Lattice Basis Reduction
Part II: Algorithms

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Outline

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Hermite reduction (size reduction)

Hermite-reduced

A lattice basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is called size-reduced if its QR decomposition satisfies

$$|r_{i,i}| \geq 2|r_{i,j}|, \quad \text{for all} \quad 1 \leq i < j \leq n,$$

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Procedure $\texttt{Reduce}(i, j)$

$$\begin{bmatrix} r_{i,i} & r_{i,j} \\ & r_{j,j} \end{bmatrix} \begin{bmatrix} 1 & -\left\lfloor \frac{r_{i,j}}{r_{i,i}} \right\rceil \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{i,i} & r_{i,j} - r_{i,i} \left\lfloor \frac{r_{i,j}}{r_{i,i}} \right\rceil \\ & r_{j,j} \end{bmatrix}$$

$$|r_{i,i}| \geq 2 \left| r_{i,j} - r_{i,i} \left\lfloor \frac{r_{i,j}}{r_{i,i}} \right\rceil \right|$$

Gauss reduction

A unimodular transformation

$$\begin{bmatrix} 1 & -\mu \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ -\mu & 1 \end{bmatrix}$$

Also called

Integer Gauss transformation
Integer elementary matrix

Outline

LLL reduction

LLL-reduced

A lattice basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is called LLL-reduced if it is size-reduced and R in the QR decomposition satisfies

$$r_{i+1,i+1}^2 + r_{i,i+1}^2 \geq \omega \, r_{i,i}^2$$

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Procedure `SwapRestore`(i)

Find a Givens plane rotation G:

$$G \begin{bmatrix} r_{i-1,i-1} & r_{i-1,i} \\ 0 & r_{i,i} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \hat{r}_{i-1,i-1} & \hat{r}_{i-1,i} \\ 0 & \hat{r}_{i,i} \end{bmatrix}.$$

Unimodular transformation: Permutation

LLL algorithm

```
k = 2;
while k <= n {
   if |r(k-1,k) / r(k-1,k-1)| > 1/2

   if r(k,k)^2 + r(k-1,k)^2 < w*r(k-1,k-1)^2 {


   } else {





   }
}
```

LLL algorithm

```
k = 2;
while k <= n {
   if |r(k-1,k) / r(k-1,k-1)| > 1/2
      Reduce(k-1,k);
   if r(k,k)^2 + r(k-1,k)^2 < w*r(k-1,k-1)^2 {


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      Reduce(k-1,k);
   if r(k,k)^2 + r(k-1,k)^2 < w*r(k-1,k-1)^2 {
      SwapRestore(k);
      k = max(k-1, 2);
   } else {




   }
}
```

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   if |r(k-1,k) / r(k-1,k-1)| > 1/2
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      SwapRestore(k);
      k = max(k-1, 2);
   } else {
      for i = k-2 downto 1
         if |r(i,k) / r(i,i)| > 1/2
            Reduce(i,k);
      k = k+1;
   }
}
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   }
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```

Redundant size reductions.

An improvement: Delayed size reduction

```
k = 2;
while k <= n
    g = round(r(k-1,k) / r(k-1,k-1));
    if r(k,k)^2 + (r(k-1,k) - g*r(k-1,k-1))^2 <
                                          w*r(k-1,k-1)^2
        ReduceSwapRestore(k);
        k = max(k-1, 2);
    else
        k = k + 1;

for k = 2 to n
    for i = k-1 downto 1
        if |r(i,k) / r(i,i)| > 1/2
            Reduce(i,k);
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                                         w*r(k-1,k-1)^2
        ReduceSwapRestore(k);
        k = max(k-1, 2);
    else
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for k = 2 to n
    for i = k-1 downto 1
        if |r(i,k) / r(i,i)| > 1/2
            Reduce(i,k);
```

Produces identical results at 50% cost.

Outline

HKZ reduction

HKZ-reduced

A lattice basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is called HKZ-reduced if it is size-reduced and for each trailing $(n-i+1) \times (n-i+1)$, $1 \leq i < n$, submatrix of R in the QR decomposition, its first column is a shortest nonzero vector in the lattice generated by the submatrix.

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Two problems

- Shortest vector problem (SVP)
- Expansion to a basis

SVP

$$\min_{z} \|Bz\|_2^2$$

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Sphere decoding

Determine a search sphere

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A simple choice of ρ: the length of the first (or shortest) column of B.

Example

$$Rz = \begin{bmatrix} 4 & 1 & 5 \\ 0 & 4 & 4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$\rho = 4$

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A necessary condition for z_3: $|3z_3| \leq 4$.

Possible values of z_3: $0, -1, 1$

Example

For each possible values of z_3, say $z_3 = 0$,

$$Rz = \begin{bmatrix} 4 & 1 & 5 \\ 0 & 4 & 4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 0 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + 0 \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$$

The problem size is reduced.

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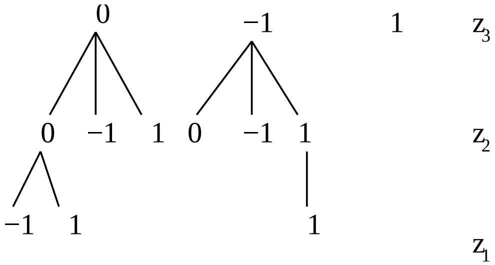
The problem size is reduced.

The necessary condition for z_2: $|4z_2| \leq 4$

Possible values of z_2: $0, -1, 1$

Example

The search tree



The solution

$$Rz = \begin{bmatrix} 4 & 1 & 5 \\ 0 & 4 & 4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$

Expanding to a basis

Problem:
Transform the basis matrix

$$A = \begin{bmatrix} 4 & 1 & 5 \\ 0 & 4 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

into a new basis matrix whose first column is the shortest vector

$$A\mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$

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That is, find a unimodular matrix Z: $A\mathbf{z} = AZ\mathbf{e}_1$ or

$$\mathbf{z} = Z\mathbf{e}_1, \qquad Z^{-1}\mathbf{z} = \mathbf{e}_1$$

Unimodular transformation that introduces zeros into an integer vector.

A plane unimodular transformation

A unimodular transformation (Luk, Zhang, and Q, 2010).

$\gcd(p, q) = \pm d$, $ap + bq = \pm d$.

Form the unimodular matrix

$$\begin{bmatrix} a & b \\ -q/d & p/d \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} d \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} a & b \\ -q/d & p/d \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} d \\ 0 \end{bmatrix}$$

Its inverse

$$\begin{bmatrix} p/d & -b \\ q/d & a \end{bmatrix}$$

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 5 \\ 0 & 4 & 4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 5 \\ 0 & 0 & 4 \\ 0 & -3 & 3 \end{bmatrix}$$

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Improving Kannan's algorithm

Kannan, 1987

Expansion method

In the kth, $k = 1, ..., n$, recursion, solve a k-dim system ($O(k^3)$).

Total $O(n^4)$

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Determine whether a set of vectors are linearly dependent.

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Our method

Efficient, $O(n^2)$

Accurate, unimodular (integer) transformations.

Properties

- Efficient.
- Exact, integer arithmetic.
- Include permutation and identity as special cases.
- Can triangularize an integer matrix.
- Any unimodular can be decomposed into a product of this plan unimodular and integer Gauss transformations.

Application

Cryptography

Find a large vector

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$$Z^{-1}v = \begin{bmatrix} 997 & -1 & 0 \\ 1234 & 0 & -543 \\ 6789 & 0 & -24989 \end{bmatrix}^{-1} v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

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Choose a diagonal A as a private key

$B = AZ$ (ill-conditioned) as public key

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Minkowski reduction

Minkowski-reduced

A lattice basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is called Minkowski-reduced if for each $\mathbf{b}_i$, $i = 1, 2, \ldots, n$, its length

$$||\mathbf{b}_i||_2 = \min(||\hat{\mathbf{b}}_i||_2, ||\hat{\mathbf{b}}_{i+1}||_2, \ldots, ||\hat{\mathbf{b}}_n||_2)$$

over all sets $\{\hat{\mathbf{b}}_i, \hat{\mathbf{b}}_{i+1}, \ldots, \hat{\mathbf{b}}_n\}$ of lattice points such that $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{i-1}, \hat{\mathbf{b}}_i, \hat{\mathbf{b}}_{i+1}, \ldots, \hat{\mathbf{b}}_n\}$ form a basis for the lattice.

Existing Minkowski reduction algorithms

- Lagrange, 1773, dimension two
- Semaev, 2001, dimension three
- Nguyen and Stehleé, 2009, dimension four
- Afflerbach and Grothe, 1985, up to dimension seven
- Helfrish, 1985, theoretical value, very expensive

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Zhang, Q, Wei, 2011

Problem

For $p = 1, 2, ..., n$, find $\mathbf{b}_p$: a shortest vector such that $\{\mathbf{b}_1, ..., \mathbf{b}_p\}$ can be extended to a basis for the lattice.

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Algorithm:

for $p = 1...n$
  find a shortest $\mathbf{v} = B\mathbf{z}$ such that
    $\{\mathbf{b}_1, ..., \mathbf{b}_{p-1}, \mathbf{v}\}$ is expandable to a basis;
  set $\mathbf{b}_p = \mathbf{v}$ and expand $\{\mathbf{b}_1, ..., \mathbf{b}_p\}$ to a basis;
end

Minkowski reduction algorithm

A proposition:

Let $B = [\mathbf{b}_1, ..., \mathbf{b}_n]$ be a generator matrix for a lattice L and a lattice vector $\mathbf{v} = B\mathbf{z}$, then $\{\mathbf{b}_1, ..., \mathbf{b}_{p-1}, \mathbf{v}\}$ is expandable to a basis for L if and only if $\gcd(z_p, ..., z_n) = \pm 1$.

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Constrained minimization problem:

$$\min_{\mathbf{z}} \|B\mathbf{z}\|_2 \quad \text{subject to} \quad \gcd(z_p, ..., z_n) = \pm 1$$

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Constrained minimization problem:

$$\min_{\mathbf{z}} \|B\mathbf{z}\|_2 \quad \text{subject to} \quad \gcd(z_p, ..., z_n) = \pm 1$$

Modified sphere decoding:

While searching for short lattice vectors, enforce the condition $\gcd(z_p, ..., z_n) = \pm 1$.

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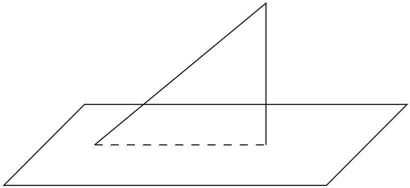
Condition # ignores intermediate singular values of $n \times n$ matrix.

An interpretation

$$\begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} & r_{1,4} \\ 0 & r_{2,2} & r_{2,3} & r_{2,4} \\ 0 & 0 & r_{3,3} & r_{3,4} \\ 0 & 0 & 0 & r_{4,4} \end{bmatrix}$$

$$\sin \theta_i = \frac{|r_{i,i}|}{\|r_{:,i}\|_2}$$



Measurement

In particular, the geometric mean σ:

$$\sigma^n = \prod_{i=1}^{n} \sin\theta_i = \prod_{i=1}^{n} \frac{|r_{i,i}|}{\|r_{:,i}\|_2} = \frac{d(L)}{\prod_{i=1}^{n} \|b_i\|_2}$$

Hadamard's inequality

$$\det(B) \le \prod_{i=1}^{n} \|b_i\|_2$$

The equality holds if and only if b_i are orthogonal.

Also called Hadamard ratio or orthogonality defect.

Measurement

- Note that $0 \leq \sigma \leq 1$, $\sigma = 1$ for any diagonal matrix, and $\sigma = 0$ for any singular matrix.

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- Since $V_n = \prod_{i=1}^{n} |r_{i,i}| = d(L)$ is a constant for a given L, we can improve σ by reducing $\|b_i\|_2$.
- Possible measurements other than the geometric mean?

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Questions?