Programming Abstraction in C++

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2010
Chapter 5. Introduction to Recursion
Introduction

A technique in which large problems are solved by reducing them to smaller problems of the same form.
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What is large? What is small?

A measurement of the size of the problem.
A technique in which large problems are solved by reducing them to smaller problems of the same form.

What is large? What is small?

A measurement of the size of the problem.

The smaller problems must be of the same form as the large problem.
Factorial function

The function \( f(n) = n! \).

Function prototype

\[
\text{int Fact(int n);}
\]

An iterative (nonrecursive) implementation

\[
\text{int Fact(int n) \{ int product;}
\]
\[
\text{product = 1;}
\]
\[
\text{for (int i = 1; i <= n; i++) \{ product *= i;}
\]
\[
\text{return product;}
\]}
Factorial function (cont.)

The recursive formulation: \( n! = n \times (n - 1)! \)

A large problem (size \( n \)) is reduced to a smaller problem (size \( n - 1 \)) of the same form (factorial).

Stopping point (simple case, trivial case): \( 0! = 1 \)
The recursive formulation: \( n! = n \times (n - 1)! \)

A large problem (size \( n \)) is reduced to a smaller problem (size \( n - 1 \)) of the same form (factorial).

Stopping point (simple case, trivial case): \( 0! = 1 \)

A recursive definition

\[
n! = \begin{cases} 
1 & \text{if } n = 0 \\
n(n - 1)! & \text{otherwise}
\end{cases}
\]
A recursive implementation

```c
int Fact(int n) {
    if (n == 0) {
        return 1;
    } else {
        return n * Fact(n-1);
    }
}
```
Tracing the recursive process

\[ f = \text{Fact}(4) \]

<table>
<thead>
<tr>
<th>n</th>
<th>( n \ast \text{Fact}(3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>( n \ast \text{Fact}(3) )</td>
</tr>
<tr>
<td>3</td>
<td>( n \ast \text{Fact}(2) )</td>
</tr>
<tr>
<td>2</td>
<td>( n \ast \text{Fact}(1) )</td>
</tr>
<tr>
<td>1</td>
<td>( n \ast \text{Fact}(0) )</td>
</tr>
<tr>
<td>0</td>
<td>return 1</td>
</tr>
</tbody>
</table>

A stack of frames.
Outline

1. Factorial Function
2. Fibonacci Sequence
3. Additive Sequences
4. Other Examples
5. Binary Search
6. Mutual Recursion
Fibonacci sequence

The sequence: \( t_0, t_1, t_2, \ldots \)

\[ t_n = t_{n-1} + t_{n-2}, \quad t_0 = 0, \quad t_1 = 1. \]
Fibonacci sequence

The sequence: \( t_0, t_1, t_2, \ldots \)

\[
t_n = t_{n-1} + t_{n-2}, \quad t_0 = 0, \quad t_1 = 1.
\]

A recursive definition

\[
t_n = \begin{cases} 
  n & \text{if } n \text{ is 0 or 1} \\
  t_{n-1} + t_{n-2} & \text{otherwise}
\end{cases}
\]
Function prototype

```c
int Fib(int n);
```

A recursive implementation

```c
int Fib(int n) {
    if (n < 2) {
        return n;
    } else {
        return (Fib(n - 1) + Fib(n - 2));
    }
}
```

Figure 5-1, p. 184.
Redundancy

\[ \text{Fib}(5) \text{ calls Fib}(4) \text{ and Fib}(3) \]
\[ \text{Fib}(4) \text{ calls Fib}(3) \text{ and Fib}(2) \]
\[ \text{Fib}(3) \text{ calls Fib}(2) \text{ and Fib}(1) \ldots \]
Redundancy

\[
\begin{align*}
\text{Fib}(5) & \text{ calls } \text{Fib}(4) \text{ and } \text{Fib}(3) \\
\text{Fib}(4) & \text{ calls } \text{Fib}(3) \text{ and } \text{Fib}(2) \\
\text{Fib}(3) & \text{ calls } \text{Fib}(2) \text{ and } \text{Fib}(1) \ldots
\end{align*}
\]

one call to \(\text{Fib}(4)\)
two calls to \(\text{Fib}(3)\)
three calls to \(\text{Fib}(2)\)
five calls to \(\text{Fib}(1)\)
three calls to \(\text{Fib}(0)\)
Fib(5) calls Fib(4) and Fib(3)
Fib(4) calls Fib(3) and Fib(2)
Fib(3) calls Fib(2) and Fib(1) …

one call to Fib(4)
two calls to Fib(3)
three calls to Fib(2)
five calls to Fib(1)
three calls to Fib(0)

Is recursion inefficient?
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Additive sequence

A generalization of the Fibonacci sequence.

Given $t_0$ and $t_1$, $t_n = t_{n-1} + t_{n-2}$.

Function prototype

```cpp
AdditiveSequence(int n, int t0, int t1);
```
Additive sequence

A generalization of the Fibonacci sequence.

Given \( t_0 \) and \( t_1 \), \( t_n = t_{n-1} + t_{n-2} \).

Function prototype

\[
\text{AdditiveSequence}(\text{int } n, \text{int } t0, \text{int } t1);
\]

The Fibonacci sequence is a special case where \( t_0 = 0 \) and \( t_1 = 1 \).

Wrapper function

\[
\text{int Fib}(\text{int } n) \{
    \text{return AdditiveSequence}(n, 0, 1)
\}
\]
Additive sequence (cont.)

An observation:
The $n$th term in an additive sequence

$$t_0, t_1, t_2, t_3, ...$$

is the $(n - 1)$st term in the additive sequence

$$t_1, t_2, t_3, ... \quad t_2 = t_0 + t_1$$
Additive sequence (cont.)

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\[
t_0, t_1, t_2, t_3, \ldots
\]
is the \((n - 1)\)st term in the additive sequence
\[
t_1, t_2, t_3, \ldots \quad t_2 = t_0 + t_1
\]

Implementation

```c
int AdditiveSequence(int n, int t0, int t1) {
    if (n == 0) return t0;
    if (n == 1) return t1;
    return AdditiveSequence(n-1, t1, t0 + t1);
}
```

Still a recursion, but no redundant calls!
Additive sequence (cont.)

An observation:
The $n$th term in an additive sequence

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Implementation

```cpp
int AdditiveSequence(int n, int t0, int t1) {
    if (n == 0) return t0;
    if (n == 1) return t1;
    return AdditiveSequence(n-1, t1, t0 + t1);
}
```

Still a recursion, but no redundant calls!

Question: What happens if the `if (n == 1)` check is missing?
Additive sequence (cont.)

What makes the difference?
What makes the difference?

- \texttt{Fib(int n)} on p. 184 makes two overlapping recursive calls;
- \texttt{Fib(int n)} on p. 186 makes one recursive call.
Additive sequence (cont.)

What makes the difference?

- \( \text{Fib}(\text{int } n) \) on p. 184 makes two overlapping recursive calls;
- \( \text{Fib}(\text{int } n) \) on p. 186 makes one recursive call.

Note. Deep recursion can cause stack overflow.
A recursive formulation

- The first and last characters are the same.
- The substring generated by removing the first and last is a Palindrome.
A recursive formulation

- The first and last characters are the same.
- The substring generated by removing the first and last is a Palindrome.

Stopping points (trivial cases, simple cases):

Since we remove two characters (first and last) at a time, we end up with either a single-character string or an empty string.
An implementation

```cpp
bool IsPalindrome(string str) {
    int len = str.length();

    if (len <= 1) {
        return true;
    } else {
        return ((str[0] == str[len - 1]) &&
                 IsPalindrome(str.substr(1, len - 2)));
    }
}
```
Improving efficiency

Using

- the positions of the first and last in the currently active substring.
- a wrapper.

Advantages of CheckPalindrome, p. 189:

- Calculate the length of the input string once;
- Avoid calling substr to make copy of substring.

IsPalindrome, Figure 5-4, p. 189.
Improving efficiency

Using

- the positions of the first and last in the currently active substring.
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**Advantages of CheckPalindrome, p. 189:**

- Calculate the length of the input string once;
- Avoid calling `substr` to make copy of substring.

**IsPalindrome, Figure 5-4, p. 189.**

Why wrapper function?

Hide implementation. The interface of **IsPalindrome** is unlikely to be changed.
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Binary search

Search for an element in an integer array sorted in ascending order.
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A recursive formulation:
Split the array in the middle, search the left half or right half depending on the given value.
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Split the array in the middle, search the left half or right half depending on the given value.

Stopping point

- The mid-entry is the element.
- No elements in the active part of the array.
Binary search

Search for an element in an integer array sorted in ascending order.

A recursive formulation:
Split the array in the middle, search the left half or right half depending on the given value.

Stopping point
- The mid-entry is the element.
- No elements in the active part of the array.

Wrapper:

```c
int FindIntInSortedArray(int key, int array[], int n) {
    return BinarySearch(key, array, 0, n-1);
}
```
int BinarySearch(int key, int array[], int low, int high) {
    if (low > high) return -1;

    int mid = (low + high) / 2;
    if (key == array[mid]) return mid;
    if (key < array[mid]) {
        return BinarySearch(key, array, low, mid - 1);
    } else {
        return BinarySearch(key, array, mid + 1, high);
    }
}
Outline

1. Factorial Function
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6. Mutual Recursion
Mutual recursion

A general recursion.

Example.

$f$ calls $g$ and $g$ calls $f$.

Function $\text{IsEven}$, Figure 5-6, p. 192.
bool IsEven(unsigned int n) {
    if (n == 0) {
        return true;
    } else {
        return IsOdd(n - 1);
    }
}

bool IsOdd(unsigned int n) {
    return !IsEven(n);
}
bool IsEven(unsigned int n) {
    if (n == 0) {
        return true;
    } else {
        return IsOdd(n - 1);
    }
}

bool IsOdd(unsigned int n) {
    return !IsEven(n);
}

Questions:

What happens if if (n == 0) check is missing in IsEven?

What happens if if (n == 1) check is added to IsOdd?
Your program should look like

```java
if (test for simple case) {
    solve simple case
} else {
    call this function with smaller size
}
```
Thinking recursively (cont.)

- Find out all possible simple cases (stopping points). The recursion should end with a simple case.
- Test your program for the simple (trivial) cases.
- Determine a measurement of the size of the problem. Decompose a big problem into smaller problems of the same form. Apply the recursive leap of faith to make sure your program generates the complete solution.