative of z w.r.t. x
yprime(2) = -dcz/(ASQ*cz*cz*cz);
```